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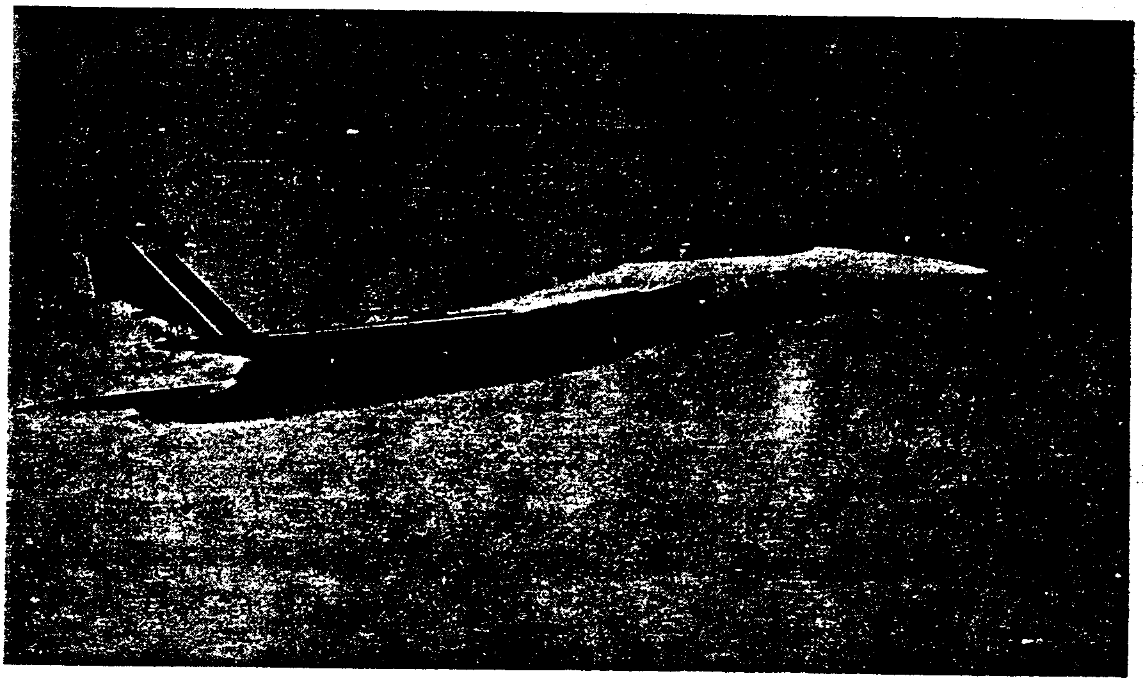
STABILITY AND CONTROL FLIGHT TEST THEORY

VOLUME I OF II



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USAF TEST PILOT SCHOOL,

EDWARDS AIR FORCE BASE, CALIFORNIA

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20. ABSTRACT (*Continue on reverse side if necessary and identify by block number*)

This handbook has been compiled by the instructors of the USAF Test Pilot School for use in the Stability and Control portion of the School's course. Most of the material in Volume I of this handbook has been extracted from several reference books and is oriented towards the test pilot. The flight test techniques and data reduction methods in Volume II have been developed at the Air Force Flight Test Center, Edwards Air Force Base, California.

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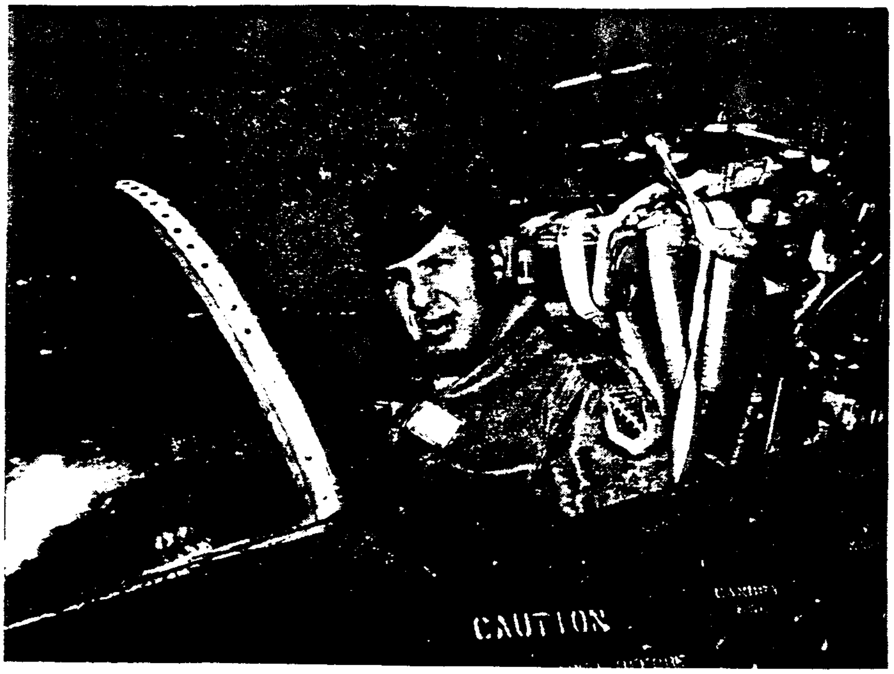

PREFACE

Stability and Control is that branch of the aeronautical sciences that is concerned with giving the pilot an aircraft with good handling qualities. As aircraft have been designed to meet greater performance specifications, new problems in Stability and Control have been encountered. The solving of these problems has advanced the science of Stability and Control to the point it is today.

This handbook has been compiled by the instructors of the USAF Test Pilot School for use in the Stability and Control portion of the School's course. Most of the material in Volume I of this handbook has been extracted from several reference books and is oriented towards the test pilot. The flight test techniques and data reduction methods in Volume II have been developed at the Air Force Flight Test Center, Edwards Air Force Base, California. This handbook is primarily intended to be used as an academic text in our School, but if it can be helpful to anyone in the conduct of Stability and Control testing, be our guest.

WARWICK H. GLASGOW
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table of contents

CHAPTER 1

DIFFERENTIAL EQUATIONS

REVISED FEBRUARY 1977

list of abbreviations and symbols

Item	Definition
x,y,z	variables
t	time in seconds
p	differential operator with dimensions of $(\text{seconds})^{-1}$
j	constant equal to $\sqrt{-1}$
ϕ, θ	angular constant in radians
e	constant equal to $\lim_{x \to 0} (1 + x)^{\frac{1}{x}} = 2.71828. . . . .$
x_t, y_t, z_t	transient solution to differential equation
x_p, y_p, z_p	particular (steady state) solution to differential equation
$\dot{x}$	the dot notation indicates differentiation with respect to time, as in $\dot{x} = \frac{dx}{dt}$.
τ	time constant in seconds
$T_{\frac{1}{2}}$	time to half amplitude in seconds
ζ	damping ratio
ω_n	undamped natural frequency in radians per second
ω_d	damped frequency in radians per second
s	Laplace variable with dimensions of $(\text{seconds})^{-1}$
L	Laplace transform
L^{-1}	inverse Laplace transform
X(s),Y(s),Z(s)	Laplace transform of x(t), y(t), z(t)
$\triangleq$	symbol used for definitions, such as $\dot{x} \triangleq \frac{dx}{dt}$ means $\dot{x}$ is defined as $\frac{dx}{dt}$

1.1 INTRODUCTION

The theory of differential equations is a subject of considerable scope, ranging from the rather simple and obvious through the abstract and not so obvious. One can spend a lifeime studying the subject, and a few people have. We have neither the time, nor perhaps the inclination for such devotions. Our purpose is to cover those aspects of the theory of differential equations which are of direct application to work at the school.

These notes deal with the tools and techniques required to analyze differential equations. Such techniques are easily extended for use in the study of aircraft dynamics. An aircraft in flight displays motions similar to a mass-spring-damper system (figure 1.1). The static stability of the airplane is similar to the spring, the moments of inertia similar to the mass, and the airflow serves to damp the aircraft motion.

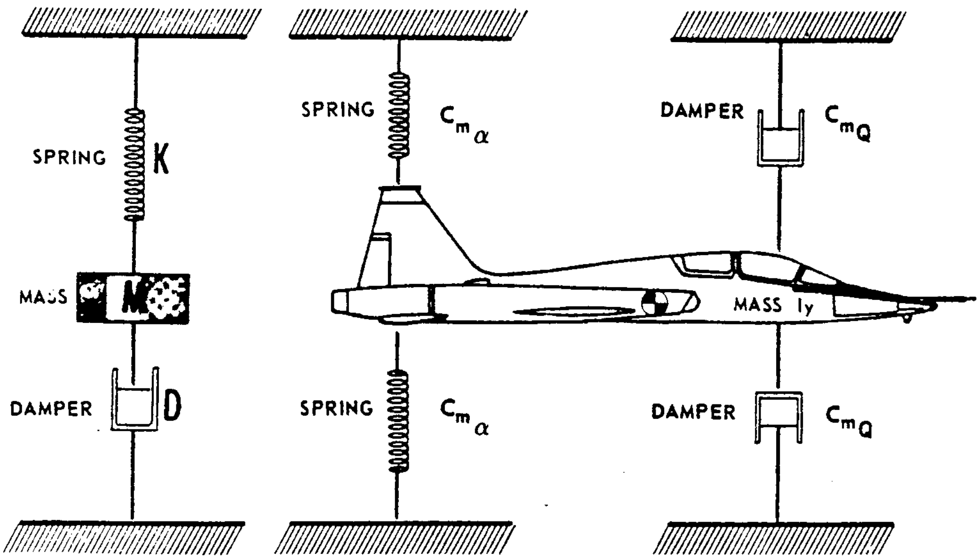


Figure 1.1

This first section provides a review of basic differential equation theory. Succeeding sections deal with operator techniques, analysis of first and second order systems, use of Laplace transforms, and solution of simultaneous equations.

Before proceeding with our study, we shall define several terms which will be used in these notes.

Differential Equation - An equation which involves a dependent variable (or variables) together with one or more of its derivatives with respect to an independent variable (or variables).

Solution - Any function, free of derivatives, which satisfies a differential equation is said to be a solution of the differential equation.

Ordinary Differential Equation - A differential equation which involves derivatives with respect to a single independent variable is called an ordinary differential equation.

Order - The n^{th} derivative of a dependent variable is called a derivative of order n, or an n^{th} order derivative. The order of a differential equation is the order of the highest order derivative present.

Degree - The exponent of the highest order derivative is called the degree of the differential equation.

Linear Differential Equation (ordinary, single dependent variable) - A differential equation in which the dependent variable and its derivatives appear in no higher than the 1st degree, and the coefficients are either constants or functions of the independent variable, is called a linear differential equation.

Linear System - Any physical system that can be described by a linear differential equation is called a linear system.

General Solution - A solution of a differential equation of order n which contains n arbitrary constants will be called a general solution of the differential equation.

1.2 REVIEW OF BASIC PRINCIPLES

Before investigating operator notation and Laplace transforms, let's review the more basic methods of solving differential equations.

1.2.1 DIRECT INTEGRATION

To solve a differential equation we seek a mathematical expression, relating the variables appearing in the differential equation, which qualifies as a solution under the definitions given above. A first thought or inspiration may be: since we are presented with an equation containing derivatives, a solution may be obtained by antidifferentiating or integration. This process removes derivatives and provides arbitrary constants.

EXAMPLE

Given

$$\frac{dy}{dx} = x + 4$$

rewriting

$$dy = (x + 4)dx$$

integrating

$$dy = (x + 4)dx$$

gives us

$$y = \frac{x^2}{2} + 4x + C \tag{1.1}$$

EXAMPLE

Given

$$\frac{d^2y}{dx^2} = x + 4$$

Assume

$$\frac{d^2y}{dx^2} = \frac{dy'}{dx}$$

where

$$y' = \frac{dy}{dx}$$

then

$$\frac{d(y')}{dx} = x + 4$$

or $d(y') = (x + 4)dx$

then

$$y' = \frac{dy}{dx} = \frac{x^2}{2} + 4x + C_1$$

integrating again

$$\int dy = \int (\frac{x^2}{2} + 4x + C_1)\ dx + C_2$$

giving

$$y = \frac{x^3}{6} + 2x^2 + C_1x + C_2 \tag{1.2}$$

Equations 1.1 and 1.2 qualify as general solutions under the definition stated earlier.

Life is full of disappointments and we would soon learn that this direct application of the integration process would fail to work in many cases.

EXAMPLE

$$2xy + (x^2 + \cos y)\frac{dy}{dx} = 0 \qquad (1.3)$$

or

$$dy = \frac{-2xy\,dx}{(x^2 + \cos y)} \qquad (1.4)$$

We cannot perform the integration of the term to the right of the equal sign in equation 1.4. Equation 1.3 can be solved, however, using straightforward techniques. ($x^2 + \sin y = c$ is a general solution.) We emphasize the word "technique" since the solution may rely upon novel approaches, special groupings, or "judicious arrangements" and, perhaps, witchcraft or conjuring. The former require extensive experience and maturity within the discipline, and the latter talents are rarely endowed by nature. We shall study a few special differential equations which are easy to solve and have wide application in the analysis of physical problems.

1.2.2 FIRST ORDER EQUATIONS

We shall consider briefly the first order ordinary differential equation. Suppose we represent such an equation by

$$F(y', y, x) = 0$$

where

$$y' = \frac{dy}{dx}$$

This is concise notation used by mathematicians to denote a differential equation containing an independent variable x, a dependent variable y, and the derivative of y with respect to x. The equation may contain the derivative in differential form.

EXAMPLES

$$\frac{dy}{dx} = x + y$$

$$3x\,dx + 4y\,dy = 0$$

$$y' = \frac{x - y}{x + y}$$

$$\frac{dy}{dx} = \frac{x - y \cos x}{\sin x + y}$$

First order differential equations may be solved by

1. Separating variables and integrating directly.
2. Recognizing exact forms and integrating directly.
3. Finding an integrating factor (fudge factor) which will make the equation exact.
4. Inspection, rearrangement of terms, etc., to use method 1 or 2, or a combination of the two.

These methods are thoroughly treated in all elementary differential equations texts. A brief review of methods 1 and 2 is given below.

● 1.2.2.1 SEPARATION OF VARIBLES

When a differential equation can be put in the form

$$f_1(x)dx + f_2(y)dy = 0 \tag{1.5}$$

where one term contains functions of x and dx only, and the other functions of y and dy only, the variables are said to be separated. A solution of equation 1.5 can then be obtained by direct integration

$$\int f_1(x)dx + \int f_2(y)dy = C \tag{1.6}$$

where C is an arbitrary constant. Note, that for a differential equation of the first order there is one arbitrary constant. In general, the number of arbitrary constants is equal to the order of the differential equation.

EXAMPLE

$$\frac{dy}{dx} = \frac{x^2 + 3x + 4}{y + 6}$$

$$(y + 6)\,dy = (x^2 + 3x + 4)\,dx$$

$$\int (y + 6)dy = \int (x^2 + 3x + 4)dx + C$$

$$\frac{y^2}{2} + 6y = \frac{x^3}{3} + \frac{3x^2}{2} + 4x + C$$

●1.2.2.2 EXACT DIFFERENTIAL EQUATION

Associated with each suitably differentiable function of two variables f(x,y) there is an expression called its total differential, namely,

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \tag{1.7}$$

Conversely, if the differential equation

$$M(x,y)dx + N(x,y)dy = 0 \tag{1.8}$$

has the property that

$$M(x,y) = \frac{\partial f}{\partial x} \text{ and } N(x,y) = \frac{\partial f}{\partial y}$$

then it can be rewritten in the form

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = df = 0$$

from which it follows that

$$f(x,y) = C$$

is a solution. Equations of this sort are said to be exact, since, as they stand, their left members are exact differentials.

A differential equation

$$M(x,y)\ dx + N(x,y)\ dy = 0$$

is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{1.9}$$

If the differential equation

$$M(x,y)\ dx + N(x,y)\ dy = 0$$

is exact, then for all values of k,

$$\int_a^x M(x,y)\ dx + \int_b^y N(a,y)\ dy = k \tag{1.10}$$

is a solution of the equation.

EXAMPLE

Show that the equation

$$(2x + 3y - 2)dx + (3x - 4y + 1)dy = 0$$

is exact and find a general solution.

Applying the test, we find

$$\frac{\partial M}{\partial y} = \frac{\partial(2x + 3y - 2)}{\partial y} = 3$$

$$\frac{\partial N}{\partial x} = \frac{\partial(3x - 4y + 1)}{\partial x} = 3$$

Since the two partial derivatives are equal, the equation is exact. Its solution can be found by means of equation 1.10.

$$\int_a^x (2x + 3y - 2)dx + \int_b^y (3a - 4y + 1)dy = k$$

$$(x^2 + 3xy - 2x)\Big]_a^x + (3ay - 2y^2 + y)\Big]_b^y = k$$

$$(x^2 + 3xy - 2x) - (a^2 + 3ay - 2a) + (3ay - 2y^2 + y) - (3ab - 2b^2 + b) = k$$

$$x^2 + 3xy - 2x - 2y^2 + y = k + a^2 - 2a + 3ab - 2b^2 + b = K$$

1.2.2.3 FIRST ORDER LINEAR DIFFERENTIAL EQUATIONS

We conclude the discussion of first order equations by considering the following form

$$\frac{dy}{dx} + R(x)\ y = 0 \qquad (1.11)$$

where R(x) may be a constant. To solve, merely separate variables.

$$\frac{dy}{y} + R(x)\ dx = 0$$

integrating

$$\int \frac{dy}{y} = -\int R(x)\ dx + C'$$

where

$$C' = \ln C$$

Thus

$$\ln y = -\int R(x)\,dx + \ln C$$

or

$$y = Ce^{-\int R(x)\,dx}$$

If R is a constant, then

$$y = Ce^{-Rx} \tag{1.12}$$

We might conclude from this result that a first order differential equation of form 1.11 with constant coefficients may be solved quite simply. This is true and the solution will always have the form of equation 1.12.

EXAMPLE

$$\frac{dy}{dx} + 2y = 0 \tag{1.13}$$

then we have directly

$$y = Ce^{-2x} \tag{1.14}$$

which is the general solution. It is quickly recognized that the solution is easily obtained by plugging the negative of the coefficient of y into the position indicated by the small square.

PROBLEMS: Set I, Nos. 1 and 2, page 1.76.

1.3 LINEAR DIFFERENTIAL EQUATIONS AND OPERATOR TECHNIQUES

A form of the differential equation that is of particular interest is

$$A_n\frac{d^ny}{dx^n} + A_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ . \ . \ . + A_1\frac{dy}{dx} + A_oy = f(x) \tag{1.15}$$

If the coefficient expressions A_n, A_{n-1}, . . . , A_o are all functions of x only, then equation 1.15 is called a linear differential equation. If the coefficient expressions A_n, . . . , A_o are all constants, then 1.15 is called a linear differential equation with constant coefficients.

EXAMPLE

$$x^2 \frac{d^2y}{dx^2} + 3 \frac{dy}{dx} + xy = \sin x$$

is a linear differential equation.

EXAMPLE

$$\frac{d^2y}{dx^2} + 6 \frac{dy}{dx} + 9y = e^x$$

is a linear differential equation with constant coefficients. Linear differential equations with constant coefficients occur frequently in the analysis of physical systems. Mathematicians and engineers have developed simple and effective techniques to solve this type of equation by using either "classical" or operational methods. When attempting to solve a linear differential equation of the form

$$A_n \frac{d^ny}{dx^n} + A_{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \ . \ . \ . + A_1 \frac{dy}{dx} + A_o y = f(x) \qquad (1.16)$$

it is helpful to examine the equation

$$A_n \frac{d^ny}{dx^n} + A_{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \ . \ . \ . + A_1 \frac{dy}{dx} + A_o y = 0 \qquad (1.17)$$

1.17 is the same as 1.16 with the right hand side zero. We shall refer to 1.16 as the general equation and equation 1.17 as the complementary or homogeneous equation. Solutions of equation 1.17 possess a useful property known as superposition, which may be briefly stated as follows: Suppose $y_1(x)$ and $y_2(x)$ are distinct solutions of 1.17. Then any linear combination of $y_1(x)$ and $y_2(x)$ is also a solution of 1.17. A linear combination would be $C_1y_1(x) + C_2y_2(x)$.

EXAMPLE

$$\frac{d^2y}{dx^2} - 5 \frac{dy}{dx} + 6y = 0$$

It can be verified that $y_1(x) = e^{3x}$ is a solution, and that $y_2(x) = e^{2x}$ is another solution which is distinct from $y_1(x)$. Using superposition, then, $y(x) = c_1e^{3x} + c_2e^{2x}$ is also a solution.

Equation 1.16 may be interpreted as representing a physical system where the left side of the equation describes the natural or designed state of the system, and where the right side of the equation represents the input or forcing function.

One might logically pursue the following line of reasoning in attempting to find a solution to the problem described by equation 1.16.

1. A general solution of 1.16 must contain n arbitrary constants and must satisfy the equation.

2. The following statements are justified by experience:

   a. It is reasonably straightforward to find a solution to the complementary equation 1.17, containing n arbitrary constants. Such a solution will be called the transient solution. Physically, it represents the response present in the system regardless of input.

   b. There are varied techniques for finding a solution of the differential equations due to this forcing function. Such solutions do not, in general, contain arbitrary constants. This solution will be called the particular or steady state solution.

3. If we take the transient solution which describes the response already existing in the system, and then add on the response due to the forcing function, it would appear that a solution so written would blend the two responses and describe the total response of the system represented by 1.16. In fact, the definition of a general solution is satisfied under such an arrangement. This is simply an extension of the principle of superposition. The transient solution contains the correct number of arbitrary constants, and the particular solution guarantees that the combined solutions satisfy the general equation 1.16. Call the transient solution y_t and the particular solution y_p. A general solution of 1.16 is then given by

$$y = y_t + y_p \tag{1.18}$$

1.3.1 TRANSIENT SOLUTION

Equation 1.13 is a complementary or homogeneous first order linear differential equation with constant coefficients. We recognized a quick and simple method of finding a solution to this equation. We also recognized that the solution was always of exponential form. We might hope that solutions of higher order equations of the same family would take the same form.

Let us examine a second order differential equation with constant coefficients to determine if

$$y = e^{mx} \tag{1.19}$$

is a solution of the equation

$$ay'' + by' + cy = 0 \tag{1.20}$$

Substituting $y = e^{mx}$ we have

$$am^2e^{mx} + bme^{mx} + ce^{mx} = 0$$

or

$$(am^2 + bm + c)e^{mx} = 0 \qquad (1.21)$$

Since $e^{mx} \neq 0$

$$am^2 + bm + c = 0 \qquad (1.22)$$

and

$$m_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad (1.23)$$

Substituting these values into our assumed solution we force it to become a solution.

$$y_t = C_1e^{m_1x} + C_2e^{m_2x} \qquad (1.24)$$

When working numerical problems it is not necessary to take the derivatives of e^{mx}, if we remember that the d^ny/dx^n is replaced by m^n. This will be true for any order differential equation with constant coefficients.

We have included the subscript "t" on y to indicate that 1.24 represents the transient solution. From the foregoing it is seen that we have succeeded in extending the method for first order complementary equations to higher order complementary or homogeneous equations. Again we note that we have traded off an integration problem for an algebra problem (solving equation 1.22 for the m's).

Differential or derivative operators can be defined and manipulated to play the same role as m above.

If we designate an operator $p, p^2, \ldots, p^n$ as follows:

$$p = \frac{d}{dx}, \; p^2 = \frac{d^2}{dx^2}, \; \ldots, \; p^n = \frac{d^n}{dx^n} \qquad (1.25)$$

$$p(y) = \frac{dy}{dx}, \; p^2(y) = \frac{d^2y}{dx^2}, \; \ldots, \; p^n(y) = \frac{d^ny}{dx^n} \qquad (1.26)$$

then 1.20 may be written

$$ap^2(y) + bp(y) + cy = 0 \quad (1.27)$$

or, since the derivative operates linearly (each term in succession),

$$(ap^2 + bp + c)y = 0 \quad (1.28)$$

and the operator expression $(ap^2 + bp + c)$ has the same algebraic structure as 1.22. The operator expression in 1.28 is a polynomial with precisely the same form as the polynomial on the left side of 1.22, hence it is often solved directly for the constants required in the solution of 1.19. In this case, the transient solution 1.24 would appear

$$y_t = c_1 e^{p_1 x} + c_2 e^{p_2 x} \quad (1.29)$$

There are cases for which 1.24 and 1.29 are not entirely satisfactory in providing a solution, but this will be discussed later. The m's or p's may be real, imaginary, or complex numbers.

EXAMPLE

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$$

Using operator notation,

$$(p^2 + p - 2)y = 0$$

$$p^2 + p - 2 = 0$$

$$p = 1, -2$$

$$y = c_1 e^x + c_2 e^{-2x}$$

We shall now consider the various cases for solutions of the complementary (homogeneous) equation.

Consider the equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \quad (1.30)$$

We have seen above that the solution of this differential equation is equivalent to solving the characteristic equation

$$ap^2 + bp + c = 0 \quad (1.31)$$

The general solution of 1.30 is of the form

$$x_t = c_1 e^{p_1 x} + c_2 e^{p_2 x} \tag{1.32}$$

where c_1 and c_2 are arbitrary constants, and p_1 and p_2 are solutions of the characteristic equation 1.31. Recall from algebra that a characteristic equation can yield complex roots, imaginary roots, or real roots, that is, $p_{1,2} = [-b \pm \sqrt{b^2 - 4\,ac}]/2a)$. We will consider the solution 1.32 for various values of the constants in equation 1.31 and consider changes in the form of the solution which may be desirable or necessary.

1.3.1.1 CASE 1: ROOTS REAL AND UNEQUAL

If p_1 and p_2 are real and unequal the desired form of solution is just as is

EXAMPLE

$$\frac{d^2y}{dx^2} + 4\,\frac{dy}{dx} - 12y = 0$$

$(p^2 + 4p - 12)y = 0$ (in operator form)

solving

$$p^2 + 4p - 12 = 0$$

gives

$$p = \frac{-4 \pm \sqrt{16 + 48}}{2}$$

$$= \frac{-4 \pm 8}{2}$$

or

$$p = -6,\ 2$$

and

$$y = c_1 e^{-6x} + c_2 e^{2x}$$

is the required solution.

1.3.1.2 CASE 2: ROOTS REAL AND EQUAL

If p_1 and p_2 are real and equal we run into trouble.

EXAMPLE

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$$

$(p^2 - 4p + 4)y = 0$ (in operator form)

solving,

$$p = \frac{4 \pm \sqrt{16 - 16}}{2} = \frac{4}{2} = 2$$

or p = 2. But this gives only one value of p. If we try to use 1.32 all we get is $y = c_1e^{2x}$ but we need two arbitrary constants to have a transient solution like 1.30. If we are really alert, we may notice that the operator expression $(p^2 - 4p + 4)$ can be written $(p - 2)(p - 2)$, or $(p - 2)^2$, which is a polynomial expression with a repeated factor. (that is, p = 2; 2 is the solution.) We can then write $y = c_1e^{2x} + c_2e^{2x}$ as the transient solution. This is really no better than our first attempt, $y = c_1e^{2x}$, since c_1 and c_2 can be combined into a single arbitrary constant.

$$y = c_1e^{2x} + c_2e^{2x} = (c_1 + c_2)e^{2x} = c_3e^{2x}$$

To solve this problem, simply multiply one of the arbitrary constants by x. Now write: $y = c_1e^{2x} + c_2xe^{2x}$. We can no longer "lump" the two coefficients of e^{2x} together. The solution now contains two arbitrary constants, and it is easily verified that

$$y_t = c_1e^{2x} + c_2xe^{2x}$$

is a transient solution of the problem above.

1.3.1.3 CASE 3: ROOTS PURELY IMAGINARY

EXAMPLE

$$\frac{d^2y}{dx^2} + y = 0$$

in operator form

$$(p^2 + 1)y = 0$$

Solving,

$$p = \frac{0 \pm \sqrt{0 - 4}}{2} = \pm \sqrt{-1}$$

In most engineering work we refer to $\sqrt{-1}$ as j. (In mathematical texts it is denoted by i.). Now,

$$p = \pm j$$

and the solution is written

$$y_t = c_1 e^{jx} + c_2 e^{-jx} \tag{1.33}$$

This is a perfectly good solution from a mathematical standpoint, but it is unwieldy and unsuggestive to engineers. A mathematician by the name of Euler worked out this puzzle for us by developing an equation called Euler's identity.

$$e^{jx} = \cos x + j \sin x \tag{1.34}$$

This equation can be restated in many ways geometrically and analytically, and can be verified by adding the series expansion of cos x to the series expansion of j sin x. Now 1.33 may be expressed

$$y_t = c_1 (\cos x + j \sin x) + c_2 [\cos (-x) + j \sin (-x)]$$

$$= (c_1 + c_2) \cos x + j (c_1 - c_2) \sin x \tag{1.35}$$

or

$$y_t = c_3 \cos x + c_4 \sin x \tag{1.36}$$

Equation 1.36 has another interesting form. Let

$$y_t = \sqrt{c_3^2 + c_4^2} \left[\frac{c_3}{\sqrt{c_3^2 + c_4^2}} \cos x + \frac{c_4}{\sqrt{c_3^2 + c_4^2}} \sin x \right] \tag{1.37}$$

Now consider a right triangle with sides labeled as follows:

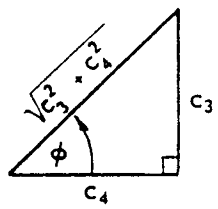


Figure 1.2

Now,

$$\frac{c_3}{\sqrt{c_3^2 + c_4^2}} = \sin \phi$$

$$\frac{c_4}{\sqrt{c_3^2 + c_4^2}} = \cos \phi$$

and

$$\sqrt{c_3^2 + c_4^2} = A.$$

A and ϕ are arbitrary constants, and 1.37 becomes

$$y_t = A\,(\sin \phi \cos x + \cos \phi \sin x)$$

or

$$y_t = A \sin (x + \phi) \qquad (1.38)$$

To summarize, if the roots of the operator polynomial are purely imaginary, they will be numerically equal but opposite in sign, and the solution will have the form 1.36 or 1.38.

1.3.1.4 CASE 4: ROOTS COMPLEX

EXAMPLE

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$$

in operator form,

$$(p^2 + 2p + 2)y = 0$$

Solving,

$$p = \frac{-2 \pm \sqrt{4 - 8}}{2} = -1 \pm \sqrt{-1}$$

or

$$p = -1 + j, \; -1 - j$$

and

$$y_t = c_1 e^{(-1 + j)x} + c_2 e^{(-1 - j)x} \tag{1.39}$$

Equation 1.39 may be written

$$y_t = e^{-x} \left[c_1 e^{jx} + c_2 e^{-jx} \right]$$

or, using the results 1.36 and 1.38,

$$y_t = e^{-x} \left[c_3 \cos x + c_4 \sin x \right] \tag{1.40}$$

or

$$y_t = e^{-x} A \sin (x + \phi) \tag{1.41}$$

Note, also, that 1.38 could be written in the form

$$y_t = A \cos (x + \theta), \text{ where } \theta = \phi - 90°$$

PROBLEMS: Set I, No. 3, page 1.76 ; Set II, a only, page 1.85.

1.3.2 PARTICULAR SOLUTION

The particular solution, for our work here, will be obtained by the method of undetermined coefficients. (There are other methods which may be used.) This method consists of assuming a solution of the same general form as the input (forcing function), but with undetermined coefficients. Substition of this assumed solution into the differential equation then enables us to evaluate these coefficients. The method of undetermined coefficients applies when the forcing function or input is a polynomial, terms of the form sin ax, cos ax, e^{ax}, or combinations of sums and products of these. The complete solution of the linear differential equation with constant coefficients is then given by 1.18 (that is, the solution to the complementary equation (transient solution), plus the particular solution).

A few remarks are appropriate regarding the second order linear differential equation with constant coefficients. Although the equation is interesting in its own right, it is of particular value to us because it is a mathematical model for several problems of physical interest.

$$a \frac{dy^2}{dx^2} + b \frac{dy}{dx} + cy = F(x) \qquad \text{(mathematical model)}$$

$$m \frac{d^2x}{dt^2} + \beta \frac{dx}{dt} + Kx = F(t) \qquad \text{(describes a mass spring damper system)} \qquad (1.42)$$

$$L \frac{d^2Q}{dt^2} + R \frac{dQ}{dt} + \frac{Q}{C} = E(t) \qquad \text{(describes a series LRC electrical circuit)}$$

Equation 1.42 are all the same mathematically, but are expressed in different notation. Different notations or symbols are employed to emphasize the physical parameters involved, or to force the solution to appear in a form that is easy to interpret. In fact, the similarity of these last two equations may suggest how one might design an electrical circuit to simulate the operation of a mechanical system.

Consider the equation

$$a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = f(x) \qquad (1.43)$$

We now must solve for the special solution (particular solution) which results from a given input, $f(x)$. This particular solution can be found by using various techniques, but we will consider only one, the method of undetermined coefficients. This method consists of assuming a solution form with unspecified constants (undetermined coefficients), and solving for the values of the constants which will satisfy the given differential equation. The method is best described by considering examples.

1.3.2.1 FORCING FUNCTION - A CONSTANT

$$\frac{d^2y}{dx^2} + 4 \frac{dy}{dx} + 3y = 6 \qquad (1.49)$$

The input is a constant (trivial polynomial), so we assume a solution of form $y_p = K$. Obviously, $d^2K/dx^2 = 0$, and $dK/dx = 0$.

Substituting,

$$0 + 4(0) + 3K = 6$$

$$y_p = K = 2$$

Therefore, $y_p = 2$ is a particular solution. We note that we can solve the equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 0$$

in operator form

$$(p^2 + 4p + 3)y = 0$$

or

$$p = -1, -3$$

and the transient solution is

$$y_t = c_1e^{-x} + c_2e^{-3x}$$

The general solution of 1.44 may be written

$$y = \underbrace{c_1e^{-x} + c_2e^{-3x}}_{\text{transient solution}} + \underbrace{2}_{\text{particular (or steady state) solution}}$$

1.3.2.2 FORCING FUNCTION = A POLYNOMIAL

EXAMPLE

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = x^2 + 2x \tag{1.45}$$

Now the form of f(x) for 1.45 is a polynomial of second degree, so we assume a particular solution for y of second degree (that is, let $y_p = Ax^2 + Bx + C$).

Then

$$\frac{dy_p}{dx} = 2Ax + B$$

and

$$\frac{d^2y_p}{dx^2} = 2A$$

Substituting into 1.45,

$$(2A) + 4\,(2Ax + B) + 3\,(Ax^2 + Bx + C) = x^2 + 2x$$

or

$$(3A)\,x^2 + (8A + 3B)\,x + (2A + 4B + 3C) = x^2 + 2x$$

Equating like powers of x,

x^2: $3A = 1$

$A = 1/3$

x: $8A + 3B = 2$

$3B = 2 - \frac{8}{3}$

$B = -\,2/9$

x^o: $2A + 4B + 3C = 0$

$3C = 8/9 - 2/3$

$C = 2/27$

Therefore,

$$y_p = 1/3\,x^2 - 2/9\,x + 2/27$$

The general solution of 1.45 is given by

$$y = c_1e^{-x} + c_2e^{-3x} + 1/3\,x^2 - 2/9\,x + 2/27$$

since the transient solution is the same as for 1.44. As a general rule, if the forcing function is a polynomial of degree n, assume a polynomial solution of degree n.

●1.3.2.3 FORCING FUNCTION = AN EXPONENTIAL

EXAMPLE

$$\frac{d^2y}{dx^2} + 4\,\frac{dy}{dx} + 3y = e^{2x} \qquad (1.46)$$

The forcing function is e^{2x} so we assume a solution of the form

$$y = Ae^{2x}$$

$$\frac{d}{dx}(Ae^{2x}) = 2Ae^{2x}$$

$$\frac{d^2}{dx^2}(Ae^{2x}) = 4Ae^{2x}$$

Substituting in 1.46,

$$4Ae^{2x} + 4(2Ae^{2x}) + 3(Ae^{2x}) = e^{2x}$$

$$e^{2x}(4A + 8A + 3A) = e^{2x}$$

The coefficients on both sides of the equation must be the same. Therefore, $4A + 8A + 3A = 1$, or $15A = 1$, and $A = 1/15$. The particular solution of 1.46 then is $y_p = 1/15\, e^{2x}$. The transient solution is still the same as for 1.44. A final example will illustrate a pitfall sometimes encountered using this method.

1.3.2.4 FORCING FUNCTION = AN EXPONENTIAL (SPECIAL CASE)

EXAMPLE

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = e^{-x} \tag{1.47}$$

The forcing function is e^{-x}, so we assume a solution of the form $y = Ae^{-x}$.

Then

$$\frac{d}{dx}(Ae^{-x}) = -Ae^{-x}$$

and

$$\frac{d^2}{dx^2}(Ae^{-x}) = Ae^{-x}$$

Substituting

$$Ae^{-x} + 4(-Ae^{-x}) + 3(Ae^{-x}) = e^{-x}$$

$$(A - 4A + 3A)e^{-x} = e^{-x}$$

$$(0)e^{-x} = e^{-x}$$

Obviously, this is an incorrect statement. To find where we made our mistake, let's review our procedures.

To solve an equation of the form

$$(p + a)\ (p + b)y = e^{-ax}$$

we solve the homogeneous equation to get

$$(p + a)\ (p + b)y = 0$$

$$p = -a,\ -b$$

$$y_t = c_1e^{-ax} + c_2e^{-bx}$$

If we assume $y_p = Ae^{-ax}$

then

$$y = y_t + y_p = c_1e^{-ax} + c_2e^{-bx} + Ae = (c_1 + A)e^{-ax} + c_2e^{-bx}$$

$$= c_3e^{-ax} + c_2e^{-bx}$$

$$= y_t$$

However, we have already seen that y_t is the solution only when the right side of the equation is zero, and will not solve the equation when we have a forcing function. Therefore, we assume a particular solution.

$$y_p = Axe^{-ax}$$

then

$$y = y_p + y_t = c_1e^{-ax} + c_2e^{-bx} + Axe^{-ax} = (c_1 + Ax)e^{-ax} + c_2e^{-bx} \neq y_t$$

Similarly, we could have the equation

$$(p + aj)\ (p - aj)\ y = \sin ax$$

with transient solution

$$y_t = c_1 \sin ax + c_2 \cos ax$$

If we assume $y_p = A \sin ax + B \cos ax$

then

$$y = y_t + y_p = (c_1 + A) \sin ax + (c_2 + B) \cos ax$$

$$+ (c_2 + B) \cos ax$$

$$= c_3 \sin ax + c_4 \cos ax = y_t$$

Therefore, we assume

$$y_p = Ax \sin ax + Bx \cos ax$$

and

$$y = (c_1 + Ax) \sin ax + (c_2 + Bx) \cos ax \neq y_t$$

Note the following, however, with the equation

$$(p + a - jb)(p + a + jb)y = \sin bx$$

$$y_t = e^{-ax}(c_1 \sin bx + c_2 \cos bx)$$

we can assume $y_p = B \sin bx + C \cos bx$

then

$$y = c_1e^{-ax} \sin bx + c_2e^{-ax} \cos bx + B \sin bx + C \cos bx$$

$$y = (c_1e^{-ax} + B) \sin bx + (c_2e^{-ax} + C) \cos bx \neq y_t$$

Similarly, if

$$(p + a - jb)(p + a + jb)y = e^{-ax}$$

we could assume

$$y_p = Ae^{-ax}$$

In our example above, equation 1.47, a valid solution can be found by assuming $Y_p = Axe^{-x}$, then

$$\frac{d}{dx}(Axe^{-x}) = A(-xe^{-x} + e^{-x})$$

and

$$\frac{d^2}{dx}(Axe^{-x}) = A(xe^{-x} - 2e^{-x})$$

Substituting

$$A(xe^{-x} - 2e^{-x}) + 4A(-xe^{-x} + e^{-x}) + 3(Axe^{-x}) = e^{-x}$$

$$(A - 4A + 3A)xe^{-x} + (-2A + 4A)e^{-x} = e^{-x}$$

$$(0)xe^{-x} + 2Ae^{-x} = e^{-x}$$

and

$$A = 1/2$$

Thus,

$$y_p = (1/2)xe^{-x}$$

is a particular solution of 1.47, and the general solution is given by:

$$y = c_1e^{-x} + c_2e^{-3x} + 1/2\ xe^{-x}$$

The key to successful application of the method of undetermined coefficients is to assume the proper form for a trial particular solution. Table 1 summarizes the results of this discussion.

PROBLEMS: Set II, 1-5, b only, page 1.85.

Table I

Differential equation: $a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = f(x)$	
f(x)*	Assume y_p**
1. β	A
2. βx^n (n a positive integer)	$A_o x^n + A_1 x^{n-1} + \ldots A_{n-1} x + A_n$
3. βe^{rx} (r either real or complex)	Ae^{rx}
4. β cos kx	A cos kx + B sin kx
5. β sin kx	
6. $\beta x^n e^{rx}$ cos kx	$(A_o x^n + \ldots + A_{n-1} x + A_n) e^{rx} \cos kx +$
7. $\beta x^n e^{rx}$ sin kx	$+ (B_o x^n + \ldots + B_{n-1} x + B_n) e^{rx} \sin kx$

*When f(x) consists of a sum of several terms, the appropriate choice for y_p is the sum of y_p expressions corresponding to these terms individually.

**Whenever a term in any of the y_p's listed in this column duplicates a term already in the complementary function, all terms in that y_p must be multiplied by the lowest positive integral power of x sufficient to eliminate the duplication.

1.3.3 SOLVING FOR CONSTANTS OF INTEGRATION

As discussed paragraph 1.2, the number of arbitrary constants in the solution of our linear differential equation is equal to the order of the equation. These constants of integration may be determined by initial or boundary conditions. That is, we must know the physical state (position, velocity, etc.) of the system at some time in order to evaluate these constants. Many times these conditions are given at t = 0 (initial conditions), which is frequently called a quiescent system.

It should be emphasized at this point, that the arbitrary constants of the solution are evaluated from the complete solution (transient plus steady state) of the equation.

We shall illustrate this method with an example.

EXAMPLE

$$\ddot{x} + 4\dot{x} + 13x = 3 \tag{1.48}$$

where the dot notation indicates derivatives with respect to time (that is, $\dot{x} = dx/dt$, $\ddot{x} = d^2x/dt^2$. We will assume that the boundary conditions are $x(0) = 5$, and $\dot{x}(0) = 8$. The transient solution is given by

$$p^2 + 4p + 13 = 0$$

$$p = -2 \pm \sqrt{4 - 13} = -2 \pm j3$$

$$x_t = e^{-2t} (A \cos 3t + B \sin 3t)$$

We assume

$$x_p = D$$

$$\dot{x} = \frac{dx_p}{dt} = 0$$

$$\ddot{x}_p = 0$$

Substituting into 1.48, we get $D = 3/13$

for a complete solution

$$x(t) = e^{-2t} (A \cos 3t + B \sin 3t) + 3/13$$

To solve for A and B, we will use the initial conditions specified above.

$$x(0) = 5 = A + 3/13$$

or

$$A = 62/13$$

Differentiating the complete solution, we get

$$\dot{x}(t) = e^{-2t} (3B \cos 3t - 3A \sin 3t) - 2e^{-2t} (A \cos 3t + B \sin 3t)$$

Substituting the second initial condition

$$\dot{x}(0) = 8 = 3B - 2A$$

$$B = \frac{76}{13}$$

Therefore, the complete solution to 1.48 with the given initial conditions is

$$x(t) = e^{-2t} [(62/13) \cos 3t + (76/13) \sin 3t] + 3/13$$

We have discussed the first and second order differential equation in some detail. It is of great importance to note that many higher order systems quite naturally decompose into first and second order systems. For example, the study of a third order equation (or system) may be conducted by examining a first and a second order system, a fourth order system analyzed by examining two second order systems, etc. All these cases are handled by solving the characteristic equation to get a transient solution and then obtaining the particular solution by any convenient method.

PROBLEMS: Set II, Nos. 1-5, c only, page 1.85.

1.4 APPLICATIONS

Up to this point, we have considered differential equations in general and linear differential equations with constant coefficients in greater detail. We have developed methods for solving first and second order equations of the following type:

$$a \frac{dx}{dt} + bx = f(t) \tag{1.49}$$

$$a \frac{d^2x}{dt^2} + b \frac{dx}{dt} + cx = f(t) \tag{1.50}$$

These two equations are mathematical models or forms. These same forms may be used to describe diverse physical systems. In this section we shall concentrate on the transient response of the systems under investigation, since this area is of primary interest in future studies.

1.4.1 FIRST ORDER EQUATION

Consider the following example:

EXAMPLE

$$4\dot{x} + x = 3 \tag{1.51}$$

where

$$\dot{x} = \frac{dx}{dt}$$

Physically, we can let x represent distance or displacement, and t represent time. To solve this equation, we find the transient solution by using the homogeneous equation

$$4\dot{x} + x = 0$$

$$(4p + 1)x = 0$$

$$4p + 1 = 0$$

$$p = -1/4$$

Thus

$$x_t = ce^{-t/4}$$

The particular solution is found by assuming

$$x_p = A$$

$$\frac{dx_p}{dt} = 0$$

Substituting

$$A = 3$$

or

$$x_p = 3$$

The complete solution is then

$$x = ce^{-t/4} + 3 \qquad (1.52)$$

The first term on the right of 1.52 represents the transient response of the physical system described by equation 1.51, and the second term represents the steady state response if the transient decays. A term useful in describing the physical effect of a negative exponential term is <u>time constant</u> which is denoted by τ. We shall define τ as

$$\tau \triangleq -\frac{1}{p}$$

Thus, equation 1.52 could be rewritten as

$$x = ce^{-t/\tau} + 3 \qquad (1.53)$$

where $\tau = 4$.

Note the following points:

1. We only discuss time constants if p is negative. If p is positive, the exponent of e is positive, and the transient solution will not decay.

2. If p is negative, τ is positive.

3. τ is the negative reciprocal of p, so that small numerical values of p give large numerical values of τ (and vice versa).

4. The value of τ is the time, in seconds, required for the displacement to decay to 1/e of its original displacement from equilibrium or steady value. To get a better understanding of this statement, let's look at 1.53.

$$x = ce^{-t/\tau} + 3$$

and let $t = \tau$. Then

$$x = ce^{-1} + 3 = c\,\frac{1}{e} + 3$$

Thus, when $t = \tau$, the exponential portion of the solution has decayed to 1/e of its original displacement (figure 1.3).

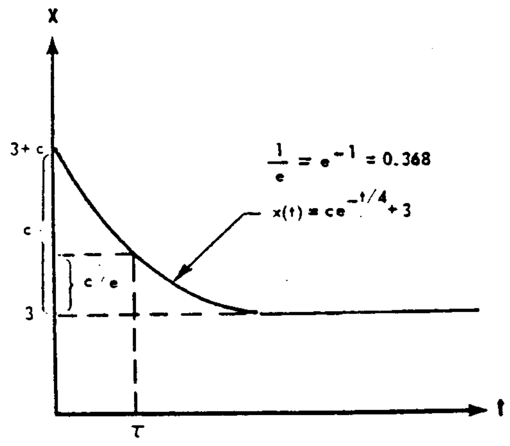


Figure 1.3

Other measures of time are sometimes used to describe the decay of the exponential of a solution. If we let T_1 denote the time it takes for the transient to decay to one-half it's original amplitude, then

$$T_1 = 0.693\,\tau \qquad (1.54)$$

This relationship can be easily shown by investigating

$$x = c_1 e^{-at} + c_2 \qquad (1.55)$$

From our definition, $\tau = 1/a$. We are looking for T_1, the value of t at which $x_t = 1/2\ x_t(0)$. Solving

$$x_t = c_1 e^{-at}$$

$$1/2\ x_t(0) = 1/2\ c_1 = c_1 e^{-aT_1}$$

$$e^{-aT_1} = 1/2$$

$$\ln 1/2 = aT_1$$

$$T_1 = \frac{-\ln 1/2}{a} = \frac{.693}{a} = .693\tau$$

Let's complete our solution of 1.51 by specifying a boundary condition and evaluating the arbitrary constant. Let x = 0 at t = 0.

$$x = ce^{-t/4} + 3$$

$$x(0) = 0 = c + 3$$

$$c = -3$$

Our complete solution for this boundary condition is

$$x = -3e^{-t/4} + 3$$

See figure 1.4.

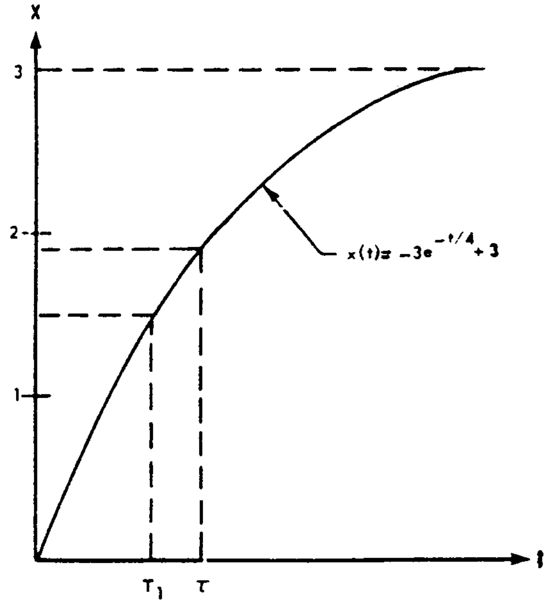


Figure 1.4

1.4.2 SECOND ORDER EQUATION

Consider an equation of the form 1.50. The characteristic equation (operator equation) can be written:

$$a\,p^2 + b\,p + c = 0 \qquad (1.56)$$

The roots of this quadratic equation determine the form of the transient solution as we have seen in paragraph 1.3. We will now discuss physical implications of the algebraic property of the roots.

1.4.2.1 ROOTS REAL AND UNEQUAL

When the roots are real and unequal, the transient solution has the form

$$x_t = c_1 e^{p_1 t} + c_2 e^{p_2 t} \qquad (1.57)$$

1.4.2.1.1 Case 1

When p_1 and p_2 are both negative, the system decays and there will be a time constant associated with each exponential (figure 1.5).

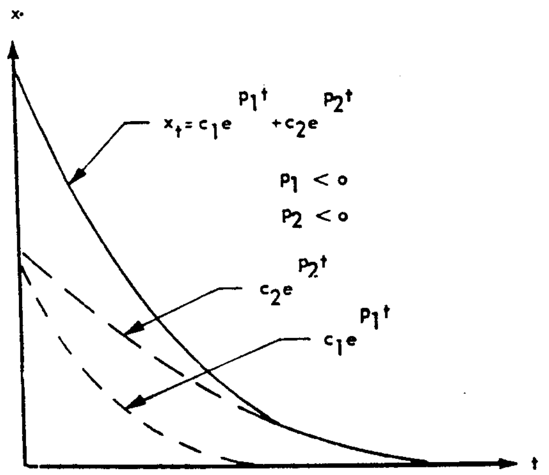


Figure 1.5

1.4.2.1.2 Case 2

When p_1 or p_2 (or both) is positive, the system will generally diverge (figures 1.6 and 1.7).

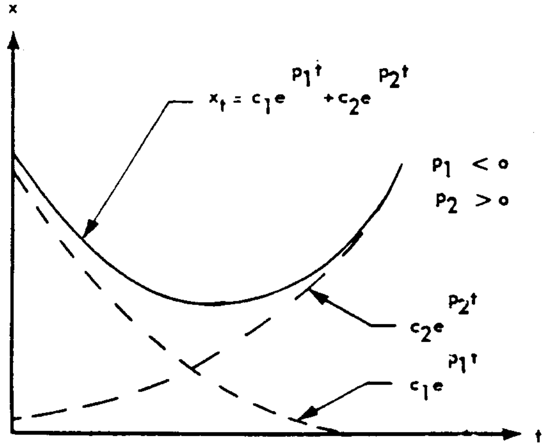


Figure 1.6

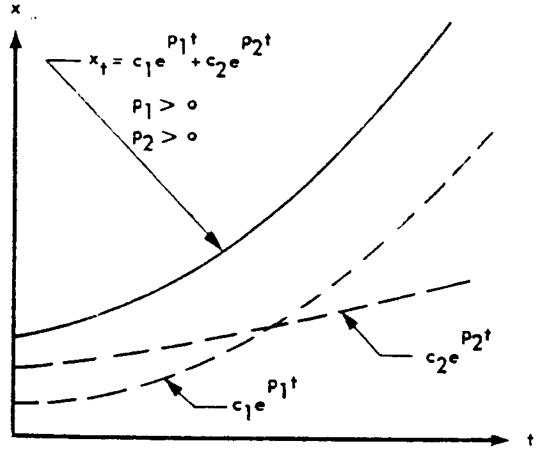


Figure 1.7

1.4.2.1.3 Case 3

Examples where p_1 or p_2 (or both) are zero, are usually not observed in practical cases.

● 1.4.2.2 ROOTS REAL AND EQUAL

When $p_1 = p_2$, the transient solution has the form

$$x_t = c_1e^{pt} + c_2te^{pt} \tag{1.58}$$

1.4.2.2.1 Case 1

When p is negative, the system will usually decay (figure 1.8). (If p is very small, the system may initially exhibit divergence.)

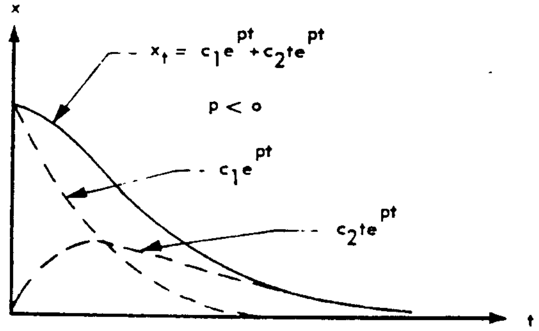


Figure 1.8

1.4.2.2.2 Case 2

When p is positive, the system will diverge.

● 1.4.2.3 ROOTS PURELY IMAGINERY

When $p = \pm jk$, the transient solution has the form

$$x_t = c_1 \sin kt + c_2 \cos kt \tag{1.59}$$

or

$$x_t = A \sin (kt + \phi) \tag{1.60}$$

or

$$x_t = A \cos (kt + \theta) \tag{1.61}$$

The system executes oscillations of constant amplitude with a frequency k (figure 1.9).

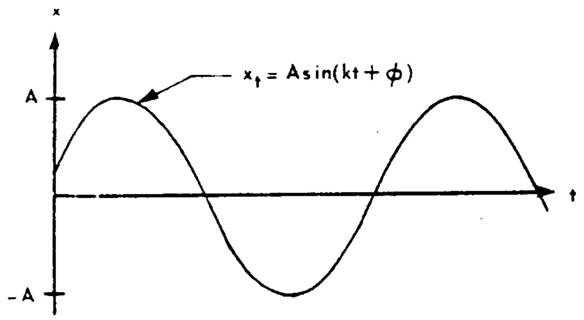


Figure 1.9

● 1.4.2.4 ROOTS COMPLEX CONJUGATES

When the roots are given by $p = k_1 \pm jk_2$, the form of the transient solution is

$$x_t = e^{k_1 t} (c_1 \cos k_2 t + c_2 \sin k_2 t) \tag{1.62}$$

or

$$x_t = A e^{k_1 t} \sin (k_2 t + \phi) \tag{1.63}$$

or

$$x_t = A e^{k_1 t} \cos (k_2 t + \theta) \tag{1.64}$$

The system executes periodic oscillations contained in an envelope given by $x = \pm A e^{k_1 t}$

1.4.2.4.1 Case 1

When k_1 is negative, the system decays (figure 1.10).

1.4.2.4.2 Case 2

When k_1 is positive, the system diverges (figure 1.11).

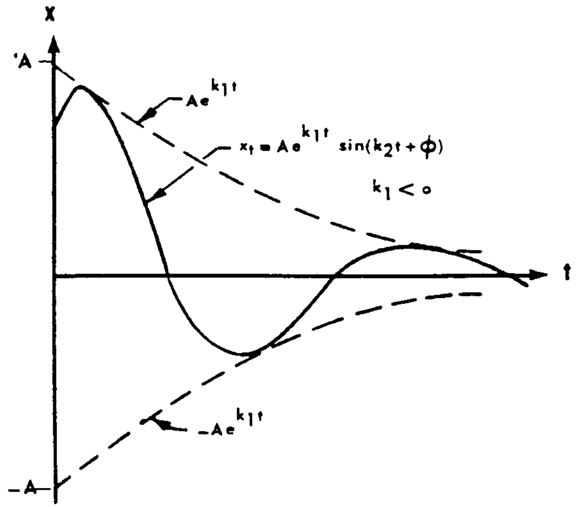


Figure 1.10

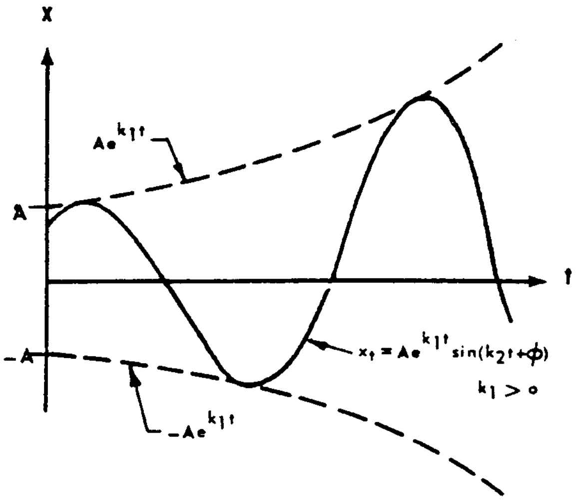


Figure 1.11

The discussion of transient solutions above reveals only part of the picture presented by equation 1.50. We still have the input for forcing function to consider, that is, f(t). In practice, a linear system that possesses a divergence (without input) may be changed to a damped system by carefully selecting or controlling the input. Conversely, a nondivergent linear system with weak damping may be made divergent by certain types of inputs.

1.4.3 SECOND ORDER LINEAR SYSTEMS

Consider the physical model shown in figure 1.12. The system consists of an object suspended by a spring, with a spring constant of k. The mass may move vertically and is subject to gravity, input, and damping, with the total viscous damping constant equal to c.

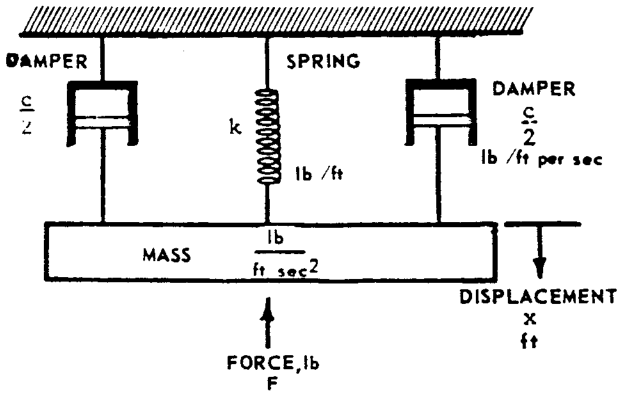


Figure 1.12

The equation for this vibrating system is given by

$$m\ddot{x} + c\dot{x} + kx = f(t) \tag{1.65}$$

The characteristic equation is given by

$$mp^2 + cp + k = 0 \tag{1.66}$$

and the roots of this equation are

$$p_{1,2} = \frac{-c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

$$= \frac{-c}{2m} \pm \sqrt{\frac{k}{m}}\sqrt{\frac{c^2}{4km} - 1} \tag{1.67}$$

Let us, for simplicity, and for reasons that will be obvious later define three constants

$$\zeta \overset{\Delta}{=} \frac{c}{2\sqrt{mk}} \tag{1.68}$$

the term ζ is called the damping ratio, and is a value which indicates the damping strength in the system.

$$\omega_n \overset{\Delta}{=} \sqrt{\frac{k}{m}} \tag{1.69}$$

ω_n is the undamped natural frequency of the system. This is the frequency at which the system would oscillate if there were no damping present.

$$\omega_d \triangleq \omega_n \sqrt{1 - \zeta^2} \qquad (1.70)$$

ω_d is the damped frequency of the system. It is the frequency at which the system oscillates when a damping ratio of ζ is present.

Substituting ζ and ω_n into 1.67 now gives

$$p_{1,2} = - \zeta \omega_n \pm j\omega_n \sqrt{1 - \zeta^2} \qquad (1.71)$$

With these roots, the transient solution becomes

$$x_t = c_1 e^{p_1 t} + c_2 e^{p_2 t}$$

$$= e^{-\zeta\omega_n t} [c_1 \cos \omega_n \sqrt{1 - \zeta^2} t + c_2 \sin \omega_n \sqrt{1 - \zeta^2} t] \qquad (1.72)$$

or

$$x_t = A e^{-\zeta\omega_n t} \sin \left(\omega_n \sqrt{1 - \zeta^2} + \phi\right) \qquad (1.73)$$

Note that the solution will lie within an exponentially decreasing envelope which has a time constant of $1/(\zeta \omega_n)$. This damped oscillation is shown in figure 1.13.

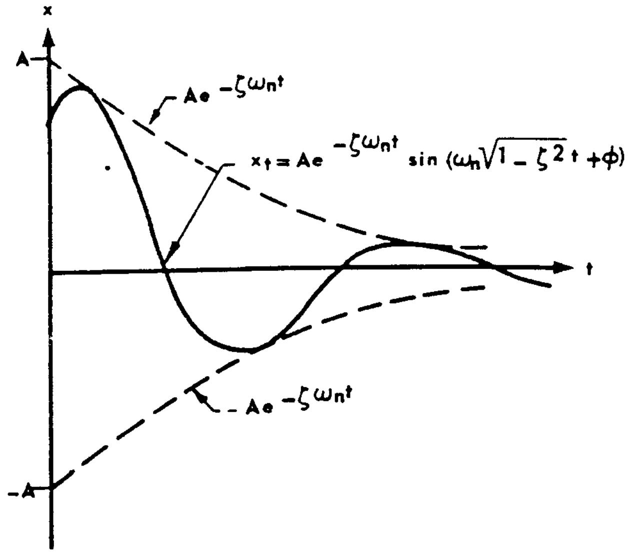


Figure 1.13

If we divide equation 1.65 by m we obtain

$$x + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{f(t)}{m}$$

or, rewriting using ω_n and ζ defined by 1.68 and 1.69

$$\ddot{x} + 2\,\zeta\,\omega_n\,\dot{x} + \omega_n^2\,x = \frac{f(t)}{m} \tag{1.74}$$

Equation 1.74 is a form of 1.65 that is most useful in analyzing the behavior of any linear system.

A general second order physical system can be compared with mass-spring-damper system. The equation defining the system was

$$m\,\ddot{x} + c\,\dot{x} + k\,x = f(t) \tag{1.65}$$

where we defined the parameters

$\omega_n = \sqrt{\frac{k}{m}}$, undamped natural frequency

$\zeta = \frac{c}{2\sqrt{mk}}$, damping ratio

From equation 1.71 we see that the numerical value of ζ is a powerful factor in determining the type of response exhibited by the system.

PROBLEMS: Set II, 1-5, d only, page 1.86.

Let us now consider the physical problem and analyze the various conditions possible. The magnitude and sign of ζ, the damping ratio, determine the response properties of the system.

There are five distinct cases which are given names descriptive of the response associated with each case.

1. $\zeta = 0$, undamped
2. $0 < \zeta < 1$, underdamped
3. $\zeta = 1$, critically damped
4. $\zeta > 1$, overdamped
5. $\zeta < 0$, unstable

We shall now examine each case, making use of equation 1.71

$$p_{1,2} = -\zeta\,\omega_n \pm j\omega_n\sqrt{1 - \zeta^2} \tag{1.71}$$

1.4.3.1 CASE 1: $\zeta = 0$, UNDAMPED

For this condition, the roots of the characteristic equation are

$$p_{1,2} = \pm j\omega_n \tag{1.75}$$

giving a transient solution of the form

$$x_t = c_1 \cos \omega_n t + c_2 \sin \omega_n t \tag{1.76}$$

or

$$x_t = A \sin (\omega_n t + \phi) \tag{1.77}$$

showing the system to have the transient response of an undamped sinusoidal oscillation with frequency ω_n. (Hence, the designation of ω_n as the "undamped natural frequency.") Figure 1.9 shows an undamped system.

Figure 1.14 illustrates typical response for differing values of damping ratios between zero and one.

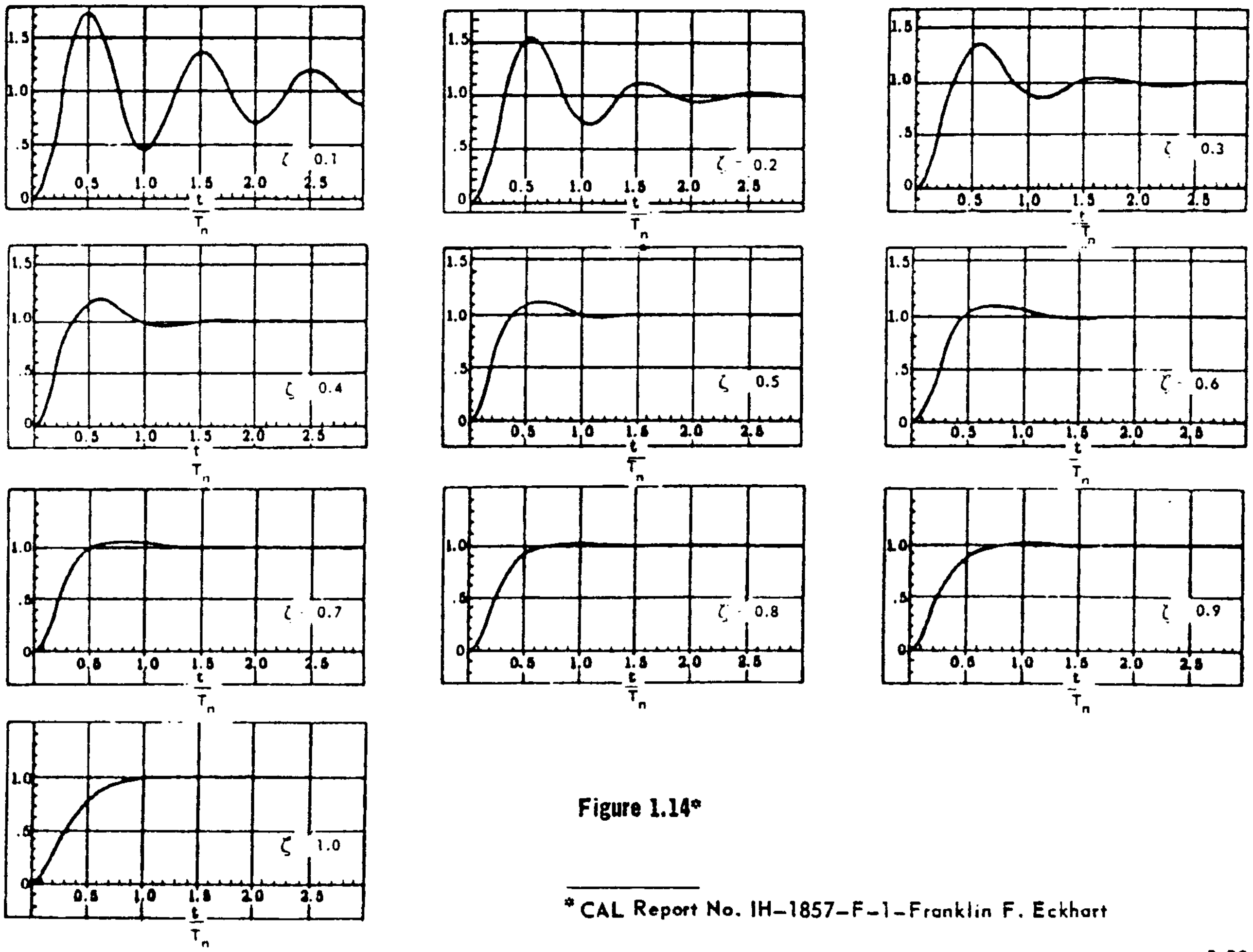


Figure 1.14*

* CAL Report No. IH-1857-F-1-Franklin F. Eckhart

1.4.3.2 CASE 2: $0 < \zeta < 1$, UNDERDAMPED

For this case, p is given by equation 1.71 and the transient solution has the form

$$x_t = A\, e^{-\zeta \omega_n t} \sin (\omega_n \sqrt{1 - \zeta^2}\; t + \phi) \tag{1.78}$$

This solution shows that the system oscillates at the damped frequency, ω_d, and is bounded by an exponentially decreasing envelope with time constant $1/(\zeta\ \omega_n)$. Figure 1.14 shows the effect of increasing the damping ratio from 0.1 to 1.0.

1.4.3.3 CASE 3: $\zeta = 1$, CRITICALLY DAMPED

For this condition, the roots of the characteristic equation are

$$p_{1,2} = -\ \omega_n \tag{1.79}$$

which gives a transient solution of the form

$$x_t = c_1 e^{-\omega_n t} + c_2 t e^{-\omega_n t} \tag{1.80}$$

This is called the critically damped case and generally will not overshoot. It should be noted, however, that large initial values of $\dot{x}$ can cause one overshoot. Figure 1.14 above shows a response when $\zeta = 1$.

1.4.3.4 CASE 4: $\zeta > 1$, OVERDAMPED

In this case, the characteristic roots are

$$p_{1,2} = -\ \zeta \omega_n \pm \omega_n \sqrt{\zeta^2 - 1} \tag{1.81}$$

which shows that both roots are real and negative. This tells us that the system will have a transient which has an exponential decay without sinusoidal motion. The transient response is given by

$$x_t = c_1 e^{-\omega_n \left[\zeta - \sqrt{(\zeta^2 - 1)}\right] t} + c_2 e^{-\omega_n \left[\zeta + \sqrt{(\zeta^2 - 1)}\right] t} \tag{1.82}$$

This response can also be written as

$$x_t = c_1 e^{-t/\tau_1} + c_2 e^{-t/\tau_2} \tag{1.83}$$

where τ_1 and τ_2 are time constants for each exponential term.

This solution is the sum of two decreasing exponentials, one with time constant τ_1 and the other with time constant τ_2. The smaller the value of τ, the quicker the transient decays. Usually the larger the value of ζ, the larger τ_1 is compared to τ_2. For the case $\zeta > 1$, τ_2 is small in comparison to τ_1 and can be neglected. The system then behaves like a first order system (that is, the effect of mass can be neglected). This can be seen most readily from equation 1.83. Figure 1.5 shows an overdamped system.

1.4.3.5 CASE 5: $-1 < \zeta < 0$, UNSTABLE

For this case, the roots of the characteristic equation are

$$p_{1,2} = - \zeta \omega_n \pm j\omega_n \sqrt{1 - \zeta^2} \tag{1.84}$$

These roots are the same as for the underdamped case, except that the exponential term in the transient solution shows an exponential increase with time.

$$x_t = e^{-\zeta\omega_n t} [c_1 \cos \omega_n \sqrt{1 - \zeta^2}\, t + c_2 \sin \omega_n \sqrt{1 - \zeta^2}\, t] \tag{1.85}$$

Whenever a term appearing in the transient solution grows with time (and especially an exponential growth), the system is generally unstable. This means that whenever the system is disturbed from equilibrium, the disturbance will increase with time. Figure 1.11 shows an unstable system.

1.4.3.6 CASE 6: $\zeta = -1$, UNSTABLE

For this case, the roots of the characteristic equation are

$$p_{1,2} = + \omega_n$$

and

$$x_t = e^{(\omega_n t)}(c_1 + c_2 t)$$

1.4.3.7 CASE 7: $\zeta < -1$, UNSTABLE

This case is similar to case 4, except that the system diverges. See figure 1.7.

$$p_{1,2} = - \zeta \omega_n \pm \omega_n \sqrt{\zeta^2 - 1}$$

EXAMPLE

Given

$$\ddot{x} + 4x = 0$$

from equation 1.74

$$\zeta = 0$$

and

$\omega_n = 2$

The system is undamped with a solution

$x_t = A \sin (2t + \phi)$

where A and ϕ are determined by substituting the boundary conditions into the complete solution.

EXAMPLE

Given

$\ddot{x} + \dot{x} + x = 0$

from equation 1.73

$\omega_n = 1$

and

$\zeta = 0.5$

we also know from equation 1.70 that

$\omega_d = \omega_n \sqrt{1 - \zeta^2} = 0.87$

The system is underdamped with a solution

$x_t = Ae^{-0.5t} \sin (0.87t + \phi)$

EXAMPLE

Given

$\frac{\ddot{x}}{4} + \dot{x} + x = 0$

We multiply 4 to get the equation in the form of equation 1.74.

Then

$\ddot{x} + 4\dot{x} + 4x = 0$

and

$\omega_n = 2$

$\zeta = 1$

The system is critically damped and has a solution given by

$$x_t = c_1e^{-2t} + c_2te^{-2t}$$

EXAMPLE

Given

$$\ddot{x} + 8\dot{x} + 4x = 0$$

we get

$$\omega_n = 2$$

and

$$\zeta = 2$$

The system is overdamped and has a solution

$$x_t = c_1e^{-7.46t} + c_2e^{-0.54t}$$

EXAMPLE

Given

$$\ddot{x} - 2\dot{x} + 4x = 0$$

From equation 1.74

$$\omega_n = 2$$

and

$$\zeta = -0.5$$

From equation 1.70

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} = 1.7$$

The solution is unstable (negative damping) and has the form

$$x_t = Ae^t \sin (1.7\ t + \phi)$$

● 14.38 DAMPING (See figure 1.14)

The best damping ratio for a system is determined by the intended use of the system. If a fast response is desired, and the size and number of overshoots is inconsequential, then we would use a small value of ζ. If it is essential that the system not overshoot, and we are not too concerned about response time, we could attempt to use a critically damped (or even an overdamped) system. The value $\zeta = 0.7$ is often referred to as the optimum damping ratio since it gives a small overshoot and a relative quick response. It should be noted that "optimum damping ratio" will change as the requirements of the physical system change.

PROBLEMS: Set II, e only, page 1.86.

1.4.4 ANALOGOUS SECOND ORDER LINEAR SYSTEMS

1.4.4 1 MECHANICAL SYSTEM

The second order equation we have been working with represents the mass-spring-damper system of figure 1.12 and has a differential equation given by

$$m\ddot{x} + c\dot{x} + kx = f(t) \tag{1.86}$$

where

m = mass

c = damping coefficient

k = spring constant

and we defined

$$\omega_n = \sqrt{\frac{k}{m}} \tag{1.87}$$

$$\zeta = \frac{c}{2\sqrt{mk}} \tag{1.88}$$

and thus

$$2\zeta\omega_n = \frac{c}{m}$$

Equation 1.86 may then be rewritten,

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = f_1(t) \tag{1.89}$$

where

$$f_1(t) = \frac{f(t)}{m}$$

or

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = f_1(t) \quad (1.90$$

● 1.4.4.2 ELECTRICAL SYSTEM

The second order equation can also be applied to the series LRC circuit shown in figure 1.15.

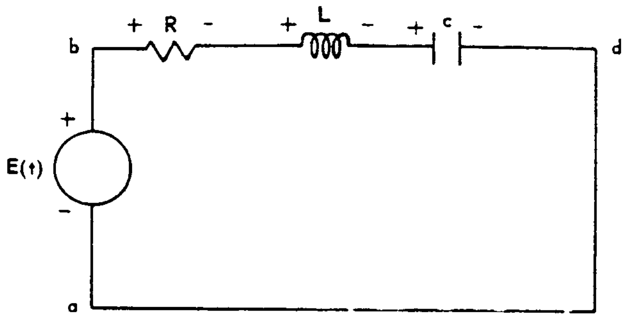


Figure 1.15

where

L = inductance

R = resistance

C = capacitance

q = charge

i = current

Assume $q(0) = \dot{q}(0) = 0$, then Kirchhoff's voltage law gives

$$\Sigma V_{abd} = 0$$

or

$$E(t) - V_R - V_L - V_C = 0$$

$$E(t) - iR - L\frac{di}{dt} - \frac{1}{C}\int_o^t idt = 0$$

Since

$$i = \frac{dq}{dt}$$

$$E(t) = Lq + R\dot{q} + \frac{q}{c} \qquad (1.91)$$

We now define

$$\omega_n = \sqrt{\frac{1}{LC}}$$

$$\zeta = \frac{R}{2\sqrt{L/C}}$$

$$2\zeta\omega_n = \frac{R}{L}$$

Using these parameters, equation 1.91 can be written

$$\ddot{q} + 2\zeta\omega_n\dot{q} + \omega_n^2 q = E_1(t) = \frac{E(t)}{L} \qquad (1.92)$$

1.4.4.3 SERVOMECHANISMS

For control systems work, the second order equation is

$$I\ddot{\theta}_o + f\dot{\theta}_o + \mu\theta_o = \mu\theta_i \qquad (1.93)$$

where

I = inertia

f = friction

μ = gain

θ_i = input

θ_o = output

Rearranging 1.93 we have

$$\ddot{\theta}_o + \frac{f}{I}\dot{\theta}_o + \frac{\mu}{I}\theta_o = \frac{\mu}{I}\theta_i \qquad (1.94)$$

or

$$\ddot{\theta}_o + 2\zeta\omega_n\dot{\theta}_o + \omega_n^2\theta_o = \omega_n^2\theta_i \qquad (1.95)$$

where we define

$$\omega_n = \sqrt{\frac{\mu}{I}}$$

$$\zeta = \frac{f}{2\sqrt{\mu I}}$$

Thus, we see that we can generally write any second order differen equation in the form

$$\ddot{x} + 2\,\zeta\,\omega_n\,\dot{x} + \omega_n^2\,x = f(t) \qquad (1$$

where each term has the same qualitative significance, but differs physical significance.

1.5 LAPLACE TRANSFORMS

We have developed a technique for solving linear differential equations with constant coefficients, with and without inputs or f functions. We have admitted that our method has limitations. It suited for differential equations with inputs of only certain form Further, the solution procedure requires that the student stay con alert for special cases that require careful handling. We accepte "bookkeeping" chores because our solution procedures had the remar property of changing or "transforming" a problem of integration int problem in algebra. (That is, solving a quadratic equation in the of second order differential equations.) This was accomplished by an assumption involving the number e, as follows:

Given

$$a\ddot{x} + b\dot{x} + cx = 0 \qquad (1.$$

Assume

$$x = e^{mt} \qquad (1.$$

Substituting

$$am^2e^{mt} + bme^{mt} + ce^{mt} = 0 \qquad (1.$$

and

$$e^{mt}\,(am^2 + bm + c) = 0 \qquad (1.$$

led us to assert that 1.98 would produce a solution if m were a roo the characteristic equation

$$am^2 + bm + c = 0 \qquad (1.$$

We then introduced an operator, $p = d/dt$, and noted a short cut (boo keeping coincidence) to writing the characteristic equation 1.101 as

$$ap^2 + bp + c = 0 \qquad (1.$$

which we then solved for p to give solution of the form

$$x = c_1e^{p_1t} + c_2e^{p_2t} \qquad (1.$$

Of course, the great shortcoming of this method was that it did not vide a solution to an equation of the form

$$a\ddot{x} + b\dot{x} + cx = f(t) \tag{1.104}$$

It only worked for the homogeneous equation. Still, we were able to patch together a solution by obtaining a particular solution (using still another technique) and adding it to the "transient" solution of the homogeneous equation. It should be appreciated that the method of undetermined coefficients also provided a solution by algebraic manipulation.

Suppose we were adventurous enough to inquire further. We ask, "Does there exist a technique which would exchange (transform) the whole differential equation, including the input, into an algebra problem?" The answer is a qualified "Yes." Fortunately, the "Yes" answer applies to the types of equationswith which we have been working.

In equation 1.104, x is a function of t. To emphasize this, we rewrite 1.104 as:

$$a\ddot{x}(t) + b\dot{x}(t) + cx(t) = f(t) \tag{1.105}$$

Suppose we multiply each term of 1.105 by e^{mt}, giving us:

$$a\ddot{x}(t)e^{mt} + b\dot{x}(t)e^{mt} + cx(t)e^{mt} = f(t)e^{mt} \tag{1.106}$$

Now, a most remarkable feature begins to emerge. It so happens that 1.106 can be integrated term by term on both sides of the equation to produce an algebraic expression in m. The algebraic expression can then be manipulated to obtain eventually the solution of 1.106.

The preceding statements have omitted many details, but express the method of solution we now seek to develop. Our new "fudge factor", e^{mt}, should be distinguished from the previous technique for solving the homogeneous equation, so we shall replace the m by the term, -s. The reason for the minus sign will become apparent later. If we are to integrate the terms of 1.106, we shall need limits of integration. In most physical problems we are interested in events that take place subsequent to a given starting time which we shall call t = 0. Since we are unsure of the duration of significant events, we shall sum up the composite of effects from time t = 0 to time t = ∞ (that should cover the field). So now equation 1.106 becomes

$$\int_0^\infty a\ddot{x}(t)\, e^{-st}\, dt + \int_0^\infty b\, \dot{x}(t)\, e^{-st}\, dt + \int_0^\infty c\, x(t)\, e^{-st}\, dt$$

$$= \int_0^\infty f(t)\, e^{-st}\, dt \tag{1.107}$$

Equation 1.107 is called the Laplace transform of equation 1.105.

There is one small problem. How do we integrate these terms? We now focus our attention upon this problem.

1.5.1 FINDING THE LAPLACE TRANSFORM OF A DIFFERENTIAL EQUATION

We now attempt to find the integrals of the terms of the differential equation 1.107. The big unanswered question posed by equation 1.107 is "What is x(t)?" (that is, x(t) is an unknown). Thus,

$$\int_0^\infty x(t)\, e^{-st}\, dt = L\left\{x(t)\right\} = X(s) \tag{1.108}$$

X(s) must, for the present, remain an unknown. (Remember that m was carried along as an unknown until the characteristic equation evolved, at which time we solved for m explicitly.) Since 1.108 transforms x(t) into a function of the variable, s, we shall say

$$\int_0^\infty c\, x(t)\, e^{-st}\, dt = c\int_0^\infty x(t)\, e^{-st}\, dt = cX(s) \tag{1.109}$$

and be content to carry along X(s) until such time that we can solve for it.

Now consider the second term, b $\dot{x}(t)$. We want to find:

$$\int_0^\infty b\, \dot{x}(t)\, e^{-st}\, dt = b\int_0^\infty \dot{x}(t)\, e^{-st}\, dt \tag{1.110}$$

To solve 1.110 we call upon a useful tool known as integration by parts.

Recall

$$\int_a^b u\,dv = uv\Big]_a^b - \int_a^b v\,du \tag{1.111}$$

Applying this tool to equation 1.110 we let

$$u = e^{-st}$$

and

$$dv = \dot{x}(t)\, dt$$

then

$$du = -se^{-st}\, dt$$

and

$$v = x(t)$$

Putting these values into 1.111 and integrating from $t = 0$ to $t = \infty$,

$$\int_0^\infty \dot{x}(t)e^{-st}\,dt = x(t)e^{-st}\Big]_0^\infty - \int_0^\infty x(t)\left[-se^{-st}\right]dt$$

$$= x(t)e^{-st}\Big]_0^\infty + s\int_0^\infty x(t)e^{-st}\,dt$$

$$= x(t)e^{-st}\Big]_0^\infty + sX(s) \qquad (1.112)$$

Now

$$x(t)e^{-st}\Big]_0^\infty = \lim_{t\to\infty} x(t)e^{-st} - x(0) \qquad (1.113)$$

and we shall assume that the term e^{-st} "dominates" the term $x(t)$ as $t \to \infty$.

Thus, $\lim x(t)e^{-st} = 0$, and equation 1.111 becomes

$$\int_0^\infty \dot{x}(t)e^{-st}dt = 0 - x(0) + sX(s) = sX(s) - x(0) \qquad (1.114)$$

Equations 1.109 and 1.114 can be abbreviated by using the letter L to signify Laplace transformations.

$$L\{\,x(t)\,\} = X(s)$$

$$L\{\,cx(t)\,\} = cX(s) \qquad (1.115)$$

$$L\{\,\dot{x}(t)\,\} = sX(s) - x(0)$$

$$L\{\,b\dot{x}(t)\,\} = b[sX(s) - x\,(0)] \qquad (1.116)$$

Equation 1.116 can be extended to higher order derivatives. Such an extension gives

$$L\{a\ddot{x}(t)\} = a\,[s^2X(s) - sx(0) - \dot{x}(0)] \qquad (1.117)$$

Returning to equation 1.107, we note that we have found the Laplace transforms of all the terms except the forcing function. To solve this transform, the forcing function must be specified. We shall consider a few typical functions and illustrate, by example, the technique for finding the Laplace transform.

EXAMPLE

$$f(t) = A = \text{constant}$$

Then

$$L\{A\} = \int_0^\infty Ae^{-st}dt = \frac{A}{-s}\int_0^\infty e^{-st}(-sdt) = -\frac{A}{s}e^{-st}\Big]_0^\infty$$

or

$$L\{A\} = \frac{A}{s} \tag{1.118}$$

[redacted]

$$f(t) = t$$

Then

$$L\{t\} = \int_0^{\infty} te^{-st}dt$$

To integrate by parts, we let

$$u = t$$

$$dv = e^{-st}dt$$

Then

$$du = dt$$

$$v = -\frac{1}{s}e^{-st}$$

Substituting into 1.11

$$\int_0^{\infty} te^{-st}dt = \frac{-te}{s}\Bigg]_0^{\infty} + \frac{1}{s}\int_0^{\infty} e^{-st}dt$$

$$= 0 - \frac{1}{s^2}e^{-st}\Bigg]_0^{\infty} = 0 + \frac{1}{s^2}$$

or

$$L\{t\} = \frac{1}{s^2} \tag{1.119}$$

EXAMPLE

$$f(t) = e^{2t}$$

Then

$$L\{e^{2t}\} = \int_0^{\infty} e^{2t}e^{-st}dt = \int_0^{\infty} e^{(2-s)t}dt = \frac{1}{s-2}$$

or

$$L\{e^{2t}\} = \frac{1}{s-2} \tag{1.120}$$

EXAMPLE

$$f(t) = \sin at$$

Then

$$L\{\sin at\} = \int_0^\infty \sin at\, e^{-st} dt$$

Integrate by parts, letting

$$u = \sin at$$

$$dv = e^{-st} dt$$

Then

$$du = a \cos at\, dt$$

$$v = -\frac{1}{s} e^{-st}$$

Substituting into 1.111

$$\int_0^\infty \sin t\, e^{-st} = \frac{-(\sin at)(e^{-st})}{s}\Bigg]_0^\infty + \frac{a}{s}\int_0^\infty \cos at\, e^{-st} dt$$

or

$$\int_0^\infty \sin at\, e^{-st} dt = 0 + \frac{a}{s}\int_0^\infty \cos at\, e^{-st} dt \qquad (1.121)$$

The expression $\cos at\, e^{-st} dt$ can also be integrated by parts, letting

$$u = \cos at$$

$$dv = e^{-st} dt$$

and

$$du = -a \sin at\, dt$$

$$v = \frac{1}{s} e^{-st}$$

Giving

$$\int_0^\infty \cos at\, e^{-st} dt = \frac{-(\cos at)(e^{-st})}{s}\Bigg]_0^\infty - \frac{a}{s}\int_0^\infty \sin at\, e^{-st} dt$$

or

$$\int_0^{\infty} \cos at\, e^{-st}dt = \frac{1}{s} - \frac{a}{s} L\{\sin t\} \qquad (1.122)$$

Substituting 1.122 into 1.121 gives

$$L\{\sin at\} = 0 + \frac{a}{s}\left[\frac{1}{s} - \frac{a}{s} L\{\sin at\}\right] = \frac{a}{s^2} - \frac{a^2}{s^2} L\{\sin at\}$$

which "obviously" yields

$$L\{\sin at\} = \frac{a}{s^2 + a^2} \qquad (1.123)$$

Also note that 1.121 may be written as

$$L\{\sin at\} = \frac{a}{s} L\{\cos at\}$$

which yields

$$L\{\cos at\} = \frac{s}{s^2 + a^2}$$

The Laplace transforms of more complicated functions may be quite tedious to derive, but the procedure is similar to that above. Fortunately, it is not necessary to derive Laplace transforms each time we use them. Extensive tables of transforms exist in most advanced mathematics and control system textbooks.

We originally asserted that the Laplace transform was going to assist in the solution of a differential equation. The technique is best described by an example.

EXAMPLE

Given

$$\ddot{x} + 4\dot{x} + 4x = 4e^{2t}$$

with conditions $x(0) = 1$, $\dot{x}(0) = -4$. Taking the Laplace transform of the equation gives

$$s^2X(s) - sx(0) - \dot{x}(0) + 4\,[sX(s) - x(0)] + 4X(s) = \frac{4}{s-2}$$

or

$$[s^2 + 4s + 4]\,X(s) + [-s + 4 - 4] = \frac{4}{s-2}$$

Solving for X(s),

$$X(s) = \frac{s^2 - 2s + 4}{(s - 2)(s + 2)^2} \tag{1.124}$$

In order to continue with our solution, it is necessary that we discuss partial fraction expansions.

PROBLEMS: Set III, page 1.100.

1.5.2 PARTIAL FRACTIONS

The method of partial fractions enables us to separate a complicated rational fraction into a sum of simpler fractions. Suppose we are given a fraction of two polynomials in a variable, s. Suppose the fraction is proper (the degree of the numerator is less than the degree of the dominator). If it is not proper, we make it proper by dividing the fraction and then consider the remainder expression. There occur several cases:

1.5.2.1 CASE 1: DISTINCT LINEAR FACTORS

To each linear factor such as (as + b), occurring once in the demoninator, there corresponds a single partial fraction of the form, A/(as + b), where A is a constant to be determined.

EXAMPLE

$$\frac{7s - 4}{s(s - 1)(s + 2)} = \frac{A}{s} + \frac{B}{s - 1} + \frac{C}{s + 2} \tag{1.125}$$

1.5.2.2 CASE 2: REPEATED LINEAR FACTORS

To each linear factor, (as + b), occurring n times in the demoninator there corresponds a set of n partial fractions.

EXAMPLE

$$\frac{s^2 - 9s + 17}{(s - 2)^2(s + 1)} = \frac{A}{(s + 1)} + \frac{B}{(s - 2)} + \frac{C}{(s - 2)^2} \tag{1.126}$$

(where A, B, and C are constants to be determined)

1.5.2.3 CASE 3: DISTINCT QUADRATIC FACTORS

To each irreducible quadratic factor, $as^2 + bs + c$, occurring once in the denominator, there corresponds a single partial fraction of the form, $(As + B)/(as^2 + bs + c)$, where A and B are constants to be determined.

EXAMPLE

$$\frac{3s^2 + 5s + 8}{(s + 2)(s^2 + 1)} = \frac{A}{s + 2} + \frac{Bs + C}{s^2 + 1} \tag{1.127}$$

1.5.2.4 CASE 4: REPEATED QUADRATIC FACTORS

To each irreducible quadratic factor, $as^2 + bs + c$, occurring n times in the denominator, there corresponds a set of n partial fractions.

EXAMPLE

$$\frac{10\,s^2 + s + 36}{(s - 4)(s^2 + 4)^2} = \frac{A}{s - 4} + \frac{Bs + C}{s^2 + 4} + \frac{Ds + E}{(s^2 + 4)^2} \tag{1.128}$$

(where A, B, C, D, and E are constants to be determined)

The "brute-force" technique for finding the constants will be illustrated by solving 1.128. Start by finding the common denominator on the right side of 1.128.

$$\frac{10\,s^2 + s + 36}{(s - 4)(s^2 + 4)^2} = \frac{A(s^2 + 4)^2 + (Bs + C)(s - 4)(s^2 + 4) + (Ds + E)(s - 4)}{(s - 4)(s^2 + 4)^2} \tag{1.129}$$

Then set the numerators equal to each other

$$10\,s^2 + s + 36 = A(s^2 + 4)^2 + (Bs + C)(s^2 + 4)(s - 4) + (Ds + E)(s - 4) \tag{1.130}$$

and, without justifying the statement, we shall assert that 1.130 must hold for all values of s. Now substitute enough values of s into 1.130 to find the constants.

1. Suppose s = 4, then 1.130 becomes

   $(10)(16) + 4 + 36 = 400A$

and

   $A = 1/2$

2. Suppose s = 2j, then 1.130 becomes

   $-40 + 2j + 36 = -4D + 2jE - 8jD - 4E$

   $-4 + 2j = -4(D + E) + 2j\,(E - 4D)$

The real and imaginary parts must be equal to their counterparts on the opposite side of the equal sign, thus

$$(D + E) = 1$$

and

$$E - 4D = 1$$

or

$$D = 0$$

and

$$E = 1$$

3. Now let $s = 0$, then 1.130 becomes

$$36 = 16A - 16\ (C) - 4E$$

and from above

$$A = 1/2,\ E = 1$$

hence

$$36 = 8 - 16C - 4$$

and

$$C = -2$$

4. Let $s = 1$, then 1.130 becomes

$$47 = 25\ (1/2) + (B - 2)(-15) - 3$$

$$94 = 25 - 30B + 60 - 6,$$

or

Th $B = -1/2$

Then 1.129 may be written

$$\frac{10\ s^2 + s + 36}{(s - 4)(s^2 + 4)^2} = 1/2\ \left(\frac{1}{s - 4}\right) - 1/2\ \left(\frac{s + 4}{s^2 + 4}\right) + \frac{1}{(s^2 + 4)^2}$$

Let's continue with our attempt to solve the differential equation

$$\ddot{x} + 4\dot{x} + 4x = 4e^{2t}$$

We have transformed the equation (and substituted initial conditions) to get

$$X(s) = \frac{s^2 - 2s + 4}{(s - 2)(s + 2)^2} \qquad (1.124)$$

We now expand by partial fractions

$$\frac{s^2 - 2s + 4}{(s - 2)(s + 2)^2} = \frac{A}{s - 2} + \frac{B}{s + 2} + \frac{C}{(s + 2)^2} \tag{1.132}$$

Taking the common demoninator, and setting numerators equal

$$s^2 - 2s + 4 = A(s + 2)^2 + B(s + 2)(s - 2) + C(s - 2) \tag{1.133}$$

We can now substitute different values of s into this equation and solve for the constants. An alternate method of solving for these constants exists, however, and we will demonstrate this new approach. If we multiply out the right side of 1.133 we get

$$s^2 - 2s + 4 = As^2 + 4As + 4A + Bs^2 - 4B + Cs - 2C$$

$$= (A + B)s^2 + (4A + C)s + (4A - 4B - 2C)$$

Now the coefficients of like powers of s on both sides of the equation must be equal (that is, the coefficient of s^2 on the left side equals the coefficient of s^2 on the right side, etc.) Equating gives

s^2 : $1 = A + B$

s^1 : $-2 = 4A + C$

s^0 : $4 = 4A - 4B - 2C$

Solving, we get

$A = 1/4$

$B = 3/4$

$C = -3$

Substituting into 1.132, we get

$$X(s) = 1/4\left(\frac{1}{s - 2}\right) + 3/4\left(\frac{1}{s + 2}\right) - 3\left(\frac{1}{s + 2}\right)^2 \tag{1.134}$$

● 1.5.3 HEAVISIDE EXPANSION THEOREMS FOR ANY F (s)

● 1.5.3.1 CASE 1 DISTINCT LINEAR FACTORS

If the denominator F(s) has a distinct linear factor, (s - a), we find the constant for that factor by multiplying F(s) by (s - a), and then evaluate the remainder of F(s) at s = a.

$$F(s) = \frac{A}{s - a} + \ldots$$

$$A = (s - a)\, F(s) \Big]_{s = a}$$

EXAMPLE

$$F(s) = \frac{7s - 4}{s(s - 1)(s + 2)} = \frac{A}{s} + \frac{B}{s - 1} + \frac{C}{s + 2}$$

$$A = sF(s)\Big]_{s = 0} = \frac{7s - 4}{(s - 1)(s + 2)}\Big]_{s = 0} = \frac{4}{(-1)(2)} = 2$$

$$B = (s - 1)F(s)\Big]_{s = 1} = \frac{7s - 4}{s(s + 2)}\Big]_{s = 1} = \frac{7 - 4}{(1)(3)} = 1$$

$$C = (s + 2)F(s)\Big]_{s = -2} = \frac{7s - 4}{s(s - 1)}\Big]_{s = -2} = \frac{-14 - 4}{(-2)(-3)} = -3$$

See case 4 also.

● 1.5.3.2 CASE 2: REPEATED LINEAR FACTORS

If the denominator of F(s) has any repeated linear factors, they must be treated in a special manner.

$$F(s) = \frac{A}{(s - a)^n} + \frac{B}{(s - a)^{n - 1}} + \frac{C}{(s - a)^{n - 2}} + \ldots + \frac{Z}{(s - a)}$$

Let $\phi(s) = (s - a)^n F(s)$

Then

$$A = \phi(s)\Big]_{s = a}$$

$$B = d\phi(s)/ds\Big]_{s = a}$$

$$C = \frac{1}{2!} d^2\phi(s)/ds^2\Big]_{s = a}$$

$$Z^* = \frac{1}{k!} \frac{d^k\phi(s)}{ds^k}\Big]_{s = a}$$

*Note: This formula is good for all constants except A above.

where $k = 1, 2, \ldots, n - 1$

where the derivatives of $\phi(s)$ are obtained by using

$$\frac{d}{ds}\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2}$$

For example,

$$F(s) = \frac{s^2 - 9s + 17}{(s - 2)^2 (s + 1)} = \frac{A}{s + 1} + \frac{B}{(s - 2)^2} + \frac{C}{s - 2}$$

$$A = (s + 1)\, F(s)\Big]_{s = -1} = \frac{s^2 - 9s + 17}{(s - 2)^2}\Big]_{s = -1} = \frac{1 + 9 + 17}{(-3)^2} = \frac{27}{9}$$

$$B = \phi(s)\Big]_{s = 2} = \frac{s^2 - 9s + 17}{s + 1}\Big]_{s = 2} = \frac{4 - 18 + 17}{3} = 1$$

$$C = \frac{d\phi(s)}{ds}\Big]_{s = 2} = \frac{(s + 1)(2s - 9) - (s^2 - 9s + 17)(1)}{(s + 1)^2}\Big]_{s = 2}$$

$$= \frac{(3)(-5) - (4 - 18 + 17)}{(3)^2} = \frac{-18}{9} = -2$$

See case 4 also.

1.5.3.3 CASE 3: DISTINCT QUADRATIC FACTORS

If the denominator of F(s) contains a distinct quadratic factor $(s + a)^2 + b^2$, we will again multiply F(s) by $(s + a)^2 + b^2$, and evaluate the remainder of F(s), $\phi(s)$, at $s = -a + jb$ and use real and imaginary parts of $\phi(s)$ to obtain the two constants.

$$F(s) = \frac{As + B}{(s + a)^2 + b^2}$$

Let

$$\phi(s) = \left[(s + a)^2 + b^2\right] F(s)$$

and compute

$$\phi_r + j\phi_i = \phi(s)\Big]_{s = -a + jb}$$

Then

$$A = \frac{\phi_i}{b} \qquad B = \frac{b\phi_r + a\phi_i}{b}$$

For example,

$$F(s) = \frac{4s^2 + 19s + 32}{(s + 2)\left[(s + 3)^2 + 4\right]} = \frac{As + B}{(s + 3)^2 + 4} + \frac{C}{s + 2}$$

$$\phi_r + j\phi_i = \phi(s)\Big]_{s = -3 + j2} = \frac{4s^2 + 19s + 32}{s + 2}\Big]_{s = -3 + j2}$$

$$= \frac{-5 - j10}{-1 + j2} \cdot \frac{-1 - j2}{-1 - j2} = -3 + j4$$

$$\phi_r = -3 \qquad \phi_i = 4 \qquad a = 3 \qquad b = 2$$

$$A = \frac{4}{2} = 2 \qquad B = \frac{-6 + 12}{2} = 3$$

See case 4 also.

1.5.3.4 CASE 4: REPEATED QUADRATIC FACTORS (AND ANY OTHER CASE)

Procedures similar to those used in the previous cases exist for this case, but they are too cumbersome for most applications. The following procedures will work for any combination of linear and quadratic factors:

$$F(s) = \frac{10s^2 + s + 36}{(s - 4)(s^2 + 4)^2} = \frac{A}{s - 4} + \frac{Bs + C}{s^2 + 4} + \frac{Ds + E}{(s^2 + 4)^2}$$

Put the right-hand side of the equation over a common denominator and then set the two numerators equal.

$$10s^2 + s + 36 = A(s^2 + 4)^2 + (Bs + C)(s - 4)(s^2 + 4) + (Ds + E)(s - 4) \tag{1.135}$$

which will be a true equation for all values of s.

1.5.3.5 PROCEDURES

The following steps may be done in any order, and in combination with the procedures in cases 1 through 3.

1. Since equation 1.135 is true for all values of s, choosing specific values of s, five in this case, and substituting into equation 1.135 will give you five equations in five unknowns, which can be solved simultaneously for the constants.

2. Expand the right hand side of equation 1.135 and find the coefficients of each power of s. These coefficients must be the same on both sides of the equation (that is, s^4: B = 0; s^0: 4A - 16C -4E = 36).

3. Let s equal an imaginary or complex number and substitute into equation 1.135. The real parts on both sides of the equation must be equal, and so must the imaginary parts.

Examining equation 1.135, we will use a combination of procedures. First, find A by using case 1

$$A = (s - 4)\ F(s)\Big]_{s=4} = \frac{10(16) + 4 + 36}{(16 + 4)^2} = \frac{200}{400} = \underline{\frac{1}{2}}$$

If we let s = j2, the only non-zero term will be the one with D and E. Letting s be a complex number will give us two equations for D and E, which we can solve simultaneously.

$$10(j2)^2 + j2 + 36 = (j2D + E)(j2 - 4)$$

$$-4 + j2 = -4D - 4E - j8D + j2E$$

Now set real and imaginary parts equal.

$$-4D - 4E = -4 \qquad\qquad -8D + 2E = 2$$

$$-2D - 2E = -2 \qquad\qquad -8D + 2E = 2$$

Adding the two equations together gives:

$$-10D = 0$$

$$\underline{D = 0}$$

Then

$$2E = 2$$

$$\underline{E = 1}$$

To find C, let s = 0 in equation 1.135

$$36 = 16A - 16C - 4E = 16(1/2) - 16C - 4(1)$$

$$16C = 8 - 4 - 36 = -32$$

$$\underline{C = -2}$$

To find B, let s = 1

$$10 + 1 + 36 = 25A + (B + C)(-3)(5) + (D + E)(-3)$$

$$47 = 25(1/2) - 15B - 15(-2) - 3$$

$$15B = 12\ 1/2 + 30 - 3 - 47 = 42\ 1/2 - 50 = -7\ 1/2$$

$$\underline{B = -\ 1/2}$$

An alternate way to find B is to calculate the coefficient of s^4 on the right-hand side of equation 1.135, and then set it equal to the coefficient of s^4 on the left hand side of the equation. Then we get

$$A + B = 0$$

$$\underline{B = -A = -\ 1/2}$$

To complete our solution, we must convert (transform) back into the time domain. The operation which converts a function X(s) back to a function of time is called the inverse Laplace transformation.

$$L^{-1}\left\{L\{x(t)\}\right\} = L^{-1}\{X(s)\} = x(t) \qquad (1.136)$$

The inverse Laplace transformation can be solved directly

$$x(t) = \frac{1}{2\pi j}\int_{C - j\infty}^{C + j\infty} X(s)e^{st}ds \qquad (1.137)$$

(where C is a real constant)

This integral, 1.137, is hardly ever used because the Laplace transform is unique and, therefore, generally X(s) can be recognized as the Laplace transform of some known x(t). In practice, tables of transform pairs (as found in most mathematics texts) will suffice to find the inverse of X(s) (see table II, page 1. ~~106,~~ 70 for some transform pairs).

Using a suitable transform table, the inverse of 1.134 can easily be found to give us a solution

$$x(t) = 1/4\ e^{2t} + 3/4\ e^{-2t} - 3te^{-2t} \qquad (1.138)$$

PROBLEMS: Set IV, A, page ~~1.106~~.

1.5.4 PROPERTIES OF LAPLACE TRANSFORMS

The Laplace transform of some f(t), a function of time, is defined as

$$L\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}dt \qquad (1.139)$$

where

$s = \sigma + j\omega$ (a complex number)

The strength of the Laplace transform is that it converts linear differential equations with constant coefficients into algebraic equations in the s-domain. All that remains to do is to take the inverse transform of the explicit solutions to return to the time domain. Although the applications at the school will consider time as the independent variable, a linear differential equation with any independent variable (such as distance) may be solved by Laplace transforms.

There are several important properties of the Laplace transform which should be included in this discussion.

In the general case it can be shown that

$$L\left\{\frac{d^n\ f(t)}{dt^n}\right\} = s^n\ F(s) - \left(s^{n-1}f(0) + s^{n-2}\ \frac{df(0)}{dt} + \ldots + \frac{d^{n-1}f(0)}{dt^{n-1}}\right) \tag{1.140}$$

It is obvious that for quiescent systems (that is, initial conditions zero)'

$$L\left\{\frac{d^n ft}{dt^n}\right\} = s^n F(s) \tag{1.141}$$

This result enables us to write down transfer function by inspection.

Another significant transform is that of an indefinite integral.

In the general case

$$L\left\{\iiint\ldots\ f(t)\,dt^n\right\} = \frac{F(s)}{s^n} + \frac{\left[\int f(t)\,dt\right]_{t\ =\ 0+}}{s^n} + \frac{\left[\iint f(t)\,dt\right]_{t\ =\ 0+}}{s^{n-1}} + \ldots$$

Equation 1.140 allows us to transform Integro-differential equations such as those arising in electrical engineering.

For the case where all integrals of f(t) evaluated at 0+ are zero, our transform becomes

$$L\left\{\iiint\ \ldots\ f(t)\,dt^n\right\} = \frac{F(s)}{s^n} \tag{1.143}$$

A third useful property of the Laplace transform arises if we consider the Laplace transform of the product of some exponential and any other function of time.

$$L\left\{e^{-at}\ f(t)\right\} = \int_0^{\infty} e^{-at}\ f(t)e^{-st}\ dt = \int_0^{\infty} f(t)e^{-(s+a)t}\ dt \tag{1.144}$$

It is apparent that this is the same form as the transform of f(t), except that the transformed independent variable is (s + a) rather than s. We conclude therefore that

$$L\{e^{-at}\ f(t)\} = L\{f(t)\}\Big]_{(s \to s + a)} = F(s + a) \tag{1.145}$$

It is important to note at this point, that the transform of the product of two functions of time is not equal to the product of the individual transforms. In symbolic form,

$$\boxed{L\{f(t)\ g(t)\} \neq F(s)\ G(s)} \tag{1.146}$$

The L{f(t) g(t)} must be solved for directly by the definition of the Laplace transform.

The last property we will consider is the Laplace transform of a pure time delay. A pure time delay of the function f(t) can be represented mathematically as

$$f(t - a)\ u(t - a) \tag{1.147}$$

where a is the length of delay and u(t - a) is the unit step defined as

$$u(t - a) = \begin{cases} 1\ , & (t - a) > 0 \\ 0\ , & (t - a) < 0 \end{cases}$$

For such a time delay

$$L\{f(t - a)\ u(t - a)\} = e^{-as}\ L\{f(t)\} \tag{1.148}$$

We shall now demonstrate the usefulness of the Laplace transform by solving several example problems.

EXAMPLE

Solve the given equation for x(t),

$$\dot{x} + 2x = 1 \tag{1.149}$$

when x(0) = 1.

By Laplace

$$L\{\dot{x}\} = sX(s) - x(0)$$

$$L\{2x\} = 2X(s)$$

$$L\{1\} = \frac{1}{s}$$

$$sX(s) - x(0) + 2X(s) = \frac{1}{s}$$

$$sX(s) + 2X(s) = \frac{1}{s} + 1$$

Thus

$$(s + 2)\ X(s) = \frac{1}{s} + 1$$

$$X(s) = \frac{s + 1}{s(s + 2)} = \frac{A}{s} + \frac{B}{s + 2}$$

Solving,

$$A = 1/2$$

and

$$B = 1 - 1/2 = 1/2$$

$$X(s) = \frac{1/2}{s} + \frac{1/2}{s + 2}$$

Inverse transforming gives

$$x(t) = 1/2 - 1/2\ e^{-2t} \qquad (1.150)$$

EXAMPLE

Given

$$\dot{x} + 2x = \sin t,\ x(0) = 5 \qquad (1.151)$$

solve for x(t).

Taking the transform of 1.151

$$sX(s) - x(0) + 2X(s) = \frac{1}{s^2 + 1}$$

and

$$X(s) = \frac{1}{(s^2 + 1)(s + 2)} + \frac{5}{s + 2} \qquad (1.152)$$

Expanding <u>the first term</u> on the right side of the equation gives

$$\frac{1}{(s^2 + 1)(s + 2)} = \frac{As + b}{s^2 + 1} + \frac{C}{s + 2}$$

Taking the common denominator and equating numerators gives

$$1 = (As + B)(s + 2) + C(s^2 + 1)$$

Substituting values of s leads to

$A = -1/5$

$B = 2/5$

$C = 1/5$

and substituting back into 1.152 gives

$$X(s) = \frac{-1/5\ s}{s^2 + 1} + \frac{2/5}{s^2 + 1} + \frac{1/5}{s + 2} + \frac{5}{s + 2}$$

Inverse transforming gives our solution

$$x(t) = -1/5 \cos t + 2/5 \sin t + 5\ 1/5\ e^{-2t} \tag{1.153}$$

EXAMPLE

Given

$$\ddot{x} + 5\dot{x} + 6x = 3e^{-3t},\ x(0) = \dot{x}(0) = 1 \tag{1.154}$$

solve for x(t).

Taking the transform of 1.154

$$s^2X(s) - sx(0) - \dot{x}(0) + 5sX(s) - 5x(0) + 6X(s) = \frac{3}{s + 3}$$

or

$$X(s) = \frac{s^2 + 9s + 21}{(s + 3)\ (s^2 + 5s + 6)}$$

Factoring the denominator,

$$X(s) = \frac{s^2 + 9s + 21}{(s + 3)(s + 2)(s + 3)} \tag{1.155}$$

$$= \frac{s^2 + 9s + 21}{(s + 3)^2(s + 2)} \tag{1.156}$$

$$= \frac{A}{s + 3} + \frac{B}{(s + 3)^2} + \frac{C}{s + 2} \tag{1.157}$$

Finding the common denominator of 1.157, and setting the resultant numerator equal to the numerator of 1.156,

$$s^2 + 9s + 21 = A(s + 3)(s + 2) + B(s + 2) + C(s + 3)^2$$

which can be solved easily for

$A = -6$

$B = -3$

$C = 7$

Now X(s) is given by

$$X(s) = \frac{-6}{s + 3} - \frac{3}{(s + 3)^2} + \frac{7}{s + 2}$$

which can be transformed to

$$x(t) = -6e^{-3t} - 3te^{-3t} + 7e^{-2t} \tag{1.158}$$

EXAMPLE

Given

$$\ddot{x} + 2\dot{x} + 10x = 3t + 6/10$$

$$x(0) = 3$$

$$\dot{x}(0) = -27/10$$

solve for x(t).

Transforming 1.159 and solving for X(s) gives

$$X(s) = \frac{3s^3 + 3.3s^2 + 0.6s + 3}{s^2 (s^2 + 2s + 10)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs + D}{s^2 + 2s + 10}$$

where

$A = 0$

$B = 0.3$

$C = 3$

$D = 3$

Thus,

$$X(s) = \frac{0.3}{s^2} + \frac{3s + 3}{s^2 + 2s + 10} \tag{1.160}$$

To make our inverse transforming a bit easier, let's rewrite 1.160 as

$$X(s) = \frac{0.3}{s^2} + 3\left[\frac{(s + 1)}{(s + 1)^2 + 3^2}\right] \tag{1.161}$$

which is readily transformable to

$$x(t) = 0.3t + 3e^{-t} \cos 3t \tag{1.162}$$

PROBLEMS: Set IV, B, page 1.106.

1.5.5 TRANSFER FUNCTION

Before beginning simultaneous differential equations, we shall define the transfer function of a system. Consider the following equation with initial conditions as shown.

$$a\ddot{x} + b\dot{x} + cx = f(t) \tag{1.163}$$

$$x(0) = \dot{x}(0) = 0$$

If we take the Laplace transform of 1.163, we get

$$as^2X(s) + bsX(s) + cX(s) = F(s) \tag{1.164}$$

or

$$\frac{X(s)}{F(s)} = \frac{1}{as^2 + bs + c}$$

Since equation 1.163 represents a system whose input is f(t) and whose output is x(t), we shall define

$$X(s) \triangleq \text{output transform}$$

$$F(s) \triangleq \text{input transform}$$

We can then define the transfer function of the system, TF, as

$$\boxed{TF \triangleq \frac{X(s)}{F(s)}} \tag{1.165}$$

For our example,

$$TF = \frac{1}{as^2 + bs + c} \tag{1.167}$$

Note that the denominator of the transfer function is algebraically the same as the characteristic equation of 1.163. We have already seen, in paragraph 1.6.1 on operator notation, that the characteristic equation completely defines the transient solution, and that the total solution is only altered by the effect of the particular solution due to the input (or forcing function). Thus, from a physical standpoint, the transfer function completely characterizes a linear system.

The transfer function has several properties which we wish to exploit. Suppose that we have two systems characterized by the differential equations

$$a\ddot{x} + b\dot{x} + cx = f(t) \tag{1.168}$$

and

$$d\ddot{y} + e\dot{y} + gy = x(t) \tag{1.169}$$

From the equations it can be seen that the first system has an input f(t), and an output x(t). The second system has an input x(t) and an output y(t). If we take Laplace transforms at 1.168 and 1.169 we get (assuming all initial conditions are equal to zero)

$$(as^2 + bs + c)\ X(s) = F(s) \tag{1.170}$$

and

$$(ds^2 + es + g)\ Y(s) = X(s) \tag{1.171}$$

Finding the transfer functions,

$$TF_1 = \frac{X(s)}{F(s)} = \frac{1}{as^2 + bs + c} \tag{1.171}$$

$$TF_2 = \frac{Y(s)}{X(s)} = \frac{1}{ds^2 + es + g} \tag{1.173}$$

Now, both of these systems can be represented schematically as shown in figure 1.16.

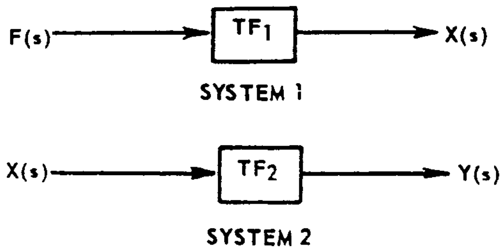
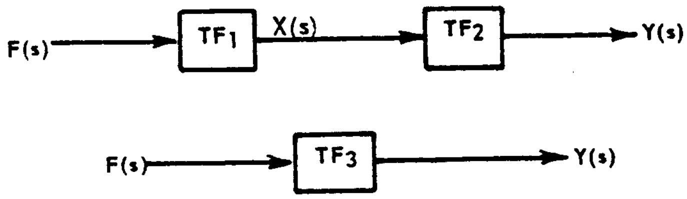


Figure 1.16

Suppose that we now wish to find the output, y(t), of system 2 due to the input, f(t), of system 1. Our first inspiration might tell us that the logical thing to do is to find x(t), but this is not necessary. We can "link" the two systems using the transfer functions, as shown in figure 1.17.



$TF_3 = (TF_1)(TF_2)$

Figure 1.17

The solution we seek, y(t), is then given by the inverse transform of Y(s), or

$$Y(s) = \left[TF_3 \right] F(s) \tag{1.174}$$

or

$$Y(s) = \left[TF_1 \right] \left[TF_2 \right] F(s) \tag{1.175}$$

This method of solution can be logically extended to include any number of systems we desire.

1.6 SIMULTANEOUS LINEAR DIFFERENTIAL EQUATIONS

In many physical problems, the mathematical description of the system can most conveniently be written as simultaneous differential equations with constant coefficients. The basic procedure for solving a system of n ordinary differential equations in n dependent variables consists in obtaining a set of equations from which all but one of the dependent variables, say x, can be eliminated. The equation resulting from the elimination is then solved for the variable x. Each of the other dependent variables is then obtained in a similar manner.

We shall consider two procedures for solution of simultaneous linear differential equations, using determinants.

Consider the system

$$2\frac{dx}{dt} + \frac{dy}{dt} - 4x - y = e^t \tag{1.177}$$

$$\frac{dx}{dt} + 3x + y = 0 \tag{1.178}$$

Using operator notation, they become

$$2(p - 2)x + (p - 1)y = e^t \tag{1.179}$$

$$(p + 3)x + y = 0 \tag{1.180}$$

● 1.6.1 SOLUTION BY MEANS OF DETERMINANTS AND OPERATOR NOTATION

Recall that for a determinant of second order the value of the determinant is given by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb \tag{1.176}$$

And then rewrite these equations in the following form

$$p_1 x + p_2 y = f_1(t) \tag{1.181}$$

$$p_3 x + p_4 y = f_2(t) \tag{1.182}$$

where the p's denote the polynomial operators which act on x and y.

Our solution for x can be given by Cramer's rule

$$\begin{vmatrix} p_1 & p_2 \\ p_3 & p_4 \end{vmatrix} x = \begin{vmatrix} f_1(t) & p_2 \\ f_2(t) & p_4 \end{vmatrix} \tag{1.183}$$

and our solution for y can be expressed as

$$\begin{vmatrix} p_1 & p_2 \\ p_3 & p_4 \end{vmatrix} y = \begin{vmatrix} p_1 & f_1(t) \\ p_3 & f_2(t) \end{vmatrix} \tag{1.184}$$

To solve the system given by equations 1.179 and 1.180, we write these equation in determinant form

$$\begin{vmatrix} 2(p - 2) & (p - 1 \\ (p + 3) & 1 \end{vmatrix} x = \begin{vmatrix} e^t & (p - 1) \\ 0 & 1 \end{vmatrix} \tag{1.185}$$

which is expanded to

$$(p^2 + 1)\ x = -e^t \tag{1.186}$$

giving a solution

$$x(t) = c_1 \cos t + c_2 \sin t - 1/2\, e^t \quad (1.187)$$

Solving for y,

$$\begin{vmatrix} 2(p-2) & (p-1) \\ (p+3) & 1 \end{vmatrix} y = \begin{vmatrix} 2(p-2) & e^t \\ (p+3) & 0 \end{vmatrix} \quad (1.188)$$

which can be expanded to

$$(p^2 + 1)y = 4e^t \quad (1.189)$$

giving a solution

$$y(t) = c_3 \cos t + c_4 \sin t + 2e^t \quad (1.190)$$

We know by examining 1.187 and 1.190 that extraneous constants are present, and to eliminate them we substitute back into equation 1.178 and see that

$$(c_2 + 3c_1 + c_3) \cos t + (3c_2 - c_1 + c_4) \sin t = 0 \quad (1.191)$$

Since 1.191 must hold for all values of t, the terms in parenthesis must vanish, giving

$$c_3 = -(3c_1 + c_2)$$

and

$$c_4 = c_1 - 3c_2$$

When these values are substituted in 1.190, we obtain the general solution.

1.6.2 SOLUTION BY MEANS OF LAPLACE TRANSFORMS

A very effective means of handling simultaneous linear differential equations is to take the Laplace transform of the set of equations and reduce the problem to a set of algebraic equations which can be solved explicitly for the dependent variable in s. This method is demonstrated below.

Given the set of equations

$$3\frac{d^2x}{dt^2} + x + \frac{d^2y}{dt^2} + 3y = f(t) \quad (1.192)$$

$$2 \frac{d^2x}{dt^2} + x + \frac{d^2y}{dt^2} + 2y = g(t) \tag{1.193}$$

where $x(0) = \dot{x}(0) = y(0) = \dot{y}(0) = 0$, find x(t) and y(t). Taking the Laplace transform of this system yields

$$(3s^2 + 1)\ X(s) + (s^2 + 3)\ Y(s) = F(s) \tag{1.194}$$

$$(2s^2 + 1)\ X(s) + (s^2 + 2)\ Y(s) = G(s) \tag{1.195}$$

From the previous section, we can solve for X(s) by rewriting these equations in determinant form, again by Cramer's rule

$$\begin{vmatrix} (3s^2 + 1) & (s^2 + 3) \\ (2s^2 + 1) & (s^2 + 2) \end{vmatrix} X(s) = \begin{vmatrix} F(s) & (s^2 + 3) \\ G(s) & (s^2 + 2) \end{vmatrix} \tag{1.196}$$

Since we are using Laplace transforms instead of operators, however, we can take this equation one step further. We can now solve explicitly for X(s), giving us

$$X(s) = \frac{\begin{vmatrix} F(s) & (s^2 + 3) \\ G(s) & (s^2 + 2) \end{vmatrix}}{\begin{vmatrix} (3s^2 + 1) & (s^2 + 3) \\ (2s^2 + 1) & (s^2 + 2) \end{vmatrix}} \tag{1.197}$$

In a similar manner,

$$Y(s) = \frac{\begin{vmatrix} (3s^2 + 1) & F(s) \\ (2s^2 + 1) & G(s) \end{vmatrix}}{\begin{vmatrix} (3s^2 + 1) & (s^2 + 3) \\ (2s^2 + 1) & (s^2 + 2) \end{vmatrix}} \tag{1.198}$$

For the particular inputs f(t) = t and g(t) = 1,

$$X(s) = \frac{\begin{vmatrix} \frac{1}{s^2} & (s^2 + 3) \\ \frac{1}{s} & (s^2 + 2) \end{vmatrix}}{(s^4 - 1)} = \frac{-s^3 + s^2 - 3s + 2}{s^2(s^4 - 1)} \qquad (1.199)$$

Expanded as a partial fraction

$$X(s) = \frac{A}{s^2} + \frac{B}{s} + \frac{Cs + D}{(s^2 + 1)} + \frac{E}{s - 1} + \frac{F}{(s + 1)} = \frac{-s^3 + s^2 - 3s + 2}{s^2(s^4 - 1)} \qquad (1.200)$$

Solving for A, B, etc., we have

$$X(s) = \frac{-2}{s^2} + \frac{3}{s} + \frac{1/2 - s}{s^2 + 1} - \frac{7/4}{s + 1} - \frac{1/4}{s - 1} \qquad (1.201)$$

which yields a solution

$$x(t) = -2t + 3 - 7/4e^{-t} - 1/4e^{t} + 1/2 \sin t - \cos t \qquad (1.202)$$

A similar approach will obtain the solution for y(t).

In the case of three simultaneous differential equations, the application of Laplace will yield the proper solutions.

$$P_1(s)\ X(s) + P_2(s)\ Y(s) + P_3(s)\ Z(s) = F_1(s) \qquad (1.203)$$

$$Q_1(s)\ X(s) + Q_2(s)\ Y(s) + Q_3(s)\ Z(s) = F_2(s) \qquad (1.204)$$

$$R_1(s)\ X(s) + R_2(s)\ Y(s) + R_3(s)\ Z(s) = F_3(s) \qquad (1.205)$$

where

$$X(s) = \frac{\begin{vmatrix} F_1 & P_2 & P_3 \\ F_2 & Q_2 & Q_3 \\ F_3 & R_2 & R_3 \end{vmatrix}}{\begin{vmatrix} P_1 & P_2 & P_3 \\ Q_1 & Q_2 & Q_3 \\ R_1 & R_2 & R_3 \end{vmatrix}} \qquad (1.206)$$

Y(s) and Z(s) will have similar forms.

PROBLEMS: Set V, page 1.119.

Table II

LAPLACE TRANSFORMS

	F(s)	f(t)
1.	$\int_0^\infty e^{-st} f(t)dt$	$f(t)$
2.	$s^n F(s) - s^{n-1} f(0+) - s^{n-2} f'(0+) - \dots - f^{(n-1)}(0+)$	$f^{(n)}(t)$
3.	$\frac{1}{s}$	1
4.	$\frac{1}{s^2}$	t
5.	$\frac{n!}{s^{n+1}}$ (n = 1, 2, ...)	t^n
6.	$\frac{1}{s + a}$	e^{-at}
7.	$\frac{1}{(s + a)^2}$	te^{-at}
8.	$\frac{n!}{(s + a)^{n+1}}$ (n = 1, 2, ...)	$t^n e^{-at}$
9.	$\frac{1}{(s + a)(s + b)}$ $a \neq b$	$\frac{1}{b - a}(e^{-at} - e^{-bt})$
10.	$\frac{s}{(s + a)(s + b)}$ $a \neq b$	$\frac{1}{a - b}(ae^{-at} - be^{-bt})$
11.	$\frac{1}{(s + a)(s + b)(s + c)}$	$\frac{(b - c)e^{-at} - (a - c)e^{-bt} + (a - b)e^{-ct}}{(a - b)(b - c)(a - c)}$
12.	$\frac{a}{s^2 + a^2}$	$\sin at$

Table II (Concluded)

	F(s)	f(t)
13.	$\frac{s}{s^2 + a^2}$	cos at
14.	$\frac{a^2}{s(s^2 + a^2)}$	1 - cos at
15.	$\frac{a^3}{s^2(s^2 + a^2)}$	at - sin at
16.	$\frac{2a^3}{(s^2 + a^2)^2}$	sin at - at cos at
17.	$\frac{2as}{(s^2 + a^2)^2}$	t sin at
18.	$\frac{2as^2}{(s^2 + a^2)^2}$	sin at + at cos at
19.	$\frac{s^2 - a^2}{(s^2 + a^2)^2}$	t cos at
20.	$\frac{(b^2 - a^2)s}{(s^2 + a^2)(s^2 + b^2)}$ $(a^2 \neq b^2)$	cos at - cos bt
21.	$\frac{b}{(s + a)^2 + b^2}$	e^{-at} sin bt
22.	$\frac{s + a}{(s + a)^2 + b^2}$	e^{-at} cos bt

1.7 PROBLEM SET I

1. Solve for y.

   a. $\frac{dy}{dx} = x^4 + 4x + \sin 6x$

   b. $\frac{d^2y}{dx^2} = e^{-x} + \sin \omega x$

   c. $\frac{d^3y}{dx^3} = x^5$

   d. $y\frac{dy}{dx} + 3x^2 = 0$

   e. $(x - 1)^2 ydx + x^2 (y + 1)dy = 0$

2. Test for exactness and solve if exact.

   a. $(y^2 - x)dx + (x^2 - y)dy = 0$

   b. $(2x^3 + 3y)dx + (3x + y - 1)dy = 0$

   c. $(2xy^4 e^y + 2xy^3 + y)dx + (x^2y^4e^y - x^2y^2 - 3x)dy = 0$

   d. Multiply c. by $1/y^4$.

3. Solve for y_t using operator notation.

   a. $5y' + 6y = 0$

   b. $y''' - 5y'' - 24y' = 0$

   c. $y'' + 12y' + 36y = 0$

   d. $y^{iv} + 25y'' = 0$

   e. $y'' + 4y' + 13y = 0$

1.8 SOLUTION TO PROBLEM SET 1

1a. $$\frac{dy}{dx} = x^4 + 4x + \sin 6x$$

By direct integration

$$\int dy = \int (x^4 + 4x + \sin 6x)dx + C$$

$$y = \frac{x^5}{5} + 2x^2 - \frac{\cos 6x}{6} + C$$

1b. $$\frac{d^2y}{dx^2} = e^{-x} + \sin \omega x$$

By direct integration

$$\int dy' = \int (e^{-x} + \sin \omega x)dx + C_1$$

$$y' = -e^{-x} - \frac{\cos \omega x}{\omega} + C_1$$

$$\int dy = \int (-e^{-x} - \frac{\cos \omega x}{\omega} + C_1)dx + C_2$$

$$y = e^{-x} - \frac{\sin \omega x}{\omega^2} + C_1 x + C_2$$

1c. $y''' = x^5$

By direct integration

$$\int dy'' = \int x^5\,dx + C_1$$

$$y'' = \frac{x^6}{6} + C_1$$

$$\int dy' = \int\left(\frac{x^6}{6} + C_1\right)dx + C_2$$

$$y' = \frac{x^7}{42} + C_1x + C_2$$

$$\int dy = \int\left(\frac{x^7}{42} + C_1x + C_2\right)dx + C_3$$

$$y = \frac{x^8}{336} + \frac{C_1x^2}{2} + C_2x + C_3$$

1d. $y\frac{dy}{dx} + 3x^2 = 0$

Separate Variables

$$\int y\,dy = \int -3x^2\,dx + C$$

$$\frac{y^2}{2} = -x^3 + C$$

1e. $(x - 1)^2 y dx + x^2 (y + 1)dy = 0$

Separate Variables

$$\frac{y+1}{y} dy = -\frac{(x-1)^2}{x^2} dx$$

$$\int (1 + \frac{1}{y}) dy = -\int (\frac{x^2 - 2x + 1}{x^2}) dx + C$$

$$\int (1 + \frac{1}{y}) dy = -\int (1 - \frac{2}{x} + \frac{1}{x^2}) dx + C$$

$$y + \ln y = -x + 2\ln x + \frac{1}{x} + C$$

2a. $(y^2 - x)dx + (x^2 - y)dy = 0$

$M = (y^2 - x) \qquad N = x^2 - y$

$\frac{\partial M}{\partial y} = 2y \qquad \frac{\partial N}{\partial x} = 2x$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial y} \Longrightarrow$ Not Exact

---

2b. $(2x^3 + 3y)dx + (3x + y - 1)dy = 0$

$M = 2x^3 + 3y \qquad N = 3x + y - 1$

$$\frac{\partial M}{\partial y} = 3 \qquad \frac{\partial N}{\partial x} = 3$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad \text{Exact}$$

$$\int_a^x (2x^3 + 3y)dx + \int_b^y (3a + y - 1)dy = k$$

$$(1/2x^4 + 3xy)\Big|_a^x + (3ay + 1/2y^2 - y)\Big|_b^y = k$$

$$(1/2x^4 + 3xy) - (1/2a^4 + 3ay) + (3ay + 1/2y^2 - y) - (3ab + 1/2b^2 - b) = k$$

$$1/2x^4 + 3xy + 1/2y^2 - y = k + 1/2a^4 + 3ab + 1/2b^2 - b = k$$

$$1/2x^4 + 3xy + 1/2y^2 - y = k$$

2c. $(2xy^4e^y + 2xy^3 + y)dx + (x^2y^4e^y - x^2y^2 - 3x)dy = 0$

$$M = 2xy^4e^y + 2xy^3 + y \qquad N = x^2y^4e^y - x^2y^2 - 3x$$

$$\frac{\partial M}{\partial y} = 2x(4y^3e^y + y^4e^y) + 6xy^2 + 1$$

$$\frac{\partial N}{\partial x} = 2xy^4e^y - 2xy^2 - 3$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \qquad \text{Not Exact}$$

2d. $(2xe^y + \frac{2x}{y} + \frac{1}{y^3})dx + (x^2e^y - \frac{x^2}{y^2} - \frac{3x}{y^4})dy = 0$

$$M = (2xe^y + \frac{2x}{y} + \frac{1}{y^3}) \qquad N = (x^2e^y - \frac{x^2}{y^2} - \frac{3x}{y^4})$$

$$\frac{\partial M}{\partial y} = 2xe^y - \frac{2x}{y^2} - \frac{3}{y^4} \qquad \frac{\partial N}{\partial x} = 2xe^y - \frac{2x}{y^2} - \frac{3}{y^4}$$

$$\frac{\partial N}{\partial y} = \frac{\partial N}{\partial x} \Longrightarrow \text{Exact}$$

$$\int_a^x (2xe^y + \frac{2x}{y} + \frac{1}{y^3})\,dx + \int_b^y (a^2e^y - \frac{a^2}{y^2} - \frac{3a}{y^4})\,dy = k$$

$$(x^2e^y + \frac{x^2}{y} + \frac{x}{y^3})\Big|_a^x + (a^2e^y + \frac{a^2}{y} + \frac{a}{y^3})\Big|_b^y = k$$

$$(x^2e^y + \frac{x^2}{y} + \frac{x}{y^3}) - (a^2e^y + \frac{a^2}{y} + \frac{a}{y^3}) + (a^2e^y + \frac{a^2}{y} + \frac{a}{y^3}) - (a^2e^b + \frac{a^2}{b} + \frac{a}{b^3}) = k$$

$$x^2e^y + \frac{x^2}{y} + \frac{x}{y^3} = k + a^2e^b + \frac{a^2}{b} + \frac{a}{b^3} = k$$

<u>3a</u>. $5y' + 6y = 0$

$$5p + 6 = 0$$

$$p = -\frac{6}{5}$$

$$y_t = Ce^{-6/5x}$$

3b. $y''' - 5y'' - 24y' = 0$

$$p^3 - 5p^2 - 24p = 0$$

$$p(p^2 - 5p - 24) = 0$$

$$p(p - 8)(p + 3) = 0$$

$$p = 0, 8, -3$$

$$y_t = C_1 + C_2e^{8x} + C_3e^{-3x}$$

3c. $y'' + 12y' + 36y = 0$

$$p^2 + 12p + 36 = 0$$

$$(p + 6)(p + 6) = 0$$

$$p = -6, -6$$

$$y_t = C_1e^{-6x} + C_2xe^{-6x}$$

3d. $y^{IV} + 25y'' = 0$

$$p^4 + 25p^2 = 0$$

$$p^2 (p^2 + 25) = 0$$

$$p = 0,\ 0,\ \pm 5j$$

$$y_t = C_1 + C_2 x + C_3 \sin (5x + \phi)$$

or

$$y_t = C_1 + C_2 x + C_4 \sin 5x + C_5 \cos 5x$$

3e. $y'' + 4y' + 13y = 0$

$$p^2 + 4p + 13 = 0$$

$$(p + 2)^2 + 9 = 0$$

$$p + 2 = \pm 3j$$

$$p = -2 \pm 3j$$

$$y_t = C_1 e^{-2x} \sin (3x + \phi)$$

or

$$y_t = e^{-2x} (C_2 \sin 3x + C_3 \cos 3x)$$

1.9 PROBLEM SET II

Given:

1. $\ddddot{y} + 36\ddot{y} = 6 + t$

2. $\ddot{y} + 5\dot{y} + 6y = 3e^{-3t}$

3. $\ddot{y} + 4\dot{y} + 4y = \cos t$

4. $2\ddot{x} + 4\dot{x} + 20x = 6t + \frac{6}{5}$

5. $3\dot{x} + 2x = -4e^{-2t}$

Find:

a. The transient solution.

b. The particular solution.

c. Substitute the following boundary conditions to eliminate arbitrary constants.

   1) For problem 1 above $\dddot{y}(0) = \frac{1}{36}$, $\ddot{y}(0) = \frac{1}{6}$, $\dot{y}(0) = y(0) = 0$

   2) For problem 2 $\dot{y}(0) = -6$, $y(0) = 1$

   3) For problem 3 $y(0) = \frac{28}{25}$, $\dot{y}(0) = -\frac{104}{25}$

   4) For problem 4 $\dot{x}(0) = -\frac{27}{10}$, $x(0) = 3$

   5) For problem 5 $x(3) = -0.14$

d. Find ω_n , ω_d , ζ , τ where applicable.

e. Describe the system damping where applicable (i.e. underdamped, overdamped, etc.).

f. SKETCH the total response (i.e. total solution).

1.10 SOLUTION TO PROBLEM SET II

1. $\overset{....}{y} + 36\ddot{y} = 6 + t$

a. $p^4 + 36p^2 = 0$

$p^2 (p^2 + 36) = 0$

$p = 0, 0, \pm 6j$

$y_t = C_1 + C_2 t + C_3 \sin (6t + \phi)$

or

$y_t = C_1 + C_2 t + C_4 \sin 6t + C_5 \cos 6t$

b. Assume

$y_p = At + B$

Checking the transient solution we see both of these same terms. We must multiply by the independent variable until we do not duplicate terms in the transient solution.

$y_p = At^3 + Bt^2$

$\dot{y}_p = 3At^2 + 2Bt$

$\ddot{y}_p = 6At + 2B$

$\dddot{y}_p = 6A$

$\overset{....}{y}_p = 0$

Substituting into the original differential equation,

$$0 + 36(6At + 2B) = 6 + t$$

$$216At + 72B = 6 + t$$

Equating like coefficients,

$$216A = 1$$

$$A = \frac{1}{216}$$

$$72B = 6$$

$$B = \frac{1}{12}$$

$$y_p = \frac{t^3}{216} + \frac{t^2}{12}$$

c) $y = y_t + y_p$

(1) $$y = C_1 + C_2 t + C_3 \sin(6t + \emptyset) + \frac{t^3}{216} + \frac{t^2}{12}$$

(2) $$y = C_1 + C_2 t + C_4 \sin 6t + C_5 \cos 6t + \frac{t^3}{216} + \frac{t^2}{12}$$

Using (1)

$$\dot{y} = C_2 + 6C_3 \cos(6t + \emptyset) + \frac{t^2}{72} + \frac{t}{6}$$

$$\ddot{y} = -36C_3 \sin(6t + \emptyset) + \frac{t}{36} + \frac{1}{6}$$

$$\dddot{y} = -216C_3 \cos(6t + \emptyset) + \frac{1}{36}$$

Substituting boundary conditions.

$$\dddot{y}(0) = -216C_3 \cos \phi + \frac{1}{36} = \frac{1}{36}$$

$$C_3 \cos \phi = 0$$

$$\ddot{y}(0) = -36C_3 \sin \phi + \frac{1}{6} = \frac{1}{6}$$

$$C_3 \sin \phi = 0$$

$$C_3 \sin \phi = C_3 \cos \phi$$

$$\sin \phi = \cos \phi$$

$$\phi = 45^\circ$$

$$C_3 = 0$$

$$C_2 = 0$$

$$C_1 = 0$$

$$y = \frac{t^3}{216} + \frac{t^2}{12}$$

Using (2)

$$y = C_1 + C_2 t + C_4 \sin 6t + C_5 \cos 6t + \frac{t^3}{216} + \frac{t^2}{12}$$

$$\dot{y} = C_2 + 6C_4 \cos 6t - 6C_5 \sin 6t + \frac{t^2}{72} + \frac{t}{6}$$

$$\ddot{y} = -36C_4 \sin 6t = 36C_5 \cos 6t + \frac{t}{36} + \frac{1}{6}$$

$$\dddot{y} = -216C_4 \cos 6t + 216C_5 \sin 6t + \frac{1}{36}$$

$$\dddot{y}(0) = -216C_4 + \frac{1}{36} = \frac{1}{36}$$

$$C_4 = 0$$

$$\ddot{y}(0) = -36C_5 + \frac{1}{6} = \frac{1}{6}$$

$$C_5 = 0$$

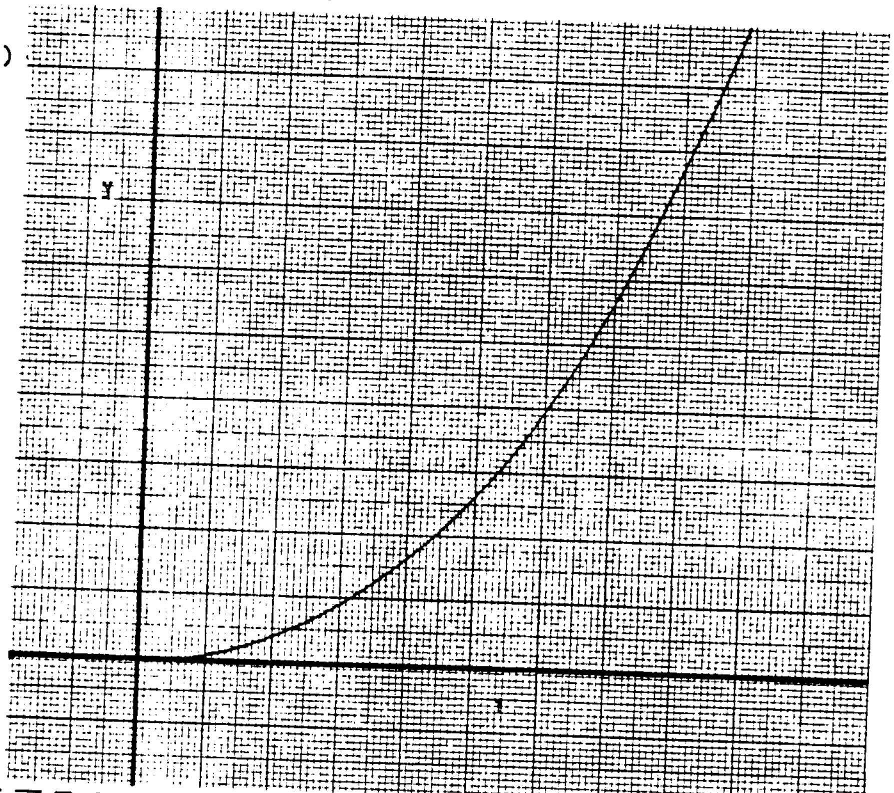
$$\dot{y}(0) = C_2 = 0$$

$$y(0) = C_1 = 0$$

$$\therefore \quad y = \frac{t^3}{216} + \frac{t^2}{12}$$

Numbers d) and e) do not apply.

f)



2. $\ddot{y} + 5\dot{y} + 6y = 3e^{-3t}$

a) $p^2 + 5p + 6 = 0$

$(p + 3)(p + 2) = 0$

$p = -3, -2$

$y_t = C_1 e^{-3t} + C_2 e^{-2t}$

b) Assume $y_p = Ae^{-3t}$ (forcing function and all its derivatives) Cross check with y_t. Since y_p appears in y_t multiply by t in order to eliminate the duplication.

$$\therefore \quad y_p = Ate^{-3t}$$

$$\dot{y}_p = -3Ate^{-3t} + Ae^{-3t}$$

$$\ddot{y}_p = 9Ate^{-3t} - 3Ae^{-3t} - 3Ae^{-3t}$$

$$= 9Ate^{-3t} - 6Ae^{-3t}$$

Substituting into the original D.E.

$$e^{-3t}\,[(9At - 6A) + 5(A - 3At) + 6At] = 3e^{-3t}$$

$$e^{-3t}\,(-A) = 3e^{-3t}$$

$$A = -3$$

$$y_p = -3te^{-3t}$$

c) $y = C_1e^{-3t} + C_2e^{-2t} - 3te^{-3t}$

$\dot{y} = -3C_1e^{-3t} - 2C_2e^{-2t} + 9te^{-3t} - 3e^{-3t}$

Substituting the boundary conditions

$$y(0) = 1 = C_1 + C_2 \Longrightarrow C_2 = 1 - C_1$$

$$\dot{y}(0) = -6 = -3C_1 - 2C_2 - 3$$

$$-3 = -3C_1 - 2 + 2C_1$$

$$C_1 = 1$$

$$C_2 = 0$$

$$y = (1 - 3t)e^{-3t}$$

From the homogeneous equation

$$y + 5\dot{y} + 6y = 0$$

$$\omega_n^2 = 6$$

$$\omega_n = \sqrt{6}$$

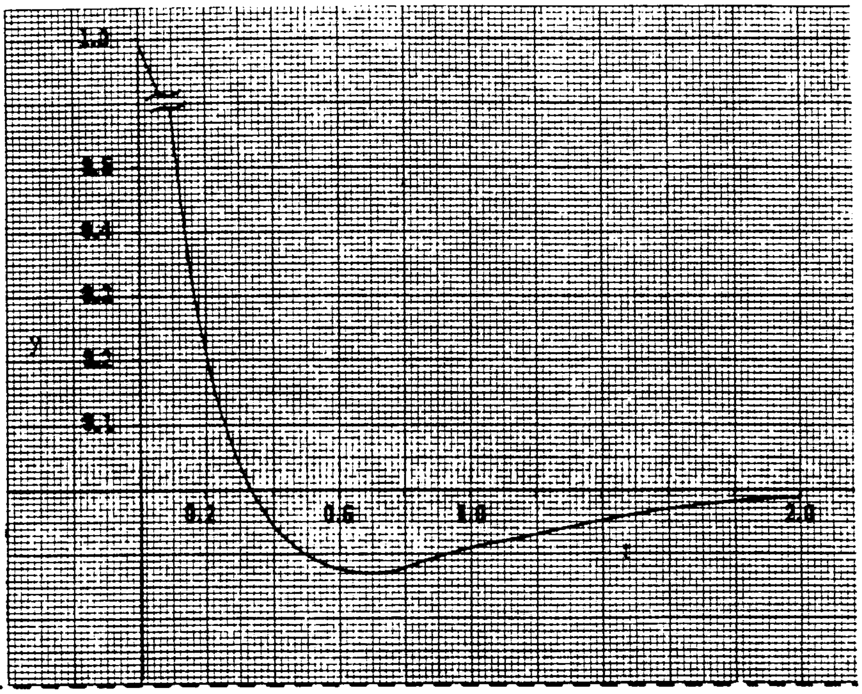
$$2\zeta\omega_n = 5$$

$$\zeta = \frac{5}{2\sqrt{6}} = 1.02$$

$$\omega_d = \text{is undefined}$$

The system is overdamped and non-oscillatory.

f)



3. $\ddot{y} + 4\dot{y} + 4y = \cos t$

a) $p^2 + 4p + 4 = 0$

$(p + 2)^2 = 0$

$p = -2, -2$

$y_t = C_1 e^{-2t} + C_2 t e^{-2t}$

b) $y_p = A_1 \sin t + A_2 \cos t$

$\dot{y}_p = A_1 \cos t - A_2 \sin t$

$\ddot{y}_p = -A_1 \sin t - A_2 \cos t$

Substituting into our original D.E.

$$(-A_1 - 4A_2 + 4A_1) \sin t \; + \; (-A_2 + 4A_1 + 4A_2) \cos t \; = \; \cos t$$

$$(3A_1 - 4A_2) \sin t \; + \; (4A_1 + 3A_2) \cos t \; = \; \cos t$$

Equating like coefficients

$$3A_1 - 4A_2 = 0 \qquad\qquad 4A_1 + 3A_2 = 1$$

Sub.

$$A_1 = \frac{4}{3} A_2 \qquad\qquad \frac{16}{3} A_2 + 3A_2 = 1$$

Sub.

$$A_1 = \frac{4}{25} \qquad\qquad A_2 = \frac{3}{25}$$

$$\therefore \; y_p = \frac{4}{25} \sin t + \frac{3}{25} \cos t$$

$$y = y_t + y_p$$

c) $$y = C_1 e^{-2t} + C_2 t e^{-2t} + \frac{4}{25} \sin t + \frac{3}{25} \cos t$$

$$y(0) = C_1 + \frac{3}{25} = \frac{28}{25}$$

$$C_1 = 1$$

$$\dot{y}(t) = -2C_1 e^{-2t} + C_2 e^{-2t} - 2C_2 t e^{-2t} + \frac{4}{25} \cos t - \frac{3}{25} \sin t$$

$$\dot{y}(0) = -2C_1 + C_2 + \frac{4}{25}$$

$$= -\frac{46}{25} + C_2 = -\frac{104}{25}$$

$$C_2 = -\frac{58}{25}$$

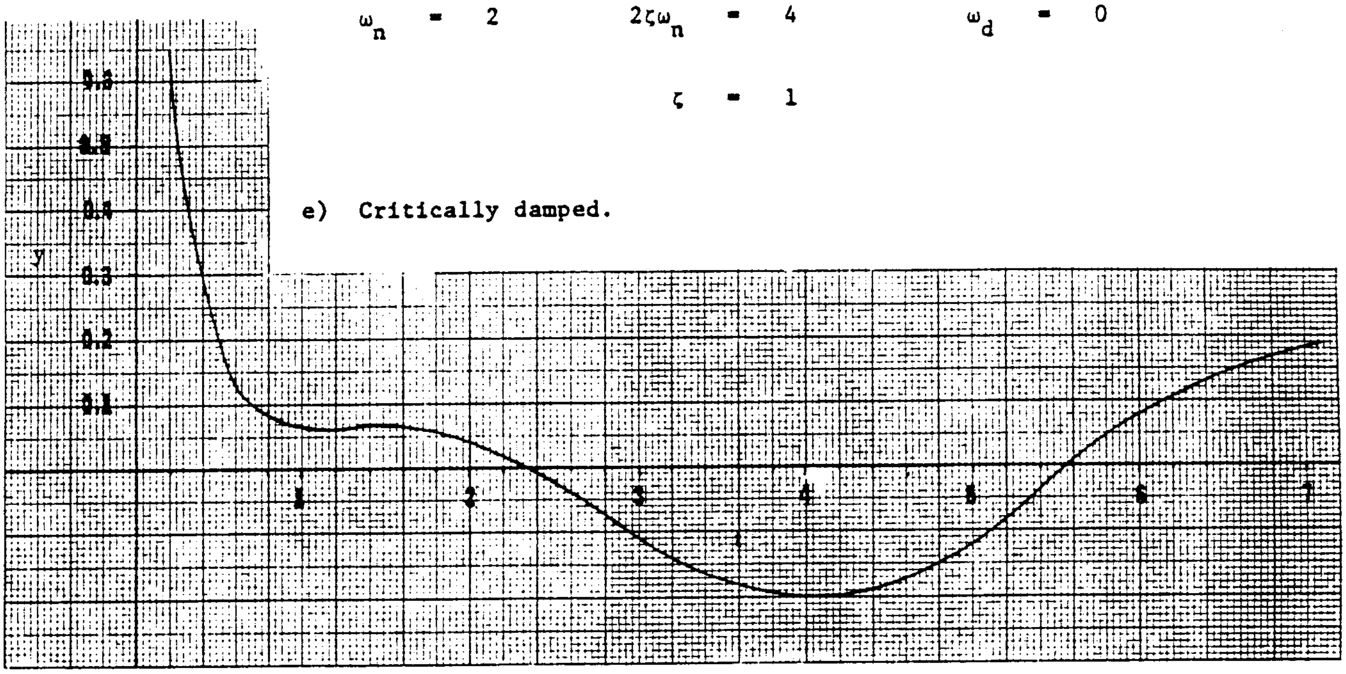
$$\therefore \; y = e^{-2t} - \frac{58}{25} t e^{-2t} + \frac{4}{25} \sin t + \frac{3}{25} \cos t$$

d) $\ddot{y} + 4\dot{y} + 4 = 0$

$\omega_n = 2 \qquad 2\zeta\omega_n = 4 \qquad \omega_d = 0$

$\zeta = 1$

e) Critically damped.



4. $2\ddot{x} + 4\dot{x} + 20x = 6t + \frac{6}{5}$

Place in "Standard Form" $\ddot{x} + 2\dot{x} + 10x = 0$

a) $\ddot{x} + 2\dot{x} + 10x = 0$

$p^2 + 2p + 10 = 0$

$(p + 1)^2 + 9 = 0$

$p = -1 \pm 3j$

$\therefore \quad x_t = e^{-t} (C_1 \sin 3t + C_2 \cos 3t)$

b) Assume $x_p = At + B$

$\dot{x}_p = A$

$\ddot{x}_p = 0$

Substituting

$$0 + 4A + 20At + 20B = 6t + \frac{6}{5}$$

$$20A = 6$$

$$A = \frac{3}{10}$$

$$4A + 20B = \frac{6}{5}$$

$$B = 0$$

$$\therefore \quad x_p = \frac{3}{10} t$$

$$x = x_t + x_p$$

c) $x(t) = e^{-t} (C_1 \sin 3t + C_2 \cos 3t) + \frac{3}{10} t$

$$\dot{x}(t) = -e^{-t} (C_1 \sin 3t + C_2 \cos 3t) + e^{-t} (3C_1 \cos 3t - 3C_2 \sin 3t) + \frac{3}{10}$$

$$x(0) - 3 = C_2$$

$$\dot{x}(0) = -\frac{27}{10} = -C_2 + 3C_1 + \frac{3}{10}$$

$$-\frac{27}{10} = -3 + 3C_1 + \frac{3}{10}$$

$$C_1 = 0$$

$$\therefore \quad x(t) = 3e^{-t}\cos 3t + \frac{3}{10}t$$

d) $\ddot{x} + 2\dot{x} + 10x = 0$

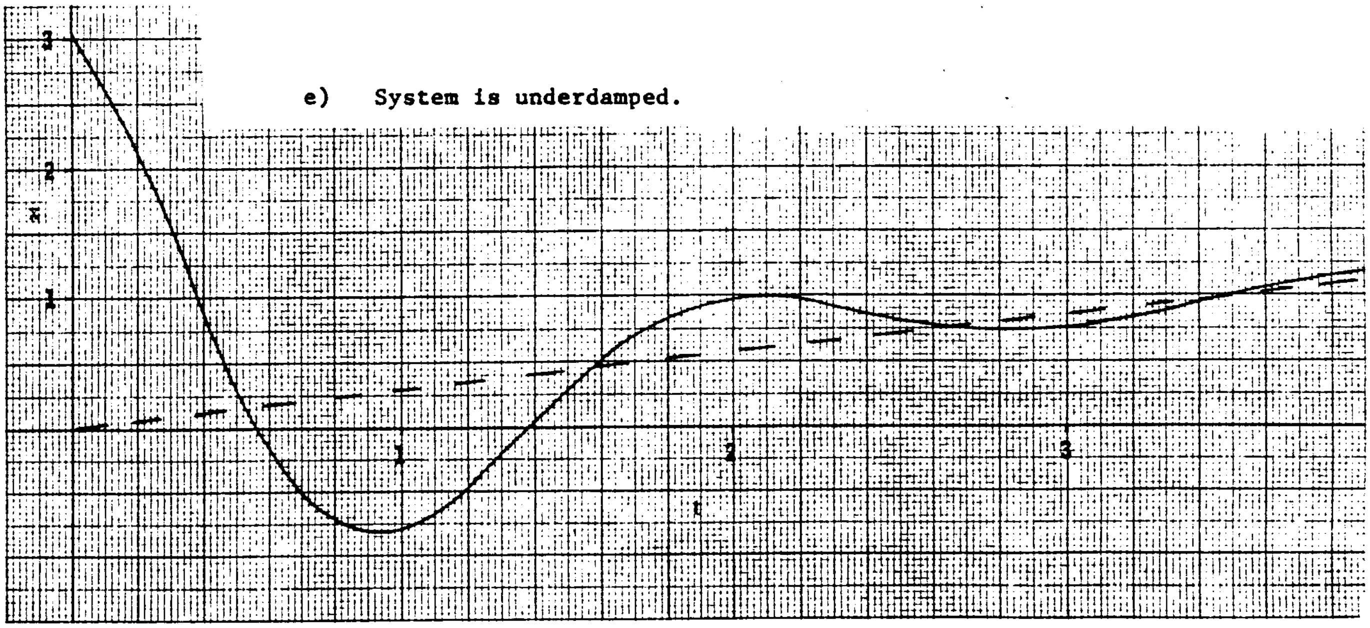
$$\omega_n = \sqrt{10}$$

$$2\zeta\omega_n = 2$$

$$\zeta = 1/\sqrt{10}$$

$$\omega_d = \omega_n\sqrt{1 - \zeta^2}$$

$$\omega_d = 3$$



e) System is underdamped.

---

5. Given

$$3\dot{x} + 2x = -4e^{-2t}$$

a) $3p + 2 = 0$

$$p = -2/3$$

$$x_t = C_1 e^{-2t/3}$$

b) Assume

$$x_p = Ae^{-2t}$$

$$\dot{x}_p = -2Ae^{-2t}$$

Substituting

$$-6Ae^{-2t} + 2Ae^{-2t} = -4e^{-2t}$$

$$A = 1$$

$$x_p = e^{-2t}$$

c) $x = C_1 e^{-2t/3} + e^{-2t}$

Substituting the boundary condition

$$x(3) = -0.14 = C_1(0.14) + .002$$

$$C_1 = \frac{-.142}{.14} = -1.01 \approx -1$$

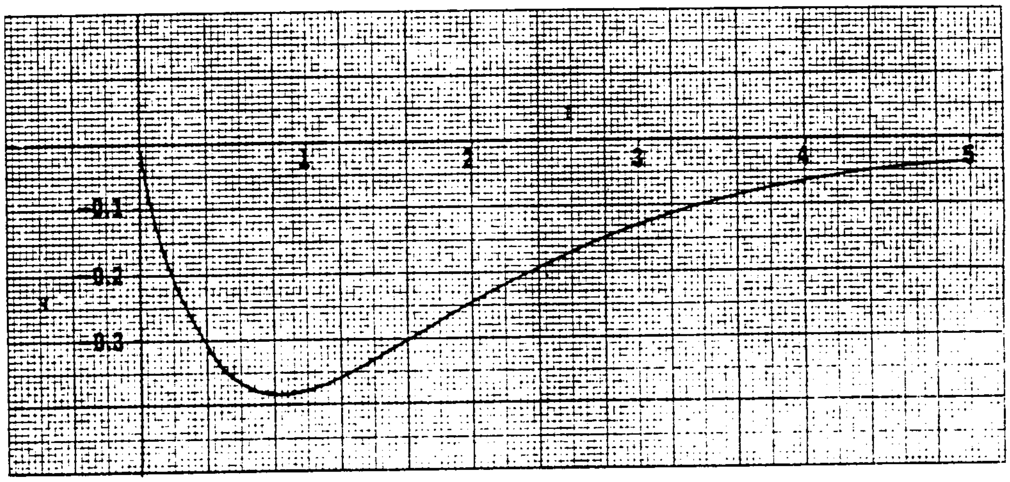
$$x(t) = -e^{-2t/3} + e^{-2t}$$

d) From the transient solution

$$x_t = C_1 e^{-2t/3}$$

$$\tau = 1.5$$

e) N/A



1.11 PROBLEM SET III

Given: $C_1\ddot{x} + C_2\dot{x} + C_3x = f(t)$

Find: $X(s)$

When:

	#1	#2	#3	#4	#5
C_1 =	3	1	4	2	1
C_2 =	1	-2	3	4	6
C_3 =	6	5	-1	7	9
$x(0)$ =	0	-1	3	-1	2
$\dot{x}(0)$ =	0	9	-2	0	-1
$f(t)$ =	$\sin 6t$	$e^{-t} \sin 3t$	$t^3 - t \sin 2t$	$6e^{-4t} + 6t \cos 4t$	$\frac{2 - \cos 2t}{4}$

■ 1.12 SOLUTION TO PROBLEM SET III

In general for a second order D.E.

$$C_1\ddot{x} + C_2\dot{x} + C_3x = f(t)$$

Take the Laplace Transform

$$C_1\,[s^2X(s) - sx(0) - \dot{x}(0)] + C_2\,[sX(s) - x(0)] + C_3X(s) = F(s)$$

$$(C_1s^2 + C_2s + C_3)X(s) = F(s) + C_1sx(0) + C_1\dot{x}(0) + C_2x(0)$$

$$X(s) = \frac{F(s) + C_1sx(0) + C_1\dot{x}(0) + C_2x(0)}{(C_1s^2 + C_2s + C_3)}$$

This equation can be used for any second order D.E. with constant coefficients.

1. Given

$$C_1 = 3$$

$$C_2 = 1$$

$$C_3 = 6$$

$$x(0) = \dot{x}(0) = 0$$

$$f(t) = \sin 6t$$

From transform pair 12, $F(s) = \frac{6}{s^2 + 36}$

$$X(s) = \frac{6}{(s^2 + 36)(3s^2 + s + 6)}$$

---

2. Given:

$$C_1 = 1$$

$$C_2 = -2$$

$$C_3 = 5$$

$$x(0) = -1$$

$$\dot{x}(0) = 9$$

$$f(t) = e^{-t} \sin 3t$$

From transform pair 21,

$$F(s) = \frac{3}{(s + 1)^2 + 9} = \frac{3}{s^2 + 2s + 10}$$

$$X(s) = \left[\frac{\left(\frac{3}{s^2 + 2s + 10}\right) - s + 11}{s^2 - 2s + 5} \right]$$

Obviously, this can be reduced to a simpler form, but that's not necessary for this exercise.

---

3. Given:

$$C_1 = 4$$

$$C_2 = 3$$

$$C_3 = -1$$

$$x(0) = 3$$

$$\dot{x}(0) = -2$$

$$f(t) = t^3 - t \sin 2t$$

From transform pairs 5 and 17,

$$F(s) = \frac{6}{s^4} - \frac{4s}{(s^2 + 4)^2}$$

$$= \frac{6s^4 - 4s^5 + 48s^2 + 96}{s^4(s^2 + 4)^2}$$

$$X(s) = \left[\frac{\left(\frac{6s^4 - 4s^5 + 48s^2 + 96}{s^4(s^2 + 4)^2} \right) + 12s + 1}{(4s^2 + 3s - 1)} \right]$$

---

4. Given:

$$C_1 = 2$$

$$C_2 = 4$$

$$C_3 = 7$$

$$x(0) = -1$$

$$\dot{x}(0) = 0$$

$$f(t) = 6e^{-4t} + 6t \cos 4t$$

From transform pairs 6 and 19,

$$F(s) = \frac{6}{s + 4} + \frac{6(s^2 - 16)}{(s^2 + 16)^2}$$

$$F(s) = \frac{6s^4 + 192s^2 + 1536 + 6s^3 + 24s^2 - 96s - 384}{(s + 4)(s^2 + 16)^2}$$

$$F(s) = \frac{6s^4 + 6s^3 + 216s^2 - 96s + 1152}{(s + 4)(s^2 + 16)^2}$$

$$X(s) = \left[\frac{F(s) - (2s + 4)}{2s^2 + 4s + 7}\right]$$

---

5. Given:

$$C_1 = 1$$

$$C_2 = 6$$

$$C_3 = 9$$

$$x(0) = 2$$

$$\dot{x}(0) = -1$$

$$f(t) = \frac{2 - \cos 2t}{4} = \frac{1}{2} - \frac{1}{4}\cos 2t$$

From transform pairs 3 and 13,

$$F(s) = \frac{1}{2}\left(\frac{1}{s}\right) - \frac{1}{4}\left(\frac{s}{s^2 + 4}\right)$$

$$F(s) = \frac{1}{4}\left[\frac{2s^2 + 8 - s^2}{s(s^2 + 4)}\right] = \frac{1}{4}\left[\frac{s^2 + 8}{s(s^2 + 4)}\right]$$

$$X(s) = \left[\frac{F(s) + (2s + 11)}{s^2 + 6s + 9}\right]$$

1.13 PROBLEM SET IV

A. Expand by partial fractions:

1) $$\frac{5s^2 + 29s + 36}{(s + 2)(s^2 + 4s + 3)}$$

2) $$\frac{2s^2 + 6s + 5}{(s^2 + 3s + 2)(s + 1)}$$

3) $$\frac{2s^4 + 7s^3 + 27s^2 + 51s + 27}{(s^3 + 9s)(s^2 + 3s + 3)}$$

4) $$\frac{7s^4 - 3s^3 + 56s^2 - 17s + 107}{(s - 1)(s^2 + 4)^2}$$

B. Solve by Laplace:

1) $\dot{y} + y = 5$, $y(0) = 0$

2) $\dot{y} + y = e^{-3t}$, $y(0) = 1$

3) $\dot{x} + 2x = \sin t$, $x(0) = 5$

4) $\ddot{x} + 2\dot{x} + 9x = 2 \sin 3t$, $x(0) = \dot{x}(0) = 0$

5) $\ddot{x} + 5\dot{x} + 6x = 3e^{-3t}$, $x(0) = \dot{x}(0) = 1$

6) $\ddot{x} + 2\dot{x} + 10x = 3t + \frac{3}{5}$, $x(0) = 3$
$\dot{x}(0) = -\frac{27}{10}$

1.14 SOLUTION TO PROBLEM SET IV

A.

1) $$\frac{5s^2 + 29s + 36}{(s+2)(s^2+4s+3)} = \frac{5s^2 + 29s + 36}{(s+2)(s+3)(s+1}$$

$$= \frac{A}{s+2} + \frac{B}{s+3} + \frac{C}{s+1}$$

$$= \frac{A(s+3)(s+1) + B(s+2)(s+1) + C(s+2)(s+3)}{(s+1)(s+2)(s+3)}$$

Setting numerators equal:

$$5s^2 + 29s + 36 = A(s+3)(s+1) + B(s+2)(s+1) + C(s+2)(s+3)$$

let $s = -3$: $45 - 87 + 36 = B(-1)(-2)$

$$2B = -6$$

$$\underline{B = -3}$$

let $s = -1$: $5 - 29 + 36 = C(1)(2)$

$$2C = 12$$

$$\underline{C = 6}$$

let $s = -2$: $20 - 58 + 36 = A(1)(-1)$

$$\underline{A = 2}$$

$$\frac{5s^2 + 29s + 36}{(s + 2)(s^2 + 4s + 3)} = \frac{2}{s + 2} + \frac{(-3)}{s + 3} + \frac{6}{s + 1}$$

---

2) $$\frac{2s^2 + 6s + 5}{(s^2 + 3s + 2)(s + 1)} = \frac{2s^2 + 6s + 5}{(s + 2)(s + 1)(s + 1)} = \frac{2s^2 + 6s + 5}{(s + 2)(s + 1)^2}$$

$$= \frac{A}{s + 2} + \frac{B}{s + 1} + \frac{C}{(s + 1)^2}$$

$$= \frac{A(s + 1)^2 + B(s + 2)(s + 1) + C(s + 2)}{(s + 2)(s + 1)^2}$$

Setting numerators equal:

$$2s^2 + 6s + 5 = A(s + 1)^2 + B(s + 2)(s + 1) + C(s + 2)$$

let $s = -1$: $2 - 6 + 5 = C$

$$\underline{C = 1}$$

let $s = -2$: $8 - 12 + 5 = A$

$$\underline{A = 1}$$

let $s = 0$: $5 = A + B(2) + C(2)$

$$5 = 1 + 2B + 2$$

$$\underline{B = 1}$$

$$\frac{2s^2 + 6s + 5}{(s + 2)(s + 1)^2} = \frac{1}{s + 2} + \frac{1}{s + 1} + \frac{1}{(s + 1)^2}$$

---

3) $$\frac{2s^4 + 7s^3 + 27s^2 + 51s + 27}{(s^3 + 9s)(s^2 + 3s + 3)} = \frac{2s^4 + 7s^3 + 27s^2 + 51s + 27}{s(s^2 + 9)(s^2 + 3s + 3)}$$

$$= \frac{A}{s} + \frac{Bs + C}{s^2 + 9} + \frac{Ds + E}{s^2 + 3s + 3}$$

Find the common denominator and set numerators equal:

$$2s^4 + 7s^3 + 27s^2 + 51s + 27 = A(s^2 + 9)(s^2 + 3s + 3) + (Bs + C)(s)(s^2 + 3s + 3) +$$

$$+ (Ds + E)(s)(s^2 + 9)$$

let $s = 0$: $27 = 27A$

$$\underline{A = 1}$$

let $s = 3j$: $162 + (-189j) - 243 + 153j + 27 = (3Bj + C)$

$$\cdot (3j)(-9 + 9j + 3)$$

$$-54 - 36j = (54B - 27C) + (-81B - 18C)j$$

Since the real part on the left must equal the real part on the right,

$$\textcircled{1} \quad -54 = 54B - 27C$$

and the imaginary parts must also be equal

$$\textcircled{2} \quad -36 = -81B - 18C$$

from ① $\quad C = +2 + 2B$

substituting into ②

$$-36 = -81B - 18(2 + 2B)$$

$$0 = -117B$$

$$\underline{B = 0}$$

$$C = 2 + 2B$$

$$\underline{C = 2}$$

let $\quad s = j \quad : \quad 2 - 7j - 27 + 51j + 27 = A(8)(2 + 3j) + (jB + C)$

$$(j)(2 + 3j) + (jD + E)(j)(8)$$

$$2 + 44j = 16 + 24j + 4j - 6 - 8D + 8jE$$

$$-8 + 16j = -8D + (8E)j$$

Setting real and imaginary parts equal,

$$-8 = -8D$$

$\underline{D = 1}$

$16 = 8E$

$\underline{E = 2}$

$$\frac{2s^4 + 7s^3 + 27s^2 + 51s + 27}{} = \frac{1}{s} + \frac{2}{s^2 + 9} + \frac{s + 2}{s^2 + 3s + 3}$$

---

4) $$\frac{7s^4 - 3s^3 + 56s^2 - 17s + 107}{(s - 1)(s^2 + 4)^2} = \frac{A}{(s - 1)} + \frac{Bs + C}{(s^2 + 4)} + \frac{Ds + E}{(s^2 + 4)^2}$$

$$= \frac{A(s^2 + 4)^2 + (Bs + C)(s - 1)(s^2 + 4) + (Ds + E)(s - 1)}{(s - 1)(s^2 + 4)^2}$$

Setting numerators equal:

$$7s^4 - 3s^3 + 56s^2 - 17s + 107 = A(s^2 + 4)^2 + (Bs + C)(s - 1)(s^2 + 4) + (Ds + E)(s - 1)$$

Let $s = +1$: $7 - 3 + 56 = 17 + 107 = 25A$

$150 = 25A$

$\underline{A = 6}$

Multiply it out

$$7s^4 - 3s^3 + 56s^2 - 17s + 107 = 6(s^4 + 8s^2 + 16) + (Bs^4 - BS^3 + 4Bs^2 -$$

$$- 4Bs - Cs^2 + 4Cs - 4C) + (Ds^2 + Es - Ds - E)$$

Equating like coefficients,

$$s^4 : \quad 7 = 6 + B \Longrightarrow \underline{B = 1}$$

$$s^3 : \quad -3 = 0 - B + C \Longrightarrow \underline{C = -2}$$

$$s^2 : \quad 56 = 48 + 4B - C + D \Longrightarrow \underline{D = 2}$$

$$s^1 : \quad -17 = -4B + 4C + E - D \Longrightarrow E - D = -5$$

$$\Longrightarrow \underline{E = -3}$$

B.

1) $\dot{y} + y = 5 \quad , \quad y(0) = 0$

$$(s + 1)\, Y(s) = \frac{5}{s}$$

$$Y(s) = \frac{5}{s(s + 1)} = \frac{A}{s} + \frac{B}{s + 1} = \frac{A(s + 1) + Bs}{s(s + 1)}$$

$$\therefore 5 = A(s + 1) + Bs$$

$$s = -1 : \quad 5 = -B$$

$$s = 0 : \quad 5 = A$$

$$Y(s) = \frac{5}{s} - \frac{5}{s + 1}$$

$$\underline{y(t) = 5 - 5e^{-t}}$$

---

2) $\dot{y} + y = e^{-3t}$, $y(0) = 1$

$$sY(s) - y(0) + Y(s) = \frac{1}{s+3}$$

$$Y(s) = \frac{1}{(s+1)(s+3)} + \frac{1}{s+1}$$

NOTE: We will use partial fractions on only part of the expression, since we have an inverse transform for $\frac{1}{s+1}$.

$$\frac{1}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3} = \frac{A(s+3) + B(s+1)}{(s+1)(s+3)}$$

$$1 = A(s+3) + B(s+1)$$

$$s = -1 \quad : \quad 1 = 2A \longrightarrow A = 1/2$$

$$s = -3 \quad : \quad 1 = -2B \longrightarrow B = -1/2$$

$$Y(s) = \frac{1}{s+1} + \frac{1/2}{s+1} - \frac{1/2}{s+3}$$

$$Y(s) = \frac{3/2}{s+1} - \frac{1/2}{s+3}$$

$$y(t) = 3/2\ e^{-t} - 1/2\ e^{-3t}$$

---

3) $\dot{x} + 2x = \sin t$ $x(0) = 5$

$$sX(s) - x(0) + 2X(s) = \frac{1}{s^2+1}$$

$$X(s) = \frac{1}{(s^2 + 1)(s + 2)} + \frac{5}{s + 2}$$

Transform available so do not include in partial fraction expansion.

$$\frac{1}{(s^2 + 1)(s + 2)} = \frac{As + B}{s^2 + 1} + \frac{C}{s + 2} = \frac{(As + B)(s + 2) + C(s^2 + 1)}{(s^2 + 1)(s + 2)}$$

$$1 = (As + B)(s + 2) + C(s^2 + 1)$$

$s = -2$: $1 = 5C \longrightarrow C = 1/5$

$s = 0$: $2B + C \Longrightarrow B = 2/5$

$s = 1$: $3A + 3B + 2C \Longrightarrow A = -1/5$

$$X(s) = \frac{-1/5\ s}{s^2 + 1} + \frac{2/5}{s^2 + 1} + \frac{5\ 1/5}{s + 2}$$

$$x(t) = -1/5 \cos t + 2/5 \sin t + 5\ 1/5\ e^{-2t}$$

---

4) $\ddot{x} + 2\dot{x} + 9x = 2 \sin 3t \qquad x(0) = \dot{x}(0) = 0$

$$(s^2 + 2s + 9)\ X(s) = \frac{6}{s^2 + 9}$$

$$X(s) \quad \frac{6}{(s^2 + 9)(s^2 + 2s + 9)} = \frac{As + B}{s^2 + 9} + \frac{Cs + D}{s^2 + 2s + 9}$$

$$6 = (As + B)(s^2 + 2s + 9) + (Cs + D)(s^2 + 9)$$

$s = 0$: $6 = 9B + 9D$

$$s = 3j \quad : \quad 6 = (3Aj + B)(-9 + 6j + 9) = (3Aj + B)(6j)$$

$$6 = -18A + 6Bj$$

$$A = -1/3$$

$$B = 0$$

$$D = 2/3$$

$$s = 1 \quad : \quad 6 = 12A + (C + D)10$$

$$6 = 12\,(-1/3) + 10C + 10\,(2/3)$$

$$C = 1/3$$

$$X(s) = \frac{-1/3\ s}{s^2 + 9} + \frac{1/3\ s + 2/3}{s^2 + 2s + 9}$$

$$X(s) = \frac{-1/3\ s}{s^2 + 9} + \frac{1/3\ (s + 2)}{(s + 1)^2 + 8}$$

$$= \frac{-1/3\ s}{s^2 + 9} + \frac{1/3\ (s + 1)}{(s + 1)^2 + 8} + \frac{1/3}{(s + 1)^2 + 8}$$

$$x(t) = -1/3 \cos 3t + 1/3\ e^{-t} \cos 2\sqrt{2}t + 1/(6\sqrt{2})\ e^{-t} \sin 2\sqrt{2}\ t\ .$$

---

5) $x + 5\dot{x} + 6x = 3e^{-3t} \quad , \quad x(0) = \dot{x}(0) = 1$

$$s^2\ X(s) - sx(0) - \dot{x}(0) + 5sX(s) - 5x(0) + 6X(s) = \frac{3}{s + 3}$$

$$(s^2 + 5s + 6)\ X(s) = \frac{3}{s + 3} + s + 6 = \frac{s^2 + 9s + 21}{s + 3}$$

$$X(s) = \frac{s^2 + 9s + 21}{(s+3)(s^2+5s+6)} = \frac{s^2 + 9s + 21}{(s+3)^2(s+2)} = \frac{A}{s+3} + \frac{B}{(s+3)^2} + \frac{C}{s+2}$$

$$s^2 + 9s + 21 = A(s+3)(s+2) + B(s+2) + C(s+3)^2$$

$s = -3 \quad : \quad 9 - 27 + 21 = -B \qquad B = -3$

$s = -2 \quad : \quad 4 - 18 + 21 = C \qquad C = 7$

$s = 0 \quad : \quad 21 = 6A + 2B + 9C \qquad A = -6$

$\underline{x(t) = -6e^{-3t} - 3te^{-3t} + 7e^{-2t}}$

---

6) $x + 2\dot{x} + 10x = 3t + \frac{3}{5}$, $x(0) = 3$, $x(0) = -\frac{27}{10}$

$$s^2X(s) = sx(0) - \dot{x}(0) + 2sX(s) = 2x(0) + 10X(s) = \frac{3}{s^2} + \frac{3/5}{s}$$

$$(s^2 + 2s + 10)\,X(s) = \frac{3}{s^2} + \frac{3/5}{s} + sx(0) + 2x(0) + \dot{x}(0)$$

$$= \frac{3}{s^2} + \frac{3/5}{s} + 3s + 6 - \frac{27}{10}$$

$$= \frac{3s^3 + 33/10\, s^2 + 6/10\, s + 3}{s^2}$$

$$X(s) = \frac{3s^3 + 33/10\, s^2 + 6/10\, s + 3}{s^2(s^2 + 2s + 10)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs + D}{(s^2 + 2s + 10)}$$

$$3s^3 + \frac{33}{10}s^2 + \frac{6}{10}s + 3 = As(s^2 + 2s + 10) + B(s^2 + 2s + 10) + (Cs + D)s^2$$

$$s = 0 \quad : \quad 3 = 10B \qquad \underline{B = \frac{3}{10}}$$

$$s = 1 \quad : \quad 3 + \frac{33}{10} + \frac{6}{10} + 3 = A(13) + \frac{3}{10}(13) + (C + D)$$

$$\text{or} \quad \underline{13A + C + D = 6}$$

$$s + j \quad : \quad -3j - \frac{33}{10} + \frac{6}{10}j + 3 = Aj(-1 + 2j + 10) +$$

$$+ \frac{3}{10}(-1 + 2j + 10) + (Cj + D)(-1)$$

Setting imaginary and real parts equal

$$-\frac{24}{10} = 9A + \frac{6}{10} - C$$

$$\underline{C = 3 + 9A}$$

and

$$-\frac{3}{10} = -2A + \frac{27}{10} - D$$

$$D = -2A + 3$$

substituting into

$$13A + C + D = 6$$

$$13A = 3 + 9A - 2A + 3 = 6$$

$$\underline{A = 0}$$

then

$$\underline{C = 3}$$

and

$$\underline{D = 3}$$

$$X(s) = \frac{3/10}{s^2} + \frac{3s + 3}{s^2 + 2s + 10} = \frac{3/10}{s^2} + 3 \frac{s + 1}{(s + 1)^2 + 3^2}$$

$$\underline{x(t) = \frac{3}{10} t + 3e^{-t} \cos 3t}$$

1.15 PROBLEM SET V

Solve the following problems using Laplace Transforms.

1. $\dot{x} + 3x - y = 1 \qquad x(0) = y(0) = 0$

   $\dot{x} + 8x + \dot{y} = 2$

2. $\dot{x} - 3x - 6y = e^{-2t} \qquad x(0) = y(0) = 0$

   $\dot{y} + \dot{x} - 3y = 1$

1.16 SOLUTION TO PROBLEM SET V

1. $\dot{x} + 3x - y = 1$

$$x(0) = y(0) = 0$$

$\dot{x} + 8x + \dot{y} = 2$

$$sX(s) - x(0) + 3X(s) - Y(s) = \frac{1}{s}$$

$$sX(s) - x(0) + 8X(s) + sY(s) - y(0) = \frac{2}{s}$$

$$(s + 3)\ X(s) - Y(s) = \frac{1}{s}$$

$$(s + 8)\ X(s) + sY(s) = \frac{2}{s}$$

$$\begin{vmatrix} s + 3 & -1 \\ s + 8 & s \end{vmatrix} X(s) = \begin{vmatrix} \frac{1}{s} & -1 \\ \frac{2}{s} & s \end{vmatrix} = 1 + \frac{2}{s} = \frac{s + 2}{s}$$

$$\begin{vmatrix} s + 3 & -1 \\ s + 8 & s \end{vmatrix} Y(s) = \begin{vmatrix} s + 3 & \frac{1}{s} \\ s + 8 & \frac{2}{s} \end{vmatrix} = \frac{2}{s}(s + 3) - \frac{1}{s}(s + 8) = \frac{s - 2}{s}$$

$$\begin{vmatrix} s + 3 & -1 \\ s + 8 & s \end{vmatrix} = s^2 + 3s + s + 8 = s^2 + 4s + 8$$

$$X(s) = \frac{s + 2}{s(s^2 + 4s + 8)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4s + 8}$$

$$A(s^2 + 4s + 8) + s(Bs + C) = s + 2$$

$s = 0: \ 8A = 2 \qquad s^2: \ A + B = 0 \qquad s: \ 4A + C = 1$

$\underline{A = 1/4} \qquad \underline{B = -1/4} \qquad \underline{C = 0}$

$$\therefore \ X(s) = \frac{1/4}{s} + \frac{-1/4\, s}{(s+2)^2 + 4} = \frac{1/4}{s} - 1/4\left[\frac{s+2}{(s+2)^2+4} - \frac{2}{(s+2)^2+4}\right]$$

$$x(t) = 1/4\left[1 - e^{-2t}(\cos 2t - \sin 2t)\right]$$

$$Y(s) = \frac{s - 2}{s(s^2 + 4s + 8)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4s + 8}$$

$$A(s^2 + 4s + 8) + s(Bs + C) = s - 2$$

$s = 0: \ 8A = -2 \qquad s^2: \ A + B = 0 \qquad s: \ 4A + C = 1$

$\underline{A = -1/4} \qquad \underline{B = 1/4} \qquad \underline{C = 2}$

$$Y(s) = 1/4\left[\frac{-1}{s} + \frac{s+8}{(s+2)^2+4}\right] = 1/4\left[\frac{-1}{s} + \frac{s+2}{(s+2)^2+4} + \frac{6}{(s+2)^2+4}\right]$$

$$y(t) = 1/4\left[-1 + e^{-2t}(\cos 2t + 3\sin 2t)\right]$$

2. $\dot{x} - 3x - 6y = e^{-2t}$

$\dot{y} + \dot{x} - 3y = 1$ $\qquad x(0) = y(0) = 0$

$$(s - 3)\,X(s) - 6Y(s) = \frac{1}{s + 2}$$

$$sX(s) + (s - 3)\,Y(s) = \frac{1}{s}$$

$$\begin{vmatrix} s-3 & -6 \\ s & s-3 \end{vmatrix} X(s) = \begin{vmatrix} \frac{1}{s+2} & -6 \\ \frac{1}{s} & s-3 \end{vmatrix} = \frac{s-3}{s+2} + \frac{6}{s} = \frac{s^2 + 3s + 12}{s(s+2)}$$

$$\begin{vmatrix} s-3 & -6 \\ s & s-3 \end{vmatrix} Y(s) = \begin{vmatrix} s-3 & \frac{1}{s+2} \\ s & \frac{1}{s} \end{vmatrix} = \frac{s-3}{s} - \frac{s}{s+2} = \frac{-s-6}{s(s+2)}$$

$$\begin{vmatrix} s-3 & -6 \\ s & s-3 \end{vmatrix} = s^2 - 6s + 9 + 6s = s^2 + 9$$

$$X(s) = \frac{s^2 + 3s + 12}{s(s+2)(s^2+9)} = \frac{A}{s} + \frac{B}{s+2} + \frac{Cs + D}{s^2 + 9}$$

$$A(s+2)(s^2+9) + Bs(s^2+9) + (Cs + D)(s)(s+2) = s^2 + 3s + 12$$

$s = 0{:}\ \ 18A = 12$ $\qquad$ $s = -2{:}\ \ -2B(4 + 9) = 4 - 6 + 12$

$\underline{A = 2/3}$ $\qquad$ $-26B = 10$

$\underline{B = -5/13}$

$s^3{:}\ \ A + B + C = 0$ $\qquad$ $s{:}\ \ 9A + 9B + 2D = 3$

$$C = 5/13 - 2/3 = \frac{15 - 26}{39}; \quad 2D = 3 - 6 + \frac{45}{13} = \frac{45 - 39}{13} = \frac{6}{13}$$

$$C = -\frac{11}{39} \qquad D = \frac{3}{13}$$

$$X(s) = \frac{2/3}{s} - \frac{5/13}{s+2} - \frac{11/39s}{s^2+9} + \frac{3/13}{s^2+9}$$

$$x(t) = 2/3 - 5/13e^{-2t} - 11/39 \cos 3t + 1/13 \sin 3t$$

$$Y(s) = \frac{-s-6}{s(s+2)(s^2+9)} = \frac{A}{s} + \frac{B}{s+2} + \frac{Cs+D}{s^2+9}$$

$$A(s+2)(s^2+9) + Bs(s^2+9) + (Cs+D)(s)(s+2) = -s-6$$

$s = 0$: $18A = -6$ $\qquad$ $s = -2$: $-2B(13) = -4$

$$A = -1/3 \qquad B = 2/13$$

$s = j3$: $(j3C + D)(j3)(s + j3) = -j3 - 6$

$$j3(j6C + 2D + j3D - 9C) = -j3 - 6$$

$$-18C - 9D - j27C + j6D = -6 - j3$$

$$+18C + 9D = +6 \qquad 6D - 27C = -3$$

$$6C + 3D = 2 \qquad 27C - 6D = 3$$

$$12C + 6D = 4$$
$$27C - 6D = 3$$
$$39C = 7$$

$$3D = 2 - \frac{42}{39} = \frac{78-42}{39} = \frac{36}{39} = \frac{12}{13}$$

$C = \frac{7}{39}$ $D = \frac{4}{13}$

$$Y(s) = \frac{-1/3}{s} + \frac{2/13}{s + 2} + \frac{7/39s}{s^2 + 9} + \frac{4/13}{s^2 + 9}$$

$$y(t) = -1/3 + 2/13e^{-2t} + 7/39 \cos 3t + 4/39 \sin 3t$$

APPENDIX I

INTEGRATING FACTOR

Given a linear first order differential equation, i.e.,

$$A(x) \frac{dy}{dx} + B(x)\ y = C(x)$$

let us rewrite the equation in the form (a standard form for the equation:

$$dy + P(x)\ y\ dx = Q(x)\ dx$$

Is it possible to find a function $v(x)$ such that

$$v(x)\ dy + v(x)\ P(x)\ y\ dx = v(x)\ Q(x)\ dx$$

is an exact equation? If this can be done, we must be able to write the equation as

$$M(x, y)\ dx + N(x, y)\ dy = 0.$$

Rewriting the equation involving $v(x)$ in this form, we get

$$\underbrace{[v(x)\ P(x)\ y - v(x)\ Q(x)]}_{M}\ dx + \underbrace{v(x)}_{N}\ dy = 0$$

Thus, $M = [v(x)\ P(x)\ y - v(x)\ Q(x)]$, and

$N = v(x)$.

For exactness, it is required that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Thus,

$$\frac{\partial M}{\partial y} = v(x)\ P(x); \quad \frac{\partial N}{\partial x} = \frac{dv(x)}{dx}$$

and

$$\frac{dv(x)}{dx} = v(x)\ P(x)$$

or

$$\int \frac{dv(x)}{v(x)} = \int P(x)\, dx$$

$$\ln (v(x)) = P(x)\, dx$$

$$\boxed{v(x) = e^{\int P(x)\, dx}}$$

The function $v(x)$ is called the integrating factor for the differential equation (in standard form). Note that as long as the D.E. is a linear first order D.E., an integrating factor exists.

INTEGRATING FACTOR - (2nd APPROACH)

For a linear differential equation of first order, i.e.,

$$\frac{dy}{dx} + P(x)\, y = Q(x)$$

there is a special procedure which always works. It is always possible to find a function $v(x)$ such that the equation can be rewritten in the form

$$\frac{d}{dx} (v(x)\, y) = v(x)\, Q(x)$$

which can be solved by direct integration. Let us examine the alternate form. It can be rewritten as

$$v(x) \frac{dy}{dx} + \left(\frac{dv(x)}{dx}\right) y = v(x)\, Q(x).$$

Dividing through by $v(x)$ results in

$$\frac{dy}{dx} + \left(\frac{1}{v(x)} \frac{dv(x)}{dx}\right) y = Q(x).$$

The original differential equation was

$$\frac{dy}{dx} + P(x)\, y = Q(x).$$

By comparing terms in the two equations, it is seen that

$$\frac{1}{v(x)} \frac{dv(x)}{dx} = P(x)$$

or, integrating,

$$\int \frac{dv(x)}{v(x)} = \int P(x)\, dx$$

$$\ln (v(x)) = \int P(x)\, dx$$

$$\boxed{v(x) = e^{\int P(x)\, dx}}$$

The quantity $v(x)$ is known as the integrating factor for the original differential equation. The solution of the alternate form, i.e.,

$$\frac{d}{dx} (v(x)\ y) = v(x)\ Q(x)$$

can be written as

$$v(x)\ y = \int v(x)\ Q(x)\, dx + C$$

or

$$\boxed{y = \frac{1}{v(x)} \int v(x)\ Q(x)\, dx + \frac{C}{v(x)}}$$

Note that if the differential equation is homogeneous, i.e., $Q(x) \equiv 0$, then

$$\boxed{y = \frac{C}{v(x)} = Ce^{-\int P(x)\, dx}}$$

This solution form is always the solution for

$$\frac{dy}{dx} + P(x)\,y = 0$$

This is a very important special case. In the case of the linear homogeneous first order differential equation,

$$\frac{dy}{dx} + P(x)\,y = 0$$

note that the variables can be separated variables and a solution can be obtained in that manner, i.e.,

$$\frac{dy}{y} + P(x)\,dx = 0$$

$$\ln(y) + \int P(x)\,dx = C' = \ln C$$

$$\ln\left(\frac{y}{C}\right) = -\int P(x)\,dx$$

$$\frac{y}{C} = e^{-\int P(x)\,dx}$$

or

$$\boxed{y = Ce^{-\int P(x)\,dx}}$$

as before.

The following examples will illustrate the solution of linear first order ordinary differential equations.

EXAMPLES:

$$\frac{dy}{dx} + 4\,y = 0$$

By inspection, the solution is

$$y = Ce^{-\int 4\,dx} = Ce^{-4x}$$ Ans.

$$\frac{dy}{dx} + 4y = 2x$$

$$v(x) = e^{\int 4dx} = e^{4x}$$

$$y = e^{-4x}\int 2\,x\,e^{4x}\,dx + Ce^{-4x}$$

$$y = e^{-4x}\left(\frac{x}{2}e^{4x} - \frac{1}{8}e^{4x} + C\right)$$

or

$$y = \frac{x}{2} - \frac{1}{8} + Ce^{-4x}$$ Ans.

EXAMPLE DIFFERENTIAL EQUATION

$$\frac{dy}{dx} + 2xy = 4x$$

a. Can it be integrated directly?

NO - the second term contains a mixture of x and y.

b. Can the variables be separated?

$$dy + 2xy\,dx = 4x\,dx$$

$$\frac{dy}{y} + 2x\,dx = \frac{4x}{y}\,dx$$

Can't separate.

c. Is the D.E. exact?

$$(2xy - 4x)\,dx + 1dy = 0$$

$M = (2xy - 4x)$, $N = 1$

$$\frac{\partial M}{\partial y} = 2x \qquad \frac{\partial N}{\partial x} = 0$$

<u>Not exact</u>.

d. Integrating factor

$$dy + \underbrace{(2x)}_{P(x)}\, y\,dx = \underbrace{(4x)}_{Q(x)}\,dx$$

$$v(x) = e^{\int 2x\,dx} = e^{x^2}$$

$$e^{x^2}\,dy + 2xe^{x^2}\,y\,dx = 4xe^{x^2}\,dx$$

$$e^{x^2}\,dy + y\,e^{x^2}(2x)\,dx = 2e^{x^2}(2x)\,dx$$

Could have gone directly to this step ⇒

$$\int d\left[e^{x^2} y\right] = 2\int e^{x^2}(2x)\,dx$$

$$y\,e^{x^2} = 2e^{x^2} + C$$

$$\boxed{y = 2 + Ce^{-x^2}}$$

Check to see if y is a solution

$$\frac{dy}{dx} = C(-2x)\,e^{-x^2}$$

$$-2C\,x\,e^{-x^2} + 2x\,(2 + Ce^{-x^2}) \stackrel{?}{=} 4x$$

$$\cancel{-2C\,x\,e^{-x^2}} + 4x + \cancel{2C\,x\,e^{-x^2}} = 4x$$

$$\underline{4x = 4x}$$

VOLUME I

CHAPTER 2
EQUATIONS OF MOTION

REVISED OCTOBER 1978

2.1 INTRODUCTION

These notes are written as a general classroom text for the theoretical approach to Flying Qualities in the course curriculum of the USAF Test Pilot School.

The theoretical discussion will, of necessity, incorporate certain simplifying assumptions. These simplifying assumptions are made in order to make the main elements of the subject more clear. The equations developed are by no means suitable for design of modern aircraft, but the basic method of attacking the problem is valid. With the aid of high speed computers, the aircraft designers' more rigorous theoretical calculations, modified by data obtained from the wind tunnel, often give results which closely predict the flying qualities of new airplanes. However, neither the theoretical nor the wind tunnel results are infallible. Therefore, there is still a valid requirement for the test pilot in the development cycle of new aircraft.

2.2 TERMS AND SYMBOLS

There will be many terms and symbols used during the stability and control phase. Some of these will be familiar, but many will be new. It will be a great asset to be able to recall at a glance the definitions represented by these symbols. Below is a condensed list of the terms and symbols used in this course:

2.2.1 Terms

Stability Derivatives - Nondimensional quantities expressing the variation of the force or moment coefficient with a disturbance from steady flight.

$$C_{m_\alpha} = \frac{\partial C_m}{\partial \alpha} \tag{2.1}$$

$$C_{n_\beta} = \frac{\partial C_n}{\partial \beta} \tag{2.2}$$

Stability Parameters - A quantity that expresses the variation of force or moment on aircraft caused by flight or by a disturbance from steady flight.

$$M_u = \frac{\rho USc}{I_y}\left[C_m + \frac{U}{2}\frac{\partial C_m}{\partial u}\right] \tag{2.3}$$

(Change in pitching moment caused by a change in velocity)

$$L_q = \frac{\rho S c}{4m} C_{L_q} \quad \text{(Change in lift caused by a change in pitch rate)} \qquad (2.4)$$

[redacted] - The initial tendency of an airplane to return to steady state flight after a disturbance.

[redacted] - The time history of the response of an airplane to a disturbance, in which the aircraft ultimately returns to a steady state flight.

[redacted] -

a) Static - The airplane would have no tendency to move from its disturbed condition.

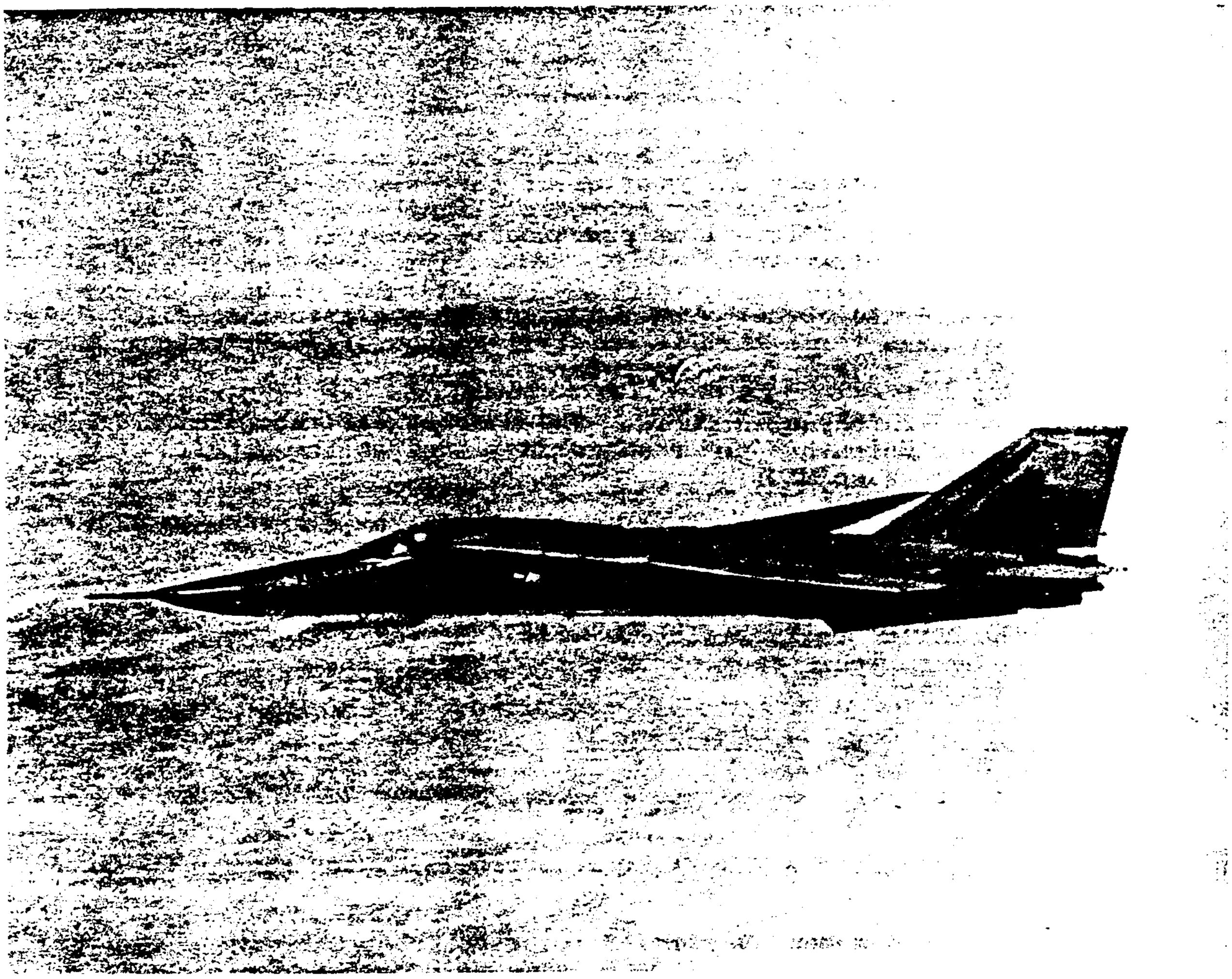
b) Dynamic - The airplane would sustain a steady oscillation caused by a disturbance.

[redacted] A characteristic of an aircraft such that when disturbed from steady flight, its tendency is to depart further or diverge from the original condition of steady flight.

[redacted] - Time history of an aircraft response to a disturbance in which the aircraft ultimately diverges to departure or destruction.

[redacted] - Any control movement or deflection that causes a positive movement or moment on the airplane shall be considered a positive control movement. This sign convention does not conform to the convention used by NASA and some reference text books. This convention is the easiest to remember and is used at the Flight Test Center, therefore, it will be used in the School.

[redacted] - The number of paths that a physical system is free to follow.



2.2.2 SYMBOLS

Symbol	Definition
a.c.	Aerodynamic Center: A point located on the wing chord (approximately one quarter of the chord length back of the leading edge for subsonic flight) about which the moment coefficient is practically constant for all angles of attack.
C	Chordwise Force: The component of the resultant aerodynamic force that is parallel to the aircraft reference axis, (i.e., fuselage reference line).
c	Mean Aerodynamic Chord: The theoretical chord for a wing which has the same force vector as the actual wing (also MAC).
c.p.	Center of Pressure: Theoretical point on the chord through which the resultant force acts.
D	Drag: The component of the resultant aerodynamic force parallel to the relative wind. It too must be specified whether this applies to a complete aircraft or to parts thereof.
$\bar{F}$	Applied force vector.
F_a, F_e, F_r	Control forces on the aileron, elevator, and rudder, respectively.
F_x, F_y, F_z	Components of applied forces on respective body axes.
$\bar{G}$	Applied moment vector.
G_x, G_y, G_z	Components of applied moments on respective body axes.
$\bar{H}$	Angular momentum vector.
HM	Hinge Moment: A moment which tends to restore or move a control surface to or from a condition to equilibrium.
H_x, H_y, H_z	Components of the angular momentum vector on the body axes.
I	Moment of Inertia: With respect to any given axis, the moment of inertia is the sum of the products of the mass of each elementary particle by the square of its distance from the axis. It is a measure of the resistance of a body to angular acceleration.
$\bar{i}, \bar{j}, \bar{k}$	Unit vectors in the body axis system.

Symbol	Definition
I_x, I_y, I_z	Moments of inertia about respective body axes.
I_{xy}, I_{yz}, I_{xz}	Products of inertia.
L	Lift: The component of the resultant aerodynamic force perpendicular to the relative wind. It must be specified whether this applies to a complete aircraft or to parts thereof.
$\mathcal{L}, \mathcal{M}, \mathcal{N}$	Aerodynamic moments about x, y, and z body axes.
N	Normal Force: The component of the resultant aerodynamic force that is perpendicular to the aircraft reference axis.
P,Q,R	Angular rates about the x, y, and z body axes, respectively.
p,q,r	Perturbed values of P,Q,R, respectively.
R	Resultant Aerodynamic Force: The vector sum of the lift and drag forces on an airfoil or airplane.
S	Wing area.
U_o	Component of velocity along the x body axis at zero time (i.e., initial condition).
U,V,W	Components of velocity along the x, y, and z body axes.
u,v,w	Perturbed values of U,V,W, respectively.
X,Y,Z	Aerodynamic force components on respective body axes (Caution: Also used as axes in "Moving Earth Axis System" in derivation of Euler angle equation. Differentiation should be obvious).
x,y,z	Axes in the body axis system.
α	Angle of attack.
β	Sideslip angle.
$\delta_a, \delta_e, \delta_r$	Deflection angle of the ailerons, elevator, and rudder, respectively.
θ, ϕ, ψ	Euler angles: pitch, roll, and yaw, respectively.
$\bar{\omega}$	Total angular velocity vector of an aircraft.
∇	Dimensionless derivative with respect to time.

2.3 OVERVIEW

The purpose of this section is to derive a set of equations that describes the motion of an airplane. An airplane has 6 degrees of freedom (i.e., it can move forward, sideways and down and it can rotate about its axes with yaw, pitch, and roll). In order to solve for these 6 unknowns, 6 simultaneous equations will be required. To derive these the following relations will be used.

START WITH NEWTON'S SECOND LAW

$$\underbrace{\bar{F}}_{\text{externally applied force}} = \frac{d}{dt}\underbrace{(M\,\bar{V})}_{\text{linear momentum}}$$

(3 linear degrees of freedom).

$$\underbrace{\bar{G}}_{\text{externally applied moment}} = \frac{d}{dt}\underbrace{(\bar{H})}_{\text{angular momentum}}$$

(3 rotational degrees of freedom).

Six Equations for the Six Degrees of Freedom of a Rigid Body.

↓

Equations are valid with respect to inertial space only.

↓

OBTAIN THE 6 AIRCRAFT EQUATIONS OF MOTION

Longitudinal

$$F_x = m\,(\dot{U} + QW - RV) \qquad (2.5)$$

$$F_z = m\,(\dot{W} + PV - QU) \qquad (2.6)$$

$$G_y = \dot{Q}\,I_y - PR\,(I_z - I_x) + (P^2 - R^2)\,I_{xz} \qquad (2.7)$$

Lateral-Directional

$$F_y = m\,(\dot{V} + RU - PW) \qquad (2.8)$$

$$G_x = \dot{P}\,I_x + QR\,(I_z - I_y) - (\dot{R} + PQ)\,I_{xz} \qquad (2.9)$$

$$G_z = \dot{R}\,I_z + PQ\,(I_y - I_x) + (QR - \dot{P})\,I_{xz} \qquad (2.10)$$

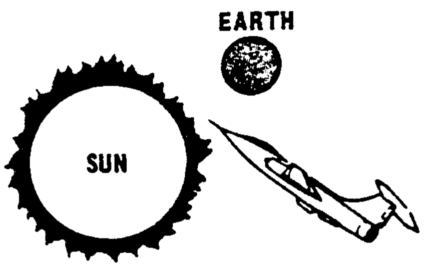
The Left-Hand Side (LHS) of the equation represents the forces and moments on the airplane while the Right-Hand Side (RHS) stands for the airplane's response to these forces and moments. Before launching into the development of these equations it will be necessary first to cover some basics.

2.4 BASICS

2.4.1 Coordinate Systems

There are many coordinate svstems that are useful in the analysis of vehicle motion. According to generally accepted notation, all coordinate systems will be right-hand orthogonal.

True Inertial Coordinate System

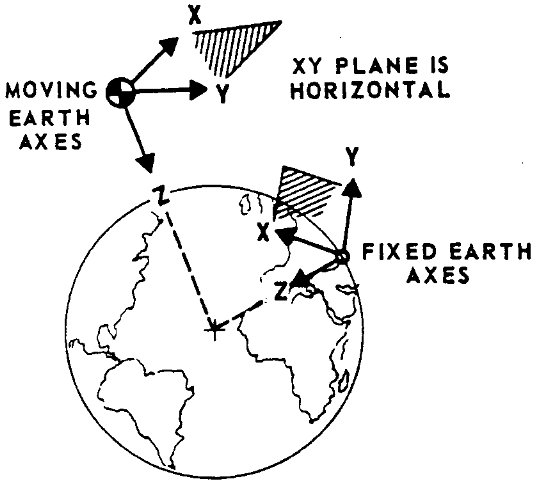


Location of origin: unknown
Approximation for space dynamics: the center of the sun.

Approximation for aircraft: the center of the earth.

Figure 2.1

The Earth Axis Systems



Location of Origin

Fixed System; arbitrary location
Moving System; at the vehicle cg

The Z axis points toward center of the earth.

The XY Plane
parallel to local horizontal.

The Orientation of the X axis is arbitrary; may be North or on the initial vehicle heading.

NOTE: There are two earth axis systems, the fixed and the moving. An example of a moving earth axis system is an inertial navigation platform. An example of a fixed earth axis is a radar site.

Figure 2.2

VEHICLE AXIS SYSTEMS

These coordinate systems are fixed to the vehicle. There are many different types, e.g.,

- Body Axis System.
- Stability Axis System.
- Principal Axis System.
- Wind Axis System.

The body and the stability axis systems are the only two that will be used during this course.

Body Axis System

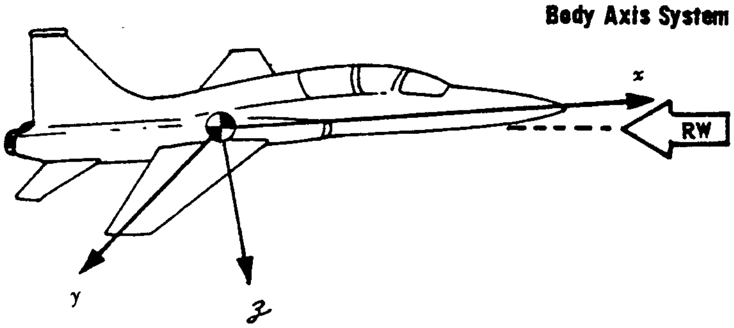


Figure 2.3

The Unit Vectors are $\bar{i}$ $\bar{j}$ $\bar{k}$.

The Origin is at the cg.

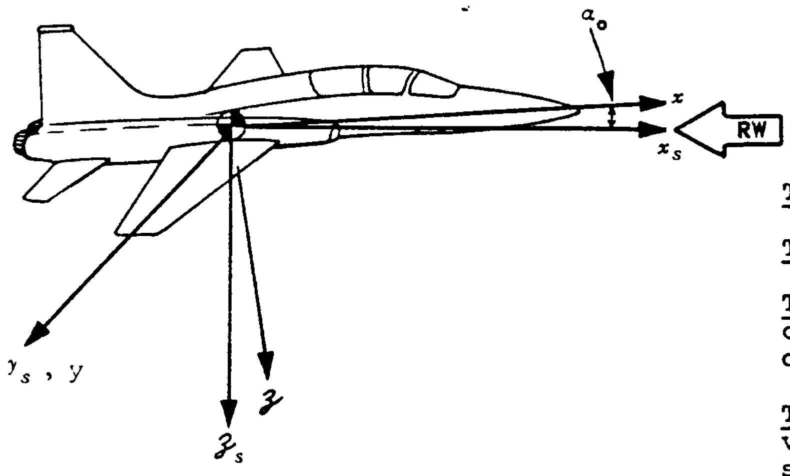
The x z plane is in the vehicle plane of symmetry.

The positive x axis points forward along the vehicle horizontal reference line.

The positive y axis points out the right wing.

The positive z axis points downward toward the bottom of the vehicle.

Stability Axis System



y BODY = y STAB
i.e., THE STABILITY xz PLANE REMAINS IN THE VEHICLE PLANE OF SYMMETRY

Figure 2.4

The unit vectors are $\bar{i}_s$, $\bar{j}_s$, $\bar{k}_s$.

The origin is at the cg.

The positive x axis points forward coincident with the initial position of the relative wind.

The x z plane must remain in the vehicle plane of symmetry, hence this stability axis system is restricted to symmetrical initial flight conditions.

The positive z axis points downward toward the bottom of the vehicle, normal to the axis. Angle of Attack is α_o at $t = 0$.

2.4.2 Vector Definitions

The Equations of Motion describe the vehicle motion in terms of four vectors ($\bar{F}$, $\bar{G}$, $\bar{V}_T$, $\bar{\omega}$). The components of these vectors resolved along the body axis system are shown below.

$\bar{F}$ - Total Linear Force (Applied)

$$\bar{F} = F_x\bar{i} + F_y\bar{j} + F_z\bar{k}$$

$\bar{G}$ - Total Moment (Applied)

$$\bar{G} = G_x\bar{i} + G_y\bar{j} + G_z\bar{k}$$

$$\bar{G} = \bar{G}_{aerodynamic} + \bar{G}_{other\ sources}$$

$$\bar{G}_{aerodynamic} = L_{aero}\bar{i} + M_{aero}\bar{j} + N_{aero}\bar{k}$$

$$\bar{G}_{aerodynamic} = \mathcal{L}\bar{i} + \mathcal{M}\bar{j} + \mathcal{N}\bar{k}$$

NOTE: Control deflections that tend to produce positive $\mathcal{L}$, $\mathcal{M}$, or $\mathcal{N}$, are defined at USAF TPS to be positive (i.e. Right δ_r is positive).

$\bar{V}_T$ - True Velocity

$$\bar{V}_T = U\bar{i} + V\bar{j} + W\bar{k}$$

where

U = forward velocity

V = side velocity

W = vertical velocity

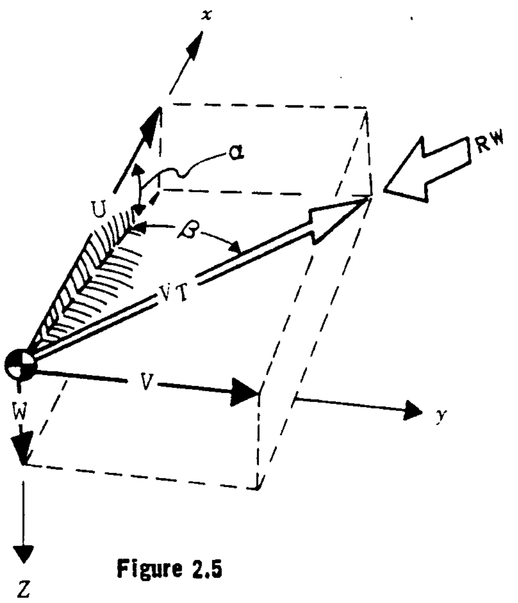


Figure 2.5

α - Angle of Attack

For small α and β

$$V_T \cos\beta \approx V_T$$

$$\therefore \alpha \approx \sin^{-1}\frac{W}{V_T}$$

or

$$\alpha \approx \frac{W}{V_T}$$

β - Sideslip Angle

$$\beta = \sin^{-1}\left(\frac{V}{V_T}\right)$$

For small β

$$\beta \approx \frac{V}{V_T}$$

CAUTION - OTHER DEFINITIONS ARE POSSIBLE

$\bar{\omega}$ - Angular Velocity

$$\bar{\omega} = P\bar{I} + Q\bar{J} + R\bar{K}$$

where

P = roll rate

Q = pitch rate

R = yaw rate

2.4.3 Euler Angles – Transformation from the Moving Earth Axis System to the Body Axis System

There are several reasons for using Euler angles in this development. Some of them are:

1) Effect of aircraft weight is related to the body axes through Euler angles.

2) When an inertial navigation system (INS) is available, data can be taken directly in Euler angles. P, Q and R can then be determined through a transformation.

Euler angles are expressed in terms of YAW (ψ), PITCH (θ) and ROLL (ϕ). The sequence (YAW, PITCH, ROLL) must be maintained to arrive at the proper set of Euler angles.

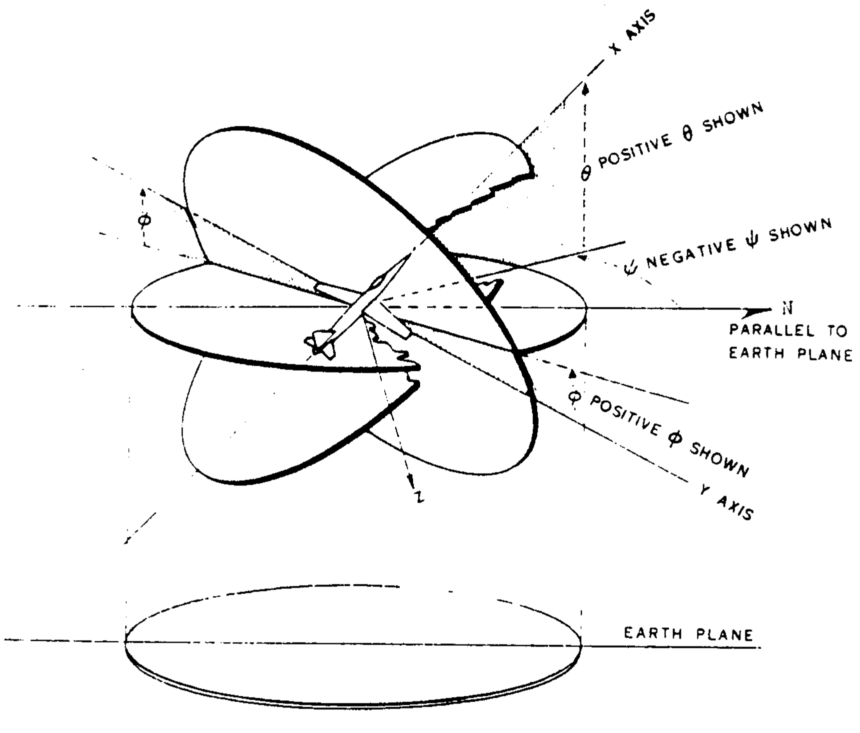


Figure 2.6

ψ - Yaw Angle - The angle between the projection of x body axes onto the horizontal plane and the initial reference position of the X earth axis. (Yaw angle is the vehicle heading only if the initial reference is North).

θ - Pitch Angle - The angle measured in a vertical plane between the x body axis and the horizontal plane.

ϕ - Bank Angle - The angle, measured in the yz plane of the body system, between the y body axis and the horizontal plane. For a given ψ and θ, bank angle is a measure of the rotation about the x axis to put the aircraft in the desired position from a wings horizontal condition.

Angular Velocity Transformation - The following relationships, derived by vector resolution, will be useful later in the study of dynamics.

$$P = \dot{\phi} - \dot{\psi} \sin \theta \tag{2.11}$$

$$Q = \dot{\theta} \cos \phi + \dot{\psi} \sin \phi \cos \theta \tag{2.12}$$

$$R = \dot{\psi} \cos \phi \cos \theta - \dot{\theta} \sin \phi \tag{2.13}$$

The above equations transform the angular rates in the moving earth axis system ($\dot{\psi}, \dot{\theta}, \dot{\phi}$) into angular rates about the body axis system (P, Q, R) for any aircraft attitude. For example, it is easy to see that when an aircraft is pitched up and banked, the vector $\bar{\dot{\psi}}$ will have components along the x, y and z body axis (figure 2.7). Remember, $\dot{\psi}$ is the angular velocity about the Z axis of the Moving Earth Axis System (it can be thought of as the rate of change of aircraft heading). Although it is not shown in figure 2.7, the aircraft may have a value of $\dot{\theta}$ and $\dot{\phi}$. In order to derive the transformation equations it is easier to analyze one vector at a time. First resolve the components of $\dot{\psi}$ on the body axes. Then do the same with $\dot{\theta}$ and $\dot{\phi}$. The components can then be added and the total transformation will result.

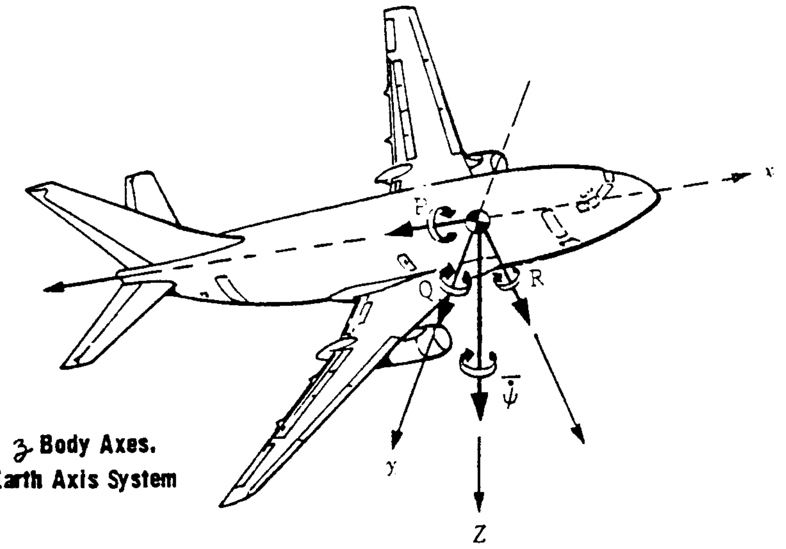


Figure 2.7 Components of $\bar{\dot{\psi}}$ Along x, y, and z Body Axes. The X and Y Axes of the Moving Earth Axis System Are Not Shown.

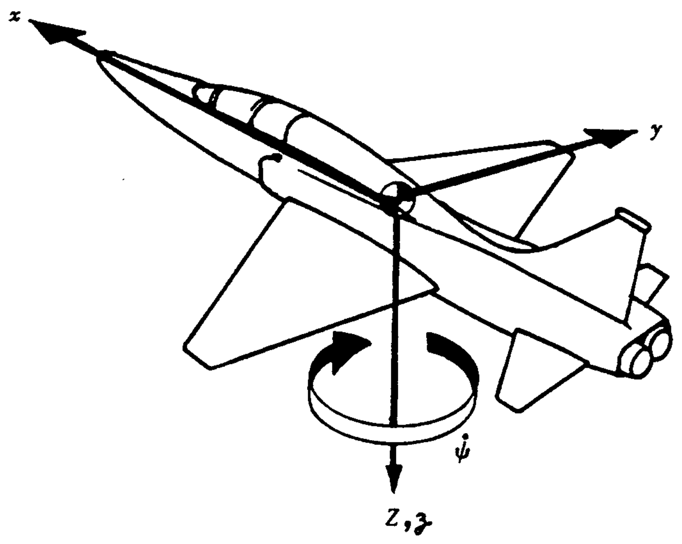
Step 1 - Resolve the components of $\dot{\psi}$ along the body axes for any aircraft attitude.

It is easy to see how $\dot{\psi}$ reflects to the body axis by starting with an aircraft in straight and level flight and changing the aircraft attitude one angle at a time. In keeping with convention, the sequence of change will be yaw, pitch and bank.

First, it can be seen that the Z axis of the Moving Earth Axis System remains aligned with z axis of the body axis system regardless of the angle ψ; therefore, the effect of $\dot{\psi}$ on P, Q, and R does not change with yaw angle,

$$\therefore R = \dot{\psi}$$

Figure 2.8



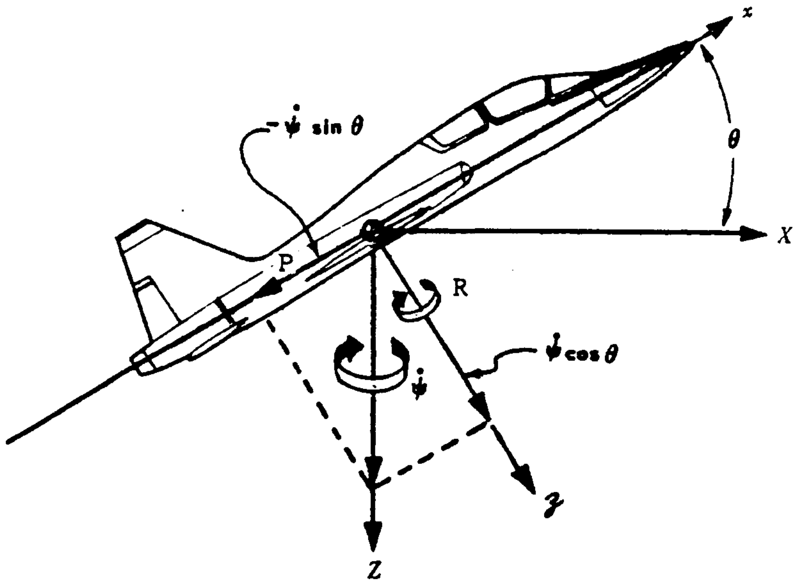


Figure 2.9

Next, consider pitch up, in this attitude, $\dot{\psi}$ has components on the x and z body axes.

$$P = -\dot{\psi}\sin\theta$$

$$R = \dot{\psi}\cos\theta$$

The Z axis is still perpendicular to the y body axis, so Q is not affected by $\dot{\psi}$ in this attitude.

Finally, bank the aircraft, leaving the pitch as it is.

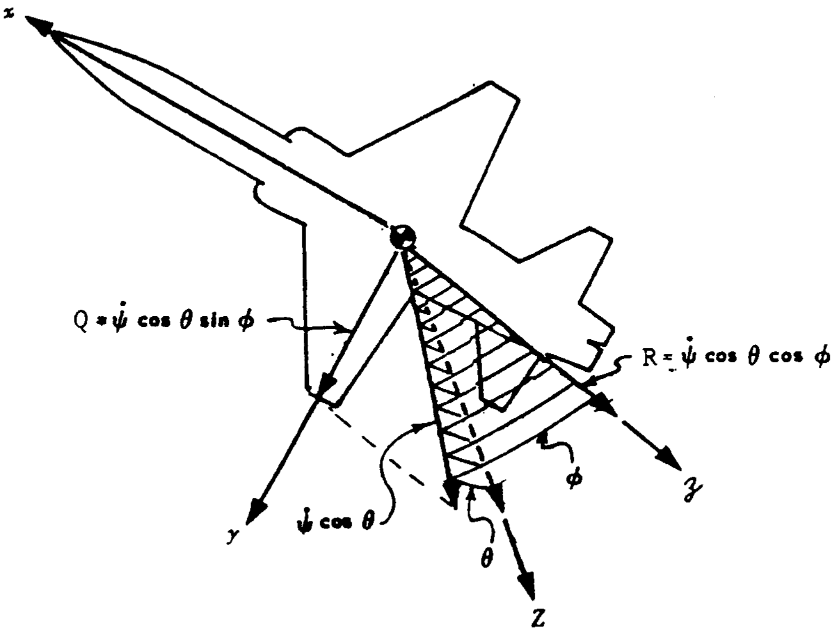


Figure 2.10

All of the components are now illustrated. Notice that roll did not change the effect of $\dot{\psi}$ on P. The components, therefore, of $\dot{\psi}$ in the body axes for any aircraft attitude are:

$$P = -\dot{\psi}\sin\theta$$

Effect of $\dot{\psi}$ only.

$$Q = \dot{\psi}\cos\theta\sin\phi$$

$$R = \dot{\psi}\cos\theta\cos\phi$$

Step 2 - Resolve the components of $\dot{\theta}$ along the body axes for any aircraft attitude.

Remember θ is the angle between the x body axis and the local horizontal. Once again, change the aircraft attitude by steps in the sequence of yaw, pitch and bank and analyze the effects of $\dot{\theta}$.

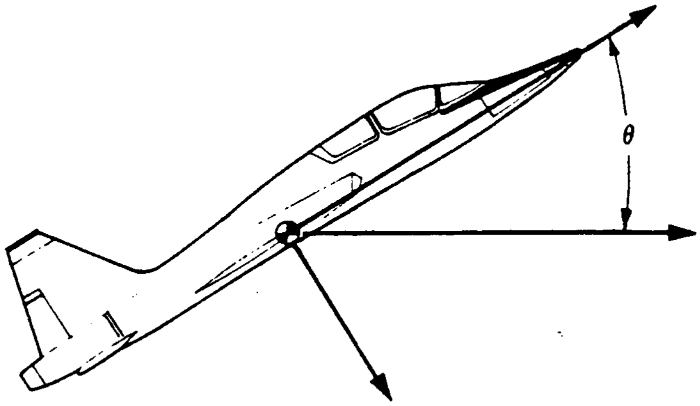


Figure 2.11

It can be seen immediately that the yaw angle has no effect. Likewise when pitched up, the y body axis remains in the horizontal plane. Therefore, $\dot{\theta}$ is the same as Q in this attitude.

$$Q = \dot{\theta}$$

Now bank the aircraft.

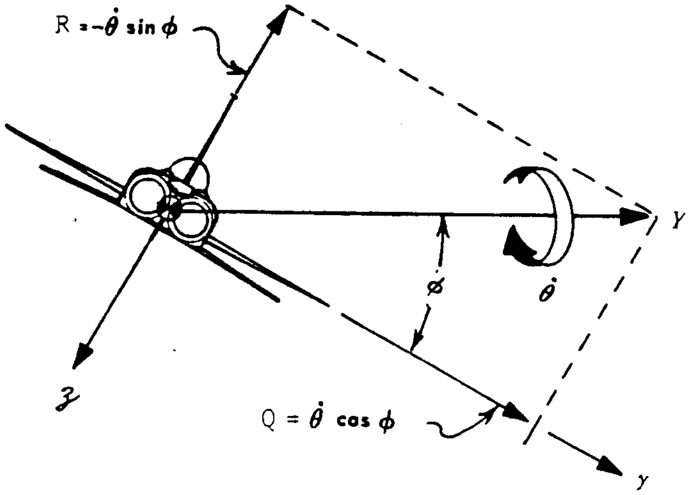


Figure 2.12

It can be seen from figure 2.12 that the components of $\dot{\theta}$ on the body axes are

$$Q = \dot{\theta} \cos \phi$$

$$R = - \dot{\theta} \sin \phi$$

Notice that P is not affected by $\dot{\theta}$ since by definition $\dot{\theta}$ is measured on an axis perpendicular to the x body axis.

Step 3 - Resolve the components of $\dot{\phi}$ along the body axes.

This one is easy since by definition $\dot{\phi}$ is measured along the x body axis. Therefore, $\dot{\phi}$ affects the value of P only, or

$$P = \dot{\phi}$$

The components of $\dot{\psi}$, $\dot{\theta}$, and $\dot{\phi}$ along the x, y, and z body axes for any aircraft attitude have been derived. These can now be summed to give the transformation equations.

$$P = \dot{\phi} - \dot{\psi} \sin \theta$$

$$Q = \dot{\theta} \cos \phi + \dot{\psi} \sin \phi \cos \theta$$

$$R = \dot{\psi} \cos \phi \cos \theta - \dot{\theta} \sin \phi$$

2.4.4 Assumptions

The following assumptions will be made to simplify the derivation of the equations of motion. The reasons for these assumptions will become obvious as the equations are derived.

Rigid Body - Aeroelastic effects must be considered separately.

Earth and Atmosphere are Assumed Fixed - Allows use of Moving Earth Axis System as an "inertial reference" so that Newton's Law can be applied.

Constant Mass - Most motion of interest in stability and control takes place in a relatively short time.

The x z Plane is a Plane of Symmetry - This restriction is made to simplify the RHS of the equation. It allows the cancellation of certain terms containing products of inertia. The restriction can easily be removed by including these terms. This causes two products of inertia, I_{xy} and I_{yz}, to be zero.

2.5 RIGHT-HAND SIDE OF EQUATION

The RHS of the equation represents the aircraft response to any forces or moments that are applied to it. Through the application of Newton's Second Law, two vector relations can be used to derive the six required equations. These are the linear force and moment relations.

2.5.1 Linear Force Relation

The vector equation for response to an applied linear force is

$$\bar{F} = \frac{d\ (m\bar{V})}{dt} \tag{2.14}$$

or the change in momentum of an object is equal to the force applied to it.

This applies, of course, only with respect to inertial space. Therefore, the motion of a body is determined by all the forces applied to it including gravitational attraction of the earth, moon, sun, and even the stars. In most cases, the practical person disregards the effect of the moon, sun, and stars since their influence is extremely small. When considering the forces on an aircraft, the motion of the earth and atmosphere can also be disregarded since the forces resulting from the earth's rotation and coriolis effects are negligible when compared with the large aerodynamic and gravitational forces involved. This simplifies the derivation considerably. The equations can now be derived using either a fixed or moving earth axis system. For graphical clarity consider a fixed earth axis system. The vehicle represented by the dot has a total velocity vector that is changing in both magnitude and direction.

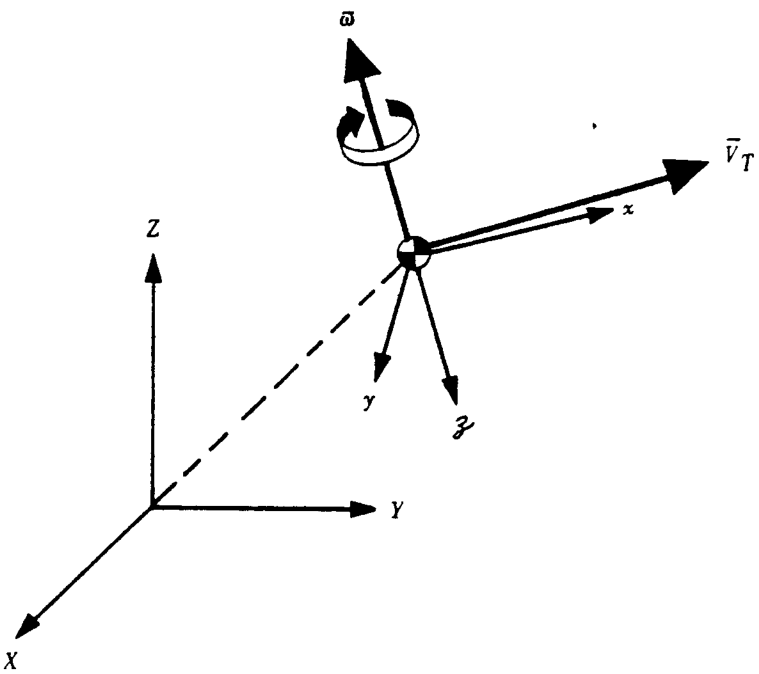


Figure 2.13

From vector analysis,

$$\left.\frac{d\bar{V}_T}{dt}\right|_{XYZ} = \left.\frac{d\bar{V}_T}{dt}\right|_{xyz} + \bar{\omega} \times \bar{V}_T$$

Substituting this into equation 2.14, and assuming mass is constant, the applied force is,

$$\bar{F} = m \left[\left.\frac{d\bar{V}_T}{dt}\right|_{xyz} + \bar{\omega} \times \bar{V}_T \right]$$

which in component form is

$$\bar{F} = m \left[\dot{U}\bar{i} + \dot{V}\bar{j} + \dot{W}\bar{k} + \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ P & Q & R \\ U & V & W \end{vmatrix} \right]$$

Expanding

$$\bar{F} = m \, [\dot{U}\bar{i} + \dot{V}\bar{j} + \dot{W}\bar{k} + (QW - RV)\bar{i} - (PW - RU)\bar{j} + (PV - QU)\bar{k}]$$

Rearranging

$$\bar{F} = m \, [(\dot{U} + QW - RV)\bar{i} + (\dot{V} + RU - PW)\bar{j} + (\dot{W} + PV - QU)\bar{k}]$$

Now since

$$\bar{F} = F_x\bar{i} + F_y\bar{j} + F_z\bar{k}$$

These three component equations result:

$$F_x = m\,(\dot{U} + QW - RV) \qquad (2.15)$$

$$F_y = m\,(\dot{V} + RU - PW) \qquad (2.16)$$

$$F_z = m\,(\dot{W} + PV - QU) \qquad (2.17)$$

2.5.2 Moment Equations

Once again from Newton's Second Law,

$$\bar{G} = \frac{d(\bar{H})}{dt} \qquad (2.18)$$

or the change in angular momentum is equal to the total applied moment.

Angular Momentum

Angular momentum should not be as difficult to understand as some people would like to make it. It can be thought of as linear momentum with a moment arm included.

Consider a ball swinging on the end of a string, at any instant of time,

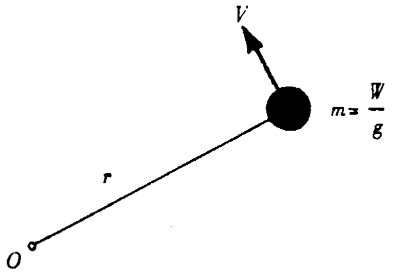


Figure 2.14

Linear momentum = $m\overline{V}$

and

Angular momentum = $mr\overline{V}$ (axis of rotation must be specified).

Therefore, they are related in the same manner that forces relate to moments.

Moment = Force · r

Angular Momentum = Linear Momentum · r

and just as a force changes linear momentum, a moment will change angular momentum.

Angular Momentum of an Aircraft

Consider a small element of mass m_1, somewhere in the aircraft, a distance $\overline{r}_1$ from the cg

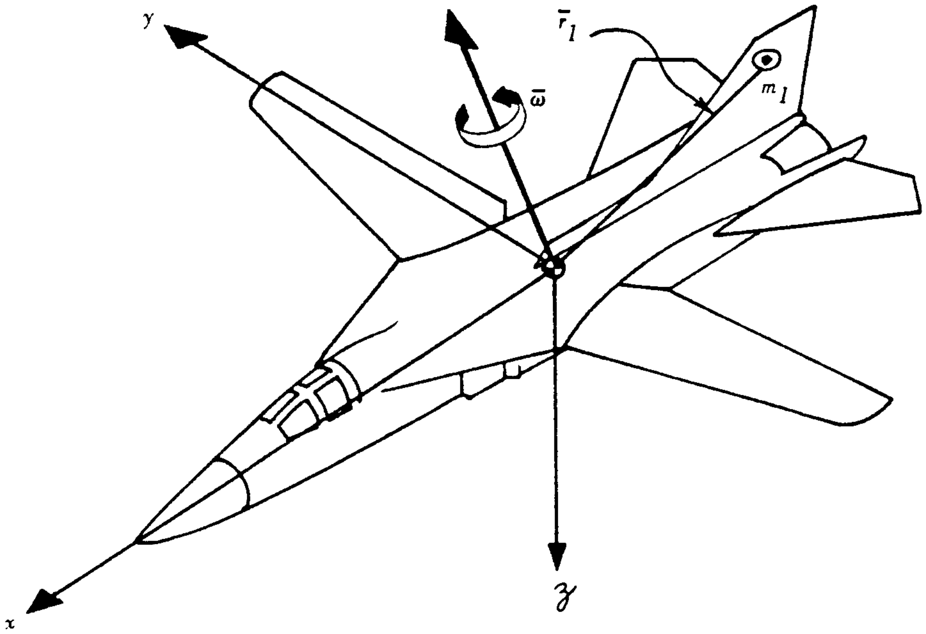


Figure 2.15

The airplane is rotating about all three axes so that

$$\bar{\omega} = P\bar{i} + Q\bar{j} + R\bar{k} \quad (2.19)$$

and

$$\bar{r}_1 = x_1\bar{i} + y_1\bar{j} + z_1\bar{k} \quad (2.20)$$

The angular momentum of m_1 is

$$\bar{H}_{m_1} = m_1 \; (\bar{r}_1 \; X \; \bar{V}_1) \quad (2.21)$$

and

$$\bar{V}_1 = \left.\frac{d\bar{r}_1}{dt}\right|_{XYZ} \qquad \text{(i.e., in the inertial frame)}$$

From vector analysis

$$\left.\frac{d\bar{r}_1}{dt}\right|_{XYZ} = \left.\frac{d\bar{r}_1}{dt}\right|_{xyz} + \bar{\omega} \; X \; \bar{r}_1 \quad (2.22)$$

Since the airplane is a rigid body $\bar{r}_1$ does not change. Therefore the first term can be excluded, and the inertial velocity of the element m_1 is

$$\bar{V}_1 = \bar{\omega} \; X \; \bar{r}_1 \quad (2.23)$$

Substituting this into equation 2.21

$$\bar{H}_{m_1} = m_1 \; (\bar{r}_1 \; X \; \bar{\omega} \; X \; \bar{r}_1) \quad (2.24)$$

This is the angular momentum of the small element of mass m_1. In order to find the angular momentum of the whole airplane, take the sum of all the elements. Using notation in which the i subscript indicates any particular element and n is the total number of elements in the airplane,

$$\bar{H} = \sum_{i=1}^{n} m_i [\bar{r}_i \; X \; \bar{\omega} \; X \; \bar{r}_i] \quad (2.25)$$

TOTAL ANGULAR MOMENTUM

where

m_i = scalar

$$\bar{r}_i = x_i\bar{i} + y_i\bar{j} + z_i\bar{k} \quad (2.26)$$

then

$$\bar{\omega} \; X \; \bar{r}_i = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ P & Q & R \\ x_i & y_i & z_i \end{vmatrix} \quad (2.27)$$

In an effort to reduce the clutter, the subscripts will be left off. The determinant can be expanded to give,

$$\bar{\omega} \times \bar{r} = (Qz - Ry)\bar{i} + (Rx - Pz)\bar{j} + (Py - Qx)\bar{k} \qquad (2.28)$$

therefore, equation 2.25 becomes

$$\bar{H} = \Sigma m \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ x & y & z \\ (Qz-Ry) & (Rx-Pz) & (Py-Qx) \end{vmatrix} \qquad (2.29)$$

So the components of $\bar{H}$ are

$$H_x = \Sigma my\ (Py - Qx) - \Sigma mz\ (Rx - Pz) \qquad (2.30)$$

$$H_y = \Sigma mz\ (Qz - Ry) - \Sigma mx\ (Py - Qx) \qquad (2.31)$$

$$H_z = \Sigma mx\ (Rx - Pz) - \Sigma my\ (Qz - Ry) \qquad (2.32)$$

Rearranging the equations

$$H_x = P\,\Sigma m\ (y^2 + z^2) - Q\,\Sigma mxy - R\,\Sigma mxz \qquad (2.33)$$

$$H_y = Q\,\Sigma m\ (z^2 + x^2) - R\,\Sigma myz - P\,\Sigma mxy \qquad (2.34)$$

$$H_z = R\,\Sigma m\ (x^2 + y^2) - P\,\Sigma mxz - Q\,\Sigma myz \qquad (2.35)$$

Define moments of inertia

$$I_x = \Sigma m\ (y^2 + z^2)$$

$$I_y = \Sigma m\ (x^2 + z^2)$$

$$I_z = \Sigma m\ (x^2 + y^2)$$

These are a measure of resistance to rotation - they are never zero.

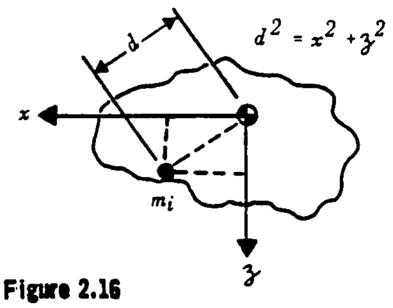


Figure 2.16

Define products of inertia

$$I_{xy} = \Sigma\ mxy$$

$$I_{yz} = \Sigma\ myz$$

$$I_{xz} = \Sigma\ mxz$$

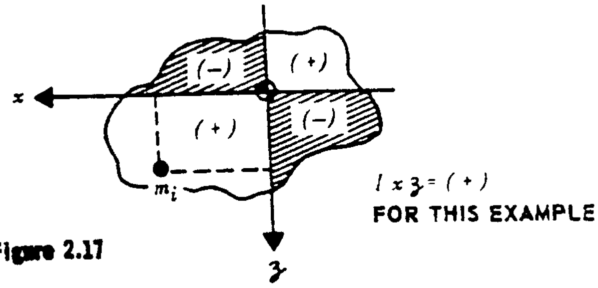


Figure 2.17

These are a measure of symmetry. They are zero for views having a line of symmetry.

The angular momentum of a rigid body is therefore:

$$\overline{H} = H_x \overline{i} + H_y \overline{j} + H_z \overline{k} \qquad (2.36)$$

So that

[redacted] (2.37)

(2.38)

(2.39)

Simplification of Angular Moment Equation for Symmetric Aircraft

A symmetric aircraft has two views that contain a line of symmetry and hence two products of inertia that are zero. The angular momentum of a symmetric aircraft therefore simplifies to:

$$\boxed{\overline{H} = (PI_x - RI_{xz})\overline{i} + QI_y\overline{j} + (RI_z - PI_{xz})\overline{k}} \qquad (2.40)$$

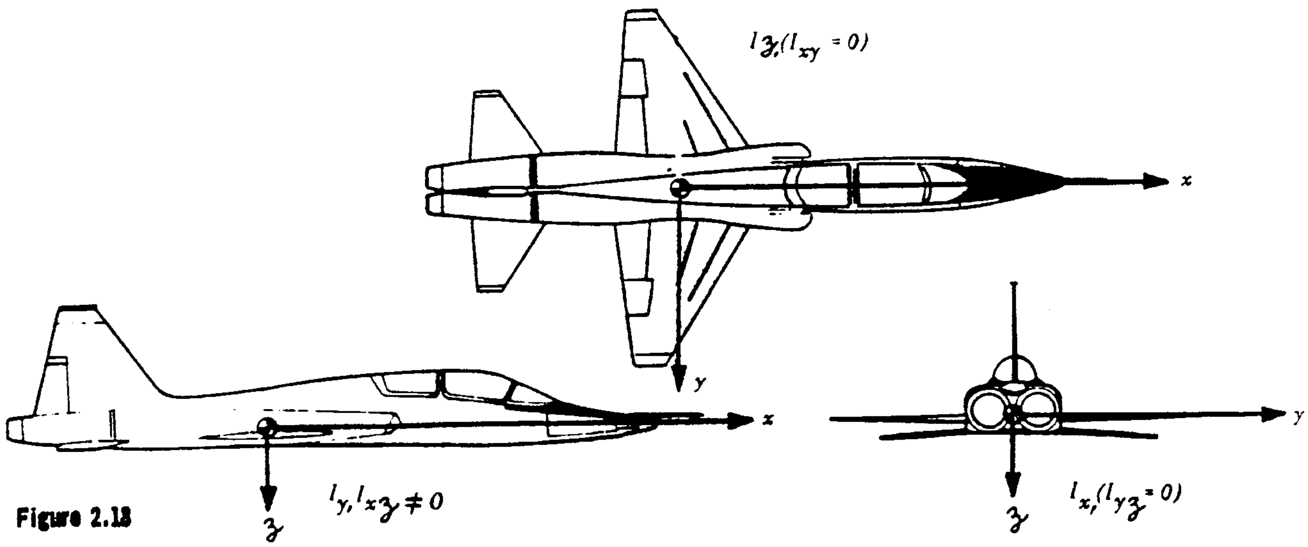


Figure 2.18

Derivation of the Three Rotational Equations

The equation for angular momentum can now be substituted into the moment equation. Remember

$$\bar{G} = \left.\frac{d\bar{H}}{dt}\right|_{XYZ} \tag{2.41}$$

applies only with respect to inertial space. Expressed in the fixed body axis system, the equation becomes:

$$\bar{G} = \left.\frac{d\bar{H}}{dt}\right|_{xyz} + \bar{\omega} \times \bar{H} \tag{2.42}$$

which is

$$\bar{G} = \dot{H}_x\bar{i} + \dot{H}_y\bar{j} + \dot{H}_z\bar{k} + \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ P & Q & R \\ H_x & H_y & H_z \end{vmatrix} \tag{2.43}$$

Remember

$$\bar{H} = (PI_x - RI_{xz})\bar{i} + QI_y\bar{j} + (RI_z - PI_{xz})\bar{k} \tag{2.40}$$

Since the body axis system is used, the moments of inertia and the products of inertia are constant. Therefore, by differentiating and substituting, the moment equation becomes

$$\bar{G} = (\dot{P}I_x - \dot{R}I_{xz})\bar{i} + \dot{Q}I_y\bar{j} + (\dot{R}I_z - \dot{P}I_{xz})\bar{k} + \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ P & Q & R \\ (PI_x - RI_{xz}) & QI_y & (RI_z - PI_{xz}) \end{vmatrix} \tag{2.45}$$

Therefore, the component equations are,

$$G_x = \dot{P}I_x + QR(I_z - I_y) - (\dot{R} + PQ)I_{xz} \tag{2.46}$$

$$G_y = \dot{Q}I_y - PR(I_z - I_x) + (P^2 - R^2)I_{xz} \tag{2.47}$$

$$G_z = \dot{R}I_z + PQ(I_y - I_x) + (QR - \dot{P})I_{xz} \tag{2.48}$$

This completes the development of the RHS of the six equations listed on page 2.6. Remember the RHS is the aircraft response or the motion of the aircraft that would result from the application of a force or a moment. The LHS of the equation represents these applied forces or moments.

2.6 LEFT-HAND SIDE OF EQUATION

2.6.1 Terminology

Before launching into the development of the LHS, it will help to clarify some of the terms used to describe the motion of the aircraft.

Steady Flight - Flight in which the existing motion remains steady with time, i.e., no transient conditions exist.

Symmetric Flight - (Longitudinal Motion) - Flight in which the vehicle plane of symmetry remains fixed in space.

$V = 0 \qquad P = R = 0$

$(\beta = 0) \qquad (\phi \text{ and } \dot{\psi} = 0)$

Asymmetric Flight - (Lateral Motion) - Flight in which the vehicle plane of symmetry does not remain fixed in space.

$V \neq 0 \qquad P \text{ and/or } R \neq 0$

$(\beta \neq 0) \qquad (\phi \text{ and/or } \dot{\psi} \neq 0)$

2.6.2 Some Special-Case Vehicle Motions

Unaccelerated Flight

(Also called straight flight or equilibrium flight.)

$F_x = 0 \qquad F_y = 0 \qquad F_z = 0$

Hence, the cg travels a straight path at constant speed. Note that equilibrium does not mean steady state. For example,

$$F_x = m(\dot{U} + QW - RV) = 0$$

could be maintained zero by fluctuation of the three terms on the right in an unsteady manner. In practice, however, it is difficult to predict that non-steady motion will remain unaccelerated and hence the straight motions most often discussed are also steady state.

Steady Straight Flight		Steady Rolls or Spins	
$F_x = 0 \quad F_y = 0$ $F_z = 0 \quad G_x = 0$ $G_y = 0 \quad G_z = 0$ On the average	$P = Q = R = 0$ Excluded by custom	$F_x = 0 \quad F_y = 0$ $F_z = 0 \quad G_x = 0$ $G_y = 0 \quad G_z = 0$ On the average	By custom this is not called straight flight even though the cg may be traveling a straight path
Trim Points, Stabilized Points		Steady Developed Spins	

Accelerated Flight (Non-Equilibrim Flight)

One or more of the linear equations is not zero, hence the cg is not traveling a straight path. Again the steady cases are of most interest.

Steady Turns	Symmetrical Pull Up
An unbalanced horizontal force results in the cg being constantly deflected inward toward the center of a curved path. This results in a constantly changing yaw angle. By the Euler angle transform, $P = -\dot{\psi}\theta$ (assumes small θ) $Q = \dot{\psi} \sin\phi \cos\theta = \dot{\psi} \sin\phi$ $R = \dot{\psi} \cos\phi \cos\theta = \dot{\psi} \cos\phi$ and hence $F_y = m(\dot{\psi} \cos\phi)U$ $F_z = -m(\dot{\psi} \sin\phi)U$ (assumes $\dot{\psi}\theta$ is very very small) Includes moderate climbs and descents.	Here an unbalanced z force is constantly deflecting the cg upward. $Q = \dot{\theta}$ $F_x \approx mQW$ and $F_z \approx -mQU$ This is a quasi-steady motion since $\dot{U}$ and $\dot{W}$ cannot long remain zero.

2.6.3 Preparation for Expansion of the Left-Hand Side

The Equations of Motion relate the vehicle motion to the applied forces and moments.

LHS		RHS
Applied Forces and Moments	=	Observed Vehicle Motion

$$F_x = m\dot{u} + \text{- - - -}$$

$$G_x = \dot{P}I_x + \text{- - - -}$$

etc.

The RHS of each of these six equations has been completely expanded in terms of easily measured quantities. The LHS must also be expanded in terms of convenient variables, to include Stability Parameters and Derivatives. Before this can be accomplished, however, the following topics must be discussed and understood.

2.6.4 Initial Breakdown of the Left-Hand Side

In general, the applied forces and moments can be broken up according to the sources shown below.

		Source						
		Aero-dynamic	Direct Thrust	Gravity	Gyro-scopic	Other		
LONGITUDINAL	F_x	X	X_T	X_g	0	X_{oth}	$= m\dot{U} + - - -$	(2.49)
	F_z	Z	Z_T	Z_g	0	Z_{oth}	$= m\dot{W} + - - -$	(2.50)
	G_y	$\mathcal{M}$	M_T	0	M_{gyro}	M_{oth}	$= \dot{Q}I_y + - - -$	(2.51)
LATERAL-DIRECTIONAL	F_y	Y	Y_T	Y_g	0	Y_{oth}	$= m\dot{V} + - - -$	(2.52)
	G_x	$\mathcal{L}$	L_T	0	L_{gyro}	L_{oth}	$= \dot{P}I_x + - - -$	(2.53)
	G_z	$\mathcal{N}$	N_T	0	N_{gyro}	N_{oth}	$= \dot{R}I_z + - - -$	(2.54)

1. Gravity Forces - These vary with orientation of the weight vector.

   $X_g = -mg \sin \theta \qquad Y_g = mg \cos \theta \sin \phi \qquad Z_g = mg \cos \theta \cos \phi$

2. Gyroscopic Moments - These occur as a result of large rotating masses such as engines and props.

3. Direct Thrust Forces and Moments - These terms include the effect of the thrust vector itself - they usually do not include the indirect or induced effects of jet flow or running propellers.

4. Aerodynamic Forces and Moments - These will be further expanded into Stability Parameters and Derivatives.

5. Other Sources - These include spin chutes, reaction controls, etc.

2.6.5 Aerodynamic Forces and Moments

By far the most important forces and moments on the LHS of the equation are the aerodynamic terms. Unfortunately they are also the most complex. As a result, certain simplifying assumptions are made and several of the smaller terms are arbitrarily excluded to simplify the analysis. Remember we are not trying to design an airplane around some critical criteria. We are only trying to derive a set of equations that will help us analyze the important factors affecting aircraft stability and control.

Choice of Axis System

Consider the aerodynamic forces on an airplane

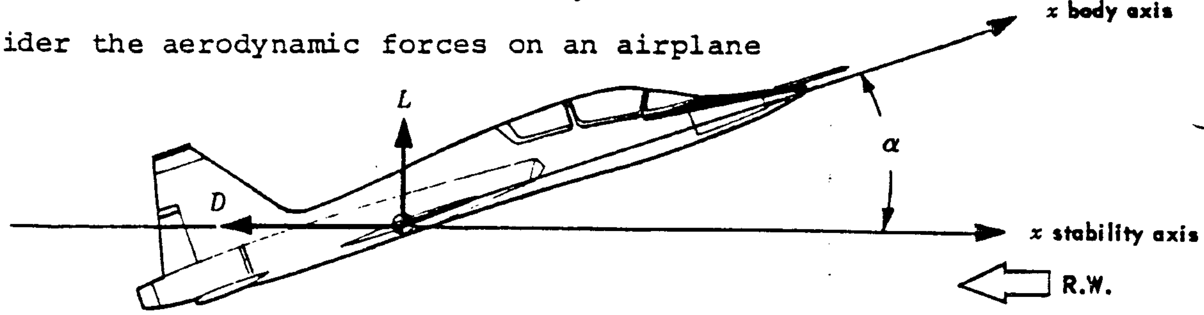


Figure 2.19

Summing forces along the x body axis

$$F_x = L \sin \alpha - D \cos \alpha \qquad (2.55)$$

Notice that if the forces were summed along the x stability axis, it would be

$$F_x = -D \qquad (2.56)$$

Obviously, it would simplify things if the stability axes were used for development of the aerodynamic forces. A small angle assumption will enable us to do this. Let's assume that α is always small enough so that

$$\cos \alpha \approx 1$$

$$\sin \alpha \approx 0$$

Using this assumption, equation 2.55 reduces to equation 2.56. Whether it be thought of as a small angle assumption or as an arbitrary choice of the stability axis system, the result is the same. The complexity of the equation is reduced. This of course would not be done for preliminary design analyses, however, for the purpose of deriving a set of equations to be used as an analytical tool in determining handling qualities, the assumption is perfectly valid, and in fact, is surprisingly accurate for relatively large values of α.

Therefore, the aerodynamic terms will be developed using the <u>stability axis system</u> so that the equations assume the form,

"DRAG" $\quad -D + X_T + X_g + X_{oth} = m\dot{U} + - - - -$ (2.57)

"LIFT" $\quad -L + Z_T + Z_g + Z_{oth} = m\dot{W} + - - - -$ (2.58)

"PITCH" $\quad \mathcal{M} + M_T + M_{gyro} + M_{oth} = \dot{Q}I_y + - - - -$ (2.59)

"SIDE" $\quad Y + Y_T + Y_g + Y_{oth} = m\dot{V} + - - - -$ (2.60)

"ROLL" $\quad \mathcal{L} + L_T + L_{gyro} + L_{oth} = \dot{P}I_x + - - - -$ (2.61)

"YAW" $\quad \mathcal{N} + N_T + N_{gyro} + N_{oth} = \dot{R}I_z + - - - -$ (2.62)

Expansion of Aerodynamic Terms

A stability and control analysis is concerned with the question of how a vehicle responds to certain perturbations or inputs. For instance, up elevator should cause the nose to come up; or if the aircraft hits some turbulence that causes a small amount of sideslip, the airplane should realign itself with the relative wind. Intuitively, it can be seen that the aerodynamic terms are going to have the most effect on the resulting motion of the aircraft. Unfortunately, the equations that result from summing forces and moments are non-linear. As a result, exact solutions to these equations are impossible. Therefore, a technique to linearize the equations must be used so that solutions can be obtained. In order to do this the small perturbation theory is introduced.

Small Perturbation Theory

The small perturbation theory is based on a simple and very popular technique used for linearizing a set of differential equations. In a nutshell, it is simply the process of expanding the equations using a Taylor series expansion and excluding the higher order terms. To fully understand the derivation some assumptions and definitions must first be established.

The Small Disturbance Assumption

The aerodynamic forces and moments are primarily a function of the following variables:

1. Temperature and Altitude

   Accounted for by ρ, M, R_e.

2. Angular Velocities

   Accounted for by P, Q, R.

3. Control Deflections

   Accounted for by δ_e, δ_a, δ_r.

4. Position and Magnitude of the Relative Wind

   Accounted for by the components U, V, W of true velocity, or alternately, by:

$$U, \qquad \alpha \approx \frac{W}{V_T}, \qquad \beta \approx \frac{V}{V_T}$$

In general, the time derivatives of these variables could also be significant. In other words:

D, L, $\mathcal{M}$, Y, $\mathcal{L}$, $\mathcal{N}$ Are a Function of:

VARIABLE	FIRST DERIVATIVE	SECOND DERIVATIVE
$U \; \alpha \; \beta$	$\dot{U} \; \dot{\alpha} \; \dot{\beta}$	$\ddot{U}$ - - - -
$P \; Q \; R$	$\dot{P}$ - -	$\ddot{P}$ - - - -
$\delta_e \; \delta_a \; \delta_r$	$\dot{\delta}_e$ - -	- - -
$\rho \; M \; R_e$ → assumed constant		

This rather formidable list can be reduced to workable proportions by making the assumption that the vehicle motion will consist only of small deviations from some initial reference condition. Fortunately, this small disturbance assumption applies to many cases of practical interest, and as a bonus, stability parameters and derivatives derived under this assumption continue to give good results for motions somewhat larger.

The variables are considered to consist of some initial value plus an incremental change, called the "perturbated value." The notation for these perturbated values is sometimes lower case and sometimes lower case with a bar.

$$P = P_o + p \qquad P = P_o + \bar{p}$$

$$U = U_o + u \qquad U = U_o + \bar{u}$$

It has been found from experience that when operating under the small disturbance assumption the vehicle motion can be thought of as two independent motions each of which is a function only of the variables shown below.

1. Longitudinal Motion

$$(D, L, \mathcal{M}) = f\,(U, \alpha, \dot{\alpha}, Q\;\; \delta_e) \qquad (2.63)$$

2. Lateral-Directional Motion

$$(Y, \mathcal{L}, \mathcal{N}) = f\,(\beta, \dot{\beta}, P, R, \delta_a, \delta_r) \qquad (2.64)$$

Initial Conditions

It will be assumed that the motion consists of small perturbations about some initial condition of steady straight symmetrical flight. From this and the definition of stability axes, the following can be stated:

$$V_T = U_o = \text{constant}$$

$$V_o = 0 \qquad \beta_o = 0$$

$$W_o = 0 \qquad \alpha_o = \text{constant}$$

$$P_o = Q_o = R_o = 0$$

$$(\rho,\ M,\ R_e,\ \text{aircraft configuration.}) = \text{constant}$$

2.6.6 Expansion by Taylor Series

As previously noted, the longitudinal motion can be assumed to be a function of five variables, U, α, $\dot{\alpha}$, Q, and δ_e. The aerodynamic forces and moments can therefore be expressed by a Taylor's expansion.

For example:

$$L = \begin{bmatrix} L_o + \frac{\partial L}{\partial U}\Delta U & + \frac{1}{2}\frac{\partial^2 L}{\partial U^2}\Delta U^2 & + - - - \\ + \frac{\partial L}{\partial \alpha}\Delta\alpha & + \frac{1}{2}\frac{\partial^2 L}{\partial \alpha^2}\Delta\alpha^2 & + - - - \\ + \frac{\partial L}{\partial \dot{\alpha}}\Delta\dot{\alpha} & + - - - - & \\ + \frac{\partial L}{\partial Q}\Delta Q & + - - - - & \\ + \frac{\partial L}{\partial \delta_e}\Delta\delta_e & + - - - - & \end{bmatrix} \qquad (2.65)$$

But we have decided to express the variables as the sum of an initial value plus a small perturbated value

$$U = U_o + u \quad \text{where} \quad u = U - U_o = \Delta U \qquad (2.66)$$

Therefore

$$\frac{\partial L}{\partial U} = \frac{\partial L}{\partial U_o}\cdot\frac{\partial U_o}{\partial U}^{\text{Zero}} + \frac{\partial L}{\partial u}\frac{\partial u}{\partial U}^{1.0} = \frac{\partial L}{\partial u} \qquad (2.67)$$

and the first term of the expansion becomes

$$\frac{\partial L}{\partial U}\Delta U = \frac{\partial L}{\partial u}u \qquad (2.68)$$

Similarly:

$$\frac{\partial L}{\partial Q} \Delta Q = \frac{\partial L}{\partial q} q \tag{2.69}$$

We also elect to let $\alpha = \Delta\alpha$, $\dot{\alpha} = \Delta\dot{\alpha}$ and $\delta_e = \Delta\delta_e$.

Dropping higher order terms involving u^2, q^2, etc., equation 2.65 now becomes

$$L = L_o + \frac{\partial L}{\partial u} u + \frac{\partial L}{\partial \alpha} \alpha + \frac{\partial L}{\partial \dot{\alpha}} \dot{\alpha} + \frac{\partial L}{\partial q} q + \frac{\partial L}{\partial \delta_e} \delta_e \tag{2.70}$$

The lateral-directional motion is a function of β, $\dot{\beta}$, P, R, δ_a, δ_r, and can be handled in a similar manner. For example, the aerodynamic terms for rolling moment become:

$$\mathcal{L} = \mathcal{L}_o + \frac{\partial \mathcal{L}}{\partial \beta} \beta + \frac{\partial \mathcal{L}}{\partial \dot{\beta}} \dot{\beta} + \frac{\partial \mathcal{L}}{\partial p} p + \frac{\partial \mathcal{L}}{\partial r} r + \frac{\partial \mathcal{L}}{\partial \delta_a} \delta_a + \frac{\partial \mathcal{L}}{\partial \delta_r} \delta_r \tag{2.71}$$

This development can be applied to all of the aerodynamic forces and moments. The equations are linear and account for all variables that have a significant effect on the aerodynamic forces and moments on an aircraft.

The equations resulting from this development can now be substituted into the LHS of the equations of motion.

2.6.7 Effects of Weight

The weight force acts through the cg of an airplane and as a result has no effect on the aircraft moments. It does affect the force equations as shown below.

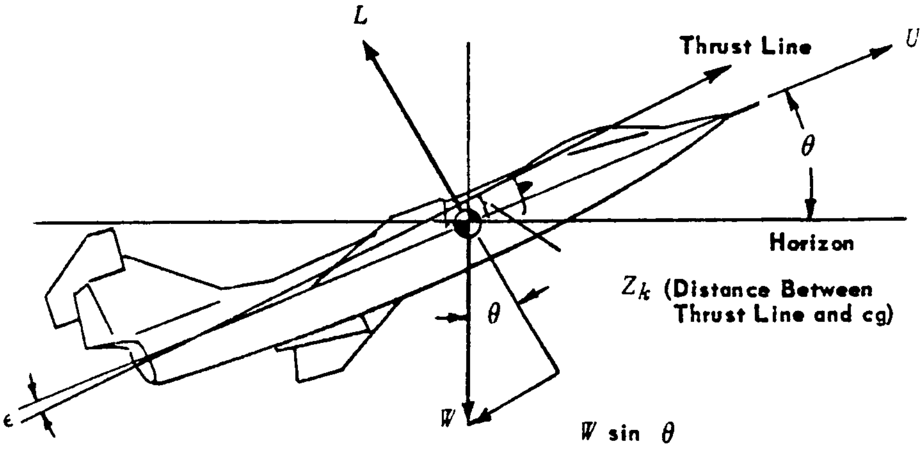


Figure 2.20

The same "small perturbation" technique can be used to analyze the effects of weight. For longitudinal motion, the only variable to consider is θ. For example, consider the effect of weight on the x axis.

$$X_g = -W \sin \theta \qquad (2.72)$$

Since weight is considered constant, θ is the only pertinent variable. Therefore, the expansion of the gravity term (X_g) can be expressed as

$$X_g = X_{g_o} + \frac{\partial X_g}{\partial \theta} \theta \quad (X_{g_o} = \text{initial condition of } X_g) \qquad (2.73)$$

For simplification and clarity, the term X_g will hereafter be referred to as drag due to weight, (D_{wt}). This in essence incorporates the same small angle assumption that was made in development of the aerodynamic terms, however, as before, the effect is negligible. Therefore, equation 2.73 becomes

$$D_{wt} = D_{o_{wt}} + \frac{\partial D}{\partial \theta} \theta$$

Likewise the Z force can be expressed as negative lift due to weight (L_{wt}), and the expanded term becomes

$$L_{wt} = L_{o_{wt}} + \frac{\partial L}{\partial \theta} \theta$$

The effect of weight on side force is dependent solely on bank angle (ϕ). Therefore,

$$Y_{wt} = Y_{o_{wt}} + \frac{\partial Y}{\partial \phi} \phi$$

These then are the component equations relating the effects of gravity to the equations of motion and can be substituted into the LHS of the equations.

2.6.8 Effects of Thrust

The thrust vector can be considered in the same way. Since thrust does not always pass through the cg its effect on the moment equation must be considered (figure 2.20). The X component would be

$$X_T = T \cos \varepsilon$$

The Z component is

$$Z_T = -T \sin \varepsilon$$

The pitching moment component is,

$$\mathcal{M}_T = T\, z_k$$

where Z_k is the perpendicular distance from the thrust line to the cg. For small disturbances, changes in thrust depend upon the change in forward speed and engine rpm. Therefore, by a small perturbation analysis

$$T = T_o + \frac{\partial T}{\partial u} u + \frac{\partial T}{\partial \delta_{rpm}} \delta_{rpm} \qquad (2.74)$$

Thrust effects will be considered in the longitudinal equations only since the thrust vector is normally in the vertical plane of symmetry and does not affect the lateral-directional motion. When considering engine-out characteristics in multi-engine aircraft; however, the asymmetric thrust effects must be considered. Once again, for clarity, X_T and Z_T will be referred to as drag due to thrust and lift due to thrust.

Thus:

$$X_{THRUST} = (T_o + \frac{\partial T}{\partial u} u + \frac{\partial T}{\partial \delta_{rpm}} \delta_{rpm}) \ (\cos \varepsilon) \qquad (2.75)$$

$$Z_{THRUST} = -(T_o + \frac{\partial T}{\partial u} u + \frac{\partial T}{\partial \delta_{rpm}} \delta_{rpm}) \ (\sin \varepsilon) \qquad (2.76)$$

$$M_{THRUST} = (T_o + \frac{\partial T}{\partial u} u + \frac{\partial T}{\partial \delta_{rpm}} \delta_{rpm}) \ (Z_k) \qquad (2.77)$$

2.6.9 Gyroscope Effects

For most analyses, gyroscopic effects are insignificant. They begin to become important as angular rates increase, (i.e., P, Q, and R become large). For static and dynamic stability analyses, angular rates are not considered large. In the area of spins and maximum roll rate maneuvers, they are large and definitely affect the motion of the airplane. Therefore, for spin and roll coupling analyses, gyroscopic effects will be considered. However, in the basic development of the equations of motion, they will not be included.

2.7 REDUCTION OF EQUATIONS TO A USABLE FORM

2.7.1 Normalization of Equations

Now that the linearized expressions have been derived, we begin the process of putting them in a more usable form. One of the first steps in this process is to "normalize" the equations. Initially, the reason for doing this will not be apparent. It is a necessary step in the simplification of the equations, however, and the rationale will become apparent later.

In order to do this each equation is multiplied through by a "normalization factor". This factor is different for each equation and is picked primarily to simplify the first term on the RHS of the equation. The following table presents the normalizing factor for each equation.

Equation	Normalizing Factor	First Term is Now Pure Accel or $\dot{\alpha}$ $\dot{\beta}$	Units	
"DRAG"	$\frac{1}{m}$	$-\frac{D}{m} + \frac{X_T}{m} + - - - - = \dot{u}$ ---	$[\frac{ft}{sec^2}]$	(2.78)
"LIFT"	$\frac{1}{mU_o}$	$-\frac{L}{mU_o} + \frac{Z_T}{mU_o} + - - - - = \frac{\dot{w}}{U_o}$ + ---	$[\frac{rad}{sec}]$	(2.79)
"PITCH"	$\frac{1}{I_y}$	$\frac{\mathcal{M}}{I_y} + \frac{M_T}{I_y} + - - - - = \dot{q}$ ---	$[\frac{rad}{sec^2}]$	(2.80)
"SIDE"	$\frac{1}{mU_o}$	$\frac{Y}{mU_o} + \frac{Y_T}{mU_o} + - - - - = \frac{\dot{v}}{U_o}$ + ---	$[\frac{rad}{sec}]$	(2.81)
"ROLL"	$\frac{1}{I_x}$	$\frac{\mathcal{L}}{I_x} + \frac{L_T}{I_x} + - - - - = \dot{p}$ + ---	$[\frac{rad}{sec^2}]$	(2.82)
"YAW"	$\frac{1}{I_z}$	$\frac{\mathcal{N}}{I_z} + \frac{N_T}{I_z} + - - - - = \dot{r}$ + ---	$[\frac{rad}{sec^2}]$	(2.83)

2.7.2 Stability Parameters

Stability parameters are simply the partial coefficients ($\frac{\partial L}{\partial u}$, etc.) multiplied by their respective normalizing factors. To demonstrate this, consider the aerodynamic terms of the lift equation. By multiplying equation 2.70 through by the factor $\frac{1}{mU_o}$, we get

$$\frac{L}{mU_o} = \frac{L_o}{mU_o} + \underbrace{\frac{1}{mU_o}\frac{\partial L}{\partial u}}_{L_u} u + \underbrace{\frac{1}{mU_o}\frac{\partial L}{\partial \alpha}}_{L_\alpha} \alpha + \underbrace{\frac{1}{mU_o}\frac{\partial L}{\partial \dot{\alpha}}}_{L_{\dot{\alpha}}} \dot{\alpha} + \underbrace{\frac{1}{mU_o}\frac{\partial L}{\partial q}}_{L_q} q + \underbrace{\frac{1}{mU_o}\frac{\partial L}{\partial \delta_e}}_{L_{\delta_e}} \delta_e \quad [\frac{rad}{sec}] \quad (2.84)$$

The indicated quantities are defined as stability parameters and the equation becomes

$$\frac{L}{mU_o} = \frac{L_o}{mU_o} + L_u u + L_\alpha \alpha + L_{\dot{\alpha}} \dot{\alpha} + L_q q + L_{\delta_e} \delta_e [\frac{rad}{sec}] \quad (2.85)$$

Stability parameters have various dimensions depending on whether they are multiplied by a linear velocity, an angle, or an angular rate

$$L_u [\frac{1}{ft}]\ u\ [\frac{ft}{sec}] = [\frac{rad}{sec}],\ L_\alpha\ [\frac{1}{sec}]\ \alpha\ [rad] = [\frac{rad}{sec}],$$

$$L_{\dot{\alpha}}\ [none]\ \dot{\alpha}\ [\frac{rad}{sec}] = [\frac{rad}{sec}]$$

The lateral-directional motion can be handled in a similar manner. For example, the normalized aerodynamic rolling moment becomes:

$$\frac{\mathcal{L}}{I_x} = \frac{\mathcal{L}_o}{I_x} + \mathcal{L}_\beta \beta + \mathcal{L}_{\dot{\beta}} \dot{\beta} + \mathcal{L}_p p + \mathcal{L}_r r + \mathcal{L}_{\delta_a} \delta_a + \mathcal{L}_{\delta_r} \delta_r\ [\frac{1}{sec^2}] \qquad (2.86)$$

where

$$\mathcal{L}_\beta = \frac{1}{I_x} \frac{\partial \mathcal{L}}{\partial \beta}\ [\frac{1}{sec^2}],\ etc.$$

These stability parameters are sometimes called "dimensional derivatives" but we will reserve the word "derivative" to indicate the non-dimensional form which can be obtained by rearrangement. This will be developed later in this chapter.

2.7.3 Simplification of the Equations

By combining all of the terms derived so far, the resulting equations are somewhat lengthy. In order to economize on effort, several simplifications can be made. For one, all "small effect" terms can be disregarded. Normally these terms are an order of magnitude less than the more predominant terms. These and other simplifications will help derive a concise and workable set of equations.

2.7.4 Longitudinal Equations

Drag Equation

The complete normalized drag equation is

$$\overbrace{-\left[\frac{D_o}{m} + D_\alpha \alpha + D_{\dot{\alpha}} \dot{\alpha} + D_u u + D_q q + D_{\delta_e} \delta_e\right]}^{\text{Aero Terms}} \quad \overbrace{-\left[\frac{D_{o_{wt}}}{m} + D_\theta \theta\right]}^{\text{Gravity Terms}}$$

$$\overbrace{+\frac{1}{m}\left[T_o + \frac{\partial T}{\partial u} u + \frac{\partial T}{\partial \delta_{rpm}} \delta_{rpm}\right](\cos \epsilon)}^{\text{Thrust Terms}} = \dot{u} + qw - rv \qquad (2.87)$$

Simplifying assumptions

1. $\frac{T_o}{m} \cos \varepsilon - \frac{D_o}{m} - \frac{D_{o_{wt}}}{m} = 0$ (Steady State)

2. $\left(\frac{\partial T}{\partial u} u + \frac{\partial T}{\partial \delta_{rpm}} \delta_{rpm}\right) \cos \varepsilon = 0$ (Constant rpm, $\frac{\partial T}{\partial u}$ is small)

3. $rv = 0$ (No lat-dir motion)

The small perturbation assumption allows us to analyze the longitudinal motion independent of lateral-directional motion.

4. $qw \approx 0$ (Order of magnitude)

5. $D_{\dot{\alpha}}$, D_q, and D_{δ_e} are all small.

The resulting equation is

$$-[D_\alpha \alpha + D_u u + D_\theta \theta] = \dot{u} \tag{2.88}$$

Rearranging

$$\dot{u} + D_\alpha \alpha + D_u u + D_\theta \theta = 0 \tag{2.89}$$

Lift Equation

The complete lift equation is

$$-\overbrace{\left[\frac{L_o}{mU_o} + L_\alpha \alpha + L_{\dot{\alpha}} \dot{\alpha} + L_u u + L_q q + L_{\delta_e} \delta_e\right]}^{\text{Aero Terms}} + \overbrace{\left[\frac{L_{o_{wt}}}{mU_o} + L_\theta \theta\right]}^{\text{Gravity Terms}}$$

$$-\overbrace{\frac{1}{mU_o}\left[T_o + \frac{\partial T}{\partial u} u + \frac{\partial T}{\partial \delta_{rpm}} \delta_{rpm}\right] (\sin \varepsilon)}^{\text{Thrust Terms}} = \frac{\dot{w} + pv - qu}{U_o}$$

Simplifying assumptions

1. $- \frac{L_o}{mU_o} + \frac{L_{o_{w_t}}}{mU_o} - \frac{T_o}{mU_o} \sin \varepsilon = 0$ (Steady State)

2. $\frac{\partial T}{\partial u} u + \frac{\partial T}{\partial \delta_{rpm}} \delta_{rpm} = 0$ (Constant rpm, $\frac{\partial T}{\partial u}$ is small)

3. $L_\theta \theta = 0$ (Order of magnitude)

4. $\frac{\dot{w}}{U_o} = \dot{\alpha}$ (α is small)

5. $pv = 0$ (No lat-dir motion)

6. $\frac{qu}{U_o} = q$ ($u = U_o$)

Thus

$$- L_\alpha \alpha - L_{\dot{\alpha}} \dot{\alpha} - L_u u - L_q q - L_{\delta_e} \delta_e = \dot{\alpha} - q \qquad (2.90)$$

Rearranging

$$- L_\alpha \alpha - (1 + L_{\dot{\alpha}}) \dot{\alpha} - L_u u + (1 - L_q) q = L_{\delta_e} \delta_e \qquad (2.91)$$

Pitch Moment Equation

$$\frac{\mathcal{M}_o}{I_y} + \mathcal{M}_\alpha \alpha + \mathcal{M}_{\dot{\alpha}} \dot{\alpha} + \mathcal{M}_u u + \mathcal{M}_{\delta_e} \delta_e + \mathcal{M}_q q + \text{Thrust Terms}$$
$$= \dot{q} - pr \frac{(I_z - I_x)}{I_y} + \frac{(p^2 - r^2)}{I_y} I_{xz} \qquad (2.92)$$

This can be simplified as before. Thus

$$\dot{q} - \mathcal{M}_\alpha \alpha - \mathcal{M}_{\dot{\alpha}} \dot{\alpha} - \mathcal{M}_u u - \mathcal{M}_q q = \mathcal{M}_{\delta_e} \delta_e \qquad (2.93)$$

Now there are three longitudinal equations that are easy to work with. Notice that there are four variables θ, α, u, and q but only three equations. To solve this problem, $\dot{\theta}$ can be substituted for q.

$q = \dot{\theta}$ and $\dot{q} = \ddot{\theta}$

Therefore the longitudinal equations become:

	(θ)	(u)	(α)		
DRAG	$D_\theta \theta$	$+ \dot{u} + D_u u$	$+ D_\alpha \alpha$	$= 0$	(2.94)
LIFT	$(1-L_q)\ \dot{\theta}$	$- L_u u$	$- (1 + L_{\dot{\alpha}})\ \dot{\alpha} - L_\alpha \alpha$	$= L_{\delta_e}\ \delta_e$	(2.95)
PITCH	$\ddot{\theta} - \mathcal{M}_q \dot{\theta}$	$- \mathcal{M}_u u$	$- \mathcal{M}_{\dot{\alpha}} \dot{\alpha} - \mathcal{M}_\alpha \alpha$	$= \mathcal{M}_{\delta_e}\ \delta_e$	(2.96)

There are now three independent equations with three variables. The terms on the RHS are the inputs or "forcing functions". Therefore, for any input δ_e, the equations can be solved to get θ, u and α at any time.

2.7.5 Lateral-Directional Equations

The complete lateral-directional equations are as follows:

Side Force

$$\frac{Y_o}{mU_o} + Y_\beta \beta + Y_{\dot{\beta}} \dot{\beta} + Y_p p + Y_r r + Y_{\delta_a} \delta_a + Y_{\delta_r} \delta_r$$

$$+ \frac{Y_{o_{wt}}}{mU_o} + Y_\phi \phi = \frac{\dot{v} + ru - pw}{U_o} \qquad (2.97)$$

Rolling Moment

$$\frac{\mathcal{L}_o}{I_x} + \mathcal{L}_\beta \beta + \mathcal{L}_{\dot{\beta}} \dot{\beta} + \mathcal{L}_p p + \mathcal{L}_r r + \mathcal{L}_{\delta_a} \delta_a + \mathcal{L}_{\delta_r} \delta_r$$

$$= \dot{p} + qr \left(\frac{I_z - I_y}{I_x}\right) - (\dot{r} + pq) \frac{I_{xz}}{I_x} \qquad (2.98)$$

Yawing Moment

$$\frac{\mathcal{N}_o}{I_z} + \mathcal{N}_\beta \beta + \mathcal{N}_{\dot{\beta}} \dot{\beta} + \mathcal{N}_p p + \mathcal{N}_r r + \mathcal{N}_{\delta_a} \delta_a + \mathcal{N}_{\delta_r} \delta_r$$

$$= \dot{r} + pq \left(\frac{I_y - I_x}{I_z}\right) + (qr - \dot{p}) \frac{I_{xz}}{I_z} \qquad (2.99)$$

In order to simplify the equations, the following assumptions are made:

1. A wings level steady state condition exists initially. Therefore, $\mathcal{L}_o$, $\mathcal{N}_o$, Y_o, and $Y_{o_{wt}}$ are zero.

2. $p = \dot{\phi}$, $\dot{p} = \ddot{\phi}$ ($\theta \approx 0$, see Euler angle transformations, pg 2.11)

3. The terms $Y_{\dot{\beta}}\dot{\beta}$, $\mathcal{N}_{\dot{\beta}}\dot{\beta}$, $\mathcal{L}_{\dot{\beta}}\dot{\beta}$ and $Y_{\delta_a}\delta_a$ are all small.

4. $\frac{\dot{v}}{U_o} = \dot{\beta}$ (β is small)

5. $\frac{ru}{U_o} = r$ ($u = U_o$)

6. $q = 0$ (no pitching motion)

Using these assumptions the lateral-directional equations reduce to:

	(β)	(ϕ)	(r)		
SIDE FORCE	$\dot{\beta} - Y_\beta \beta$	$- Y_p\dot{\phi} - Y_\phi \phi$	$+ (1 - Y_r) r$	$= Y_{\delta_r}\delta_r$	(2.100)
ROLLING MOMENT	$- \mathcal{L}_\beta \beta$	$+ \ddot{\phi} - \mathcal{L}_p \dot{\phi}$	$- \frac{I_{xz}}{I_x}\dot{r} - \mathcal{L}_r r$	$= \mathcal{L}_{\delta_a}\delta_a + \mathcal{L}_{\delta_r}\delta_r$	(2.101)
YAWING MOMENT	$- \mathcal{N}_\beta \beta$	$- \frac{I_{xz}}{I_z}\ddot{\phi} - \mathcal{N}_p \dot{\phi}$	$+ \dot{r} - \mathcal{N}_r r$	$= \mathcal{N}_{\delta_a}\delta_a + \mathcal{N}_{\delta_r}\delta_r$	(2.102)

Once again there are three unknowns and three equations. These equations may be used to analyze the lateral-directional motion of the aircraft.

2.7.6 Stability Derivatives

The parameteric equations give all the information necessary to describe the motion of any particular airplane. There is only one problem. When using a wind tunnel model for verification, a scaling factor must be used to find the values for the aircraft. In order to eliminate this requirement a set of non-dimensional equations must be derived. This can be illustrated best by an example:

EXAMPLE

Given the parametric equation for pitching moment,

$$\ddot{\theta} - \mathcal{M}_q \dot{\theta} - \mathcal{M}_u u - \mathcal{M}_{\dot{\alpha}} \dot{\alpha} - \mathcal{M}_\alpha \alpha = \mathcal{M}_{\delta_e} \delta_e$$

Derive an equation in which all terms are NON-DIMENSIONAL.

The steps in this process are

1. Take each stability parameter and substitute its coefficient relation, i.e.,

$$\mathcal{M}_q = \frac{1}{I_y} \frac{\partial \mathcal{M}}{\partial q} = \frac{1}{I_y} \frac{\partial (C_m \frac{1}{2} \rho U^2 Sc)}{\partial q} \tag{2.103}$$

C_m is the only variable that is dependent on q, therefore,

$$\mathcal{M}_q = \frac{\rho U_o^2 Sc}{2I_y} \left(\frac{\partial C_m}{\partial q}\right) \tag{2.104}$$

2. Non-dimensionalize the partial term, i.e.,

$$\frac{\partial C_m}{\partial q} \text{ has dimensions } = \frac{\text{dimensionless}}{\text{rad/sec}} = \text{sec}$$

To non-dimensionlize the partial terms, there exist certain compensating factors that will be shown later. In this case the compensating factor is

$$\frac{c}{2U_o} \quad \frac{[\text{ft}]}{[\text{ft/sec}]} = \text{sec}$$

Multiply and divide equation 2.104 by the compensating factor and get

$$\mathcal{M}_q = \frac{\rho U_o^2 \; Sc}{2I_y} \cdot \frac{\frac{c}{2U_o}}{\frac{c}{2U_o}} \cdot \frac{\partial C_m}{\partial q}$$ → This term is now dimensionless

check

$$\frac{\partial C_m}{\partial (\frac{cq}{2U_o})} = \frac{\text{dimensionless}}{\frac{\text{ft/sec}}{\text{ft/sec}}} = \text{dimensionless}$$

This is called a <u>stability</u> <u>derivative</u> and is written

$$C_{m_q} = \frac{\partial C_m}{\partial\left(\frac{cq}{2U_o}\right)} \text{ - by definition}$$

3. When the entire term as originally derived is considered, i.e.,

$$\mathcal{M}_q q = \frac{\rho U_o^2 Sc}{2I_y} \cdot \frac{c}{2U_o} \cdot C_{m_q} \cdot q$$

It can be rearranged so that

$$\mathcal{M}_q q = \frac{\rho U_o^2 Sc}{2I_y} \cdot C_{m_q} \cdot \frac{cq}{2U_o}$$

Define

$$\hat{q} = \frac{cq}{2U_o} \quad \frac{[\text{ft/sec}]}{[\text{ft/sec}]} \text{ dimensionless}$$

∴ The term becomes

$$\mathcal{M}_q q = \frac{\rho U_o^2 Sc}{2I_y} C_{m_q} \hat{q}$$

- $\hat{q}$ → Dimensionless variable
- C_{m_q} → Dimensionless stability derivative
- $\frac{\rho U_o^2 Sc}{2I_y}$ → Constants

But q is expressed as $\dot{\theta}$ in the equation. To convert this substitute $\frac{d\theta}{dt}$ for q.

$$\hat{q} = \frac{cq}{2U_o} = \frac{c}{2U_o} \frac{d(\theta)}{dt}$$

Then,

$$\text{Define } \nabla = \frac{c}{2U_o} \frac{d(\ )}{dt}$$

∇ can be considered to be a dimensionless derivative with respect to time and acts like an operator.

therefore

$$\hat{q} = \nabla\theta = \frac{c}{2U_o}\frac{d\theta}{dt} \text{ (i.e., dimensionless derivative of } \theta\text{)}$$

4. Do the same for each term in the parametric equation.

$$\mathcal{M}_u = \frac{1}{I_y}\frac{\partial \mathcal{M}}{\partial u} = \frac{1}{I_y}\frac{\partial\left(C_m \frac{1}{2}\rho U^2 Sc\right)}{\partial u}$$

Since both C_m and U are functions of u, then

$$\mathcal{M}_u = \frac{\rho U_o^2 Sc}{2I_y}\left(\frac{\partial C_m}{\partial u} + \frac{2C_{m_o}}{U_o}\right)$$

but $C_{m_o} = 0$ since initial conditions are steady state. The compensating factor for this case is $\frac{1}{U_o}$

$$\therefore \mathcal{M}_u u = \frac{\rho U_o^2 Sc}{2I_y} \cdot \underbrace{\frac{\partial C_m}{\partial \frac{u}{U_o}}}_{C_{m_u}} \cdot \underbrace{\frac{u}{U_o}}_{\hat{u}}$$

5. Once all of the terms have been derived, they are substituted into the original equation, and multiplied through by

$$\frac{2I_y}{\rho U_o^2 Sc}$$

which gives

$$\frac{2I_y}{\rho U_o^2 Sc}\ddot{\theta} - C_{m_q}\nabla\theta - C_{m_u}\hat{u} - C_{m_{\dot{\alpha}}}\nabla\alpha - C_{m_\alpha}\alpha = C_{m_{\delta_e}}\delta_e$$

The first term is non-dimensional, however, it can be changed to a more convenient form. Multiply and divide by

$$c^2/4U_o^{\,2}$$

$$\frac{2I_y}{\rho U_o^{\,2}\, Sc}\;\frac{4U_o^{\,2}}{c^2} \qquad\qquad \frac{c^2}{4U_o^{\,2}}\cdot\ddot{\theta}$$

Define $i_y = \dfrac{8I_y}{\rho S\, c^3}$

$$\frac{c^2}{4U_o^{\,2}}\;\ddot{\theta} = \frac{c}{2U_o}\,\frac{d\left(\dfrac{c}{2U_o}\,\dfrac{d(\theta)}{dt}\right)}{dt}$$

$$= \nabla^2\,\theta$$

Therefore, the term becomes,

$$i_y\,\nabla^2\,\theta$$

and the final equation is,

$$(i_y\,\nabla^2 - C_{m_q}\nabla)\,\theta - C_{m_u}\hat{u} - (C_{m_{\dot{\alpha}}}\nabla - C_{m_\alpha})\,\alpha = C_{m_{\delta_e}}\,\delta_e$$

The compensating factors for all of the variables are listed below.

Angular Rates	Compensating Factor	Non-Dimensional Angular Rates
p = rad/sec	$\dfrac{b}{2U_o}$	$\hat{p} = \dfrac{pb}{2U_o} = \nabla\,\phi$
q = rad/sec	$\dfrac{c}{2U_o}$	$\hat{q} = \dfrac{qc}{2U_o} = \nabla\,\theta$
r = rad/sec	$\dfrac{b}{2U_o}$	$\hat{r} = \dfrac{rb}{2U_o} = \nabla\,\psi$

Angular Rates	Compensating Factor	Non-Dimensional Angular Rates
$\dot{\beta}$ = rad/sec	$\frac{b}{2U_o}$	$\hat{\dot{\beta}} = \frac{\dot{\beta} b}{2U_o} = \nabla \beta$
$\dot{\alpha}$ = rad/sec	$\frac{c}{2U_o}$	$\hat{\dot{\alpha}} = \frac{\dot{\alpha} c}{2U_o} = \nabla \alpha$
u = ft/sec	$\frac{1}{U_o}$	$\hat{u} = \frac{u}{U_o}$

α - no change

β - no change

These derivations have been presented to give an understanding of their origin and what they represent. It is not necessary to be able to derive each and every one of the equations. It is important, however, to understand several facts about the non-dimensional equations.

1. Since these equations are non-dimensional, they can be used to describe any aircraft that are geometrically similar.

2. Stability derivatives can be thought of as if they were stability parameters. Therefore, C_{m_α} refers to the same aerodynamic characteristics as M_α, only it is in a non-dimensional form.

3. Most aircraft designers and builders are accustomed to speaking in terms of stability derivatives. Therefore, it is a good idea to develop a "feel" for all of the important ones.

4. These equations as well as the parametric equations describe the complete motion of an aircraft. They can be programmed directly into a computer and connected to a flight simulator. They may also be used in cursory design analyses. Due to their simplicity, they are especially useful as an analytical tool to investigate aircraft handling qualities and determine the effects of changes in aircraft design.

Basic Force Relations

To aid in developing the stability derivatives from the basic force relations, the following table is provided.

LONGITUDINAL MOTION			
Equation	Coefficient	Normalizing Factor	Parameters
Drag	$D = \frac{1}{2}\rho U^2 S C_D$	$\frac{1}{m}$	D_u (D_α requires special derivation) (D_{δ_e}, $D_{\dot\alpha}$ and D_q are insignificant)
Lift	$L = \frac{1}{2}\rho U^2 S C_L$	$\frac{1}{mU_o}$	L_u $L_{\dot\alpha}$ L_q L_{δ_e} L_α
Pitch	$\mathcal{M} = \frac{1}{2}\rho U^2 S c C_m$	$\frac{1}{I_y}$	$\mathcal{M}_u$ $\mathcal{M}_\alpha$ $\mathcal{M}_{\dot\alpha}$ $\mathcal{M}_q$ $\mathcal{M}_{\delta_e}$

	Linear Velocity	Angles	Angular Rates
Independent Variables	u	α δ_e	$\dot\alpha$ q
Compensating Factors	$\frac{1}{U_o}$	None	$\frac{c}{2U_o}$

LATERAL-DIRECTIONAL MOTION			
Equation	Coefficient	Normalizing Factor	Parameters
Side	$Y = \frac{1}{2}\rho U^2 S C_Y$	$\frac{1}{mU_o}$	Y_β Y_p Y_r Y_{δ_r} ($Y_{\dot\beta}$ and Y_{δ_a} are insignificant)
Roll	$\mathcal{L} = \frac{1}{2}\rho U^2 S b C_\ell$	$\frac{1}{I_x}$	$\mathcal{L}_\beta$ $\mathcal{L}_p$ $\mathcal{L}_r$ $\mathcal{L}_{\delta_a}$ $\mathcal{L}_{\delta_r}$ ($\mathcal{L}_{\dot\beta}$ is insignificant)
Yaw	$\mathcal{N} = \frac{1}{2}\rho U^2 S b C_n$	$\frac{1}{I_z}$	$\mathcal{N}_\beta$ $\mathcal{N}_p$ $\mathcal{N}_r$ $\mathcal{N}_{\delta_a}$ $\mathcal{N}_{\delta_r}$ ($\mathcal{N}_{\dot\beta}$ is insignificant)

	Angles	Angular Rates
Independent Variables	β δ_a δ_r	$\dot\beta$ p r
Compensating Factors	None	$\frac{b}{2U_o}$

Non-Dimensional Derivative Equations

A list of the non-dimensional derivative equations is presented below.

NON-DIMENSIONAL LONGITUDINAL EQUATIONS

	(θ)	$(\hat{u})$	(α)	
Drag	$C_{L_o}\theta$	$+\ (2\mu\nabla + C_{D_u} + 2C_{D_0})\hat{u}$	$+\ C_{D_\alpha}\alpha$	$= 0$
Lift	$(2\mu\nabla - C_{L_q}\nabla)\theta$	$-\ (C_{L_u} + 2C_{L_o})\hat{u}$	$-\ (2\mu\nabla + C_{L_{\dot{\alpha}}}\nabla + C_{L_\alpha})\alpha$	$= C_{L_{\delta_e}}\delta_e$
Pitching Moment	$(i_y\nabla^2 - C_{m_q}\nabla)\theta$	$-\ C_{m_u}\hat{u}$	$-\ (C_{m_{\dot{\alpha}}}\nabla + C_{m_\alpha})\alpha$	$= C_{m_{\delta_e}}\delta_e$

$$i_y = \frac{8I_y}{\rho S c^3} \qquad \mu = \frac{2m}{\rho Sc}$$

$$\nabla = \frac{c}{2U_o}\frac{d(\ )}{dt}$$

NON-DIMENSIONAL LATERAL-DIRECTIONAL EQUATIONS

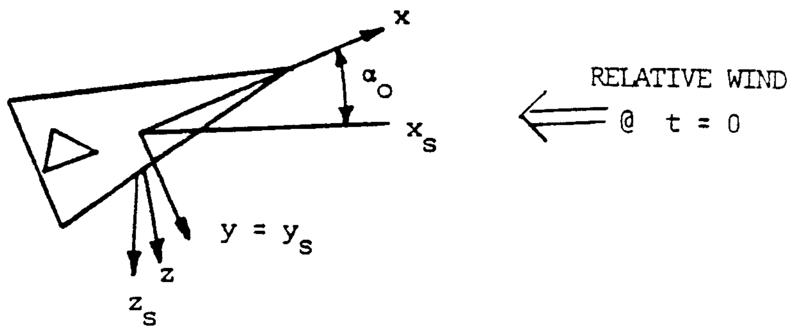
	(β)	(ϕ)	$(\hat{r})$	
Side Force	$(2\mu\nabla - C_{Y_\beta})\beta$	$-\ (C_{Y_p}\nabla + C_{L_o})\phi$	$+\ (2\mu - C_{Y_r})\hat{r}$	$= C_{Y_{\delta_r}}\delta_r$
Rolling Moment	$-\ C_{\ell_\beta}\beta$	$+\ (i_x\nabla^2 - C_{\ell_p}\nabla)\phi$	$-\ (i_{xz}\nabla + C_{\ell_r})\hat{r}$	$= C_{\ell_{\delta_a}}\delta_a + C_{\ell_{\delta_r}}\delta_r$
Yawing Moment	$-\ C_{n_\beta}\beta$	$-\ (i_{xz}\nabla^2 + C_{n_p}\nabla)\phi$	$+\ (i_z\nabla - C_{n_r})\hat{r}$	$= C_{n_{\delta_a}}\delta_a + C_{n_{\delta_r}}\delta_r$

$$i_x = \frac{8I_x}{\rho Sb^3} \qquad \nabla = \frac{b}{2U_o}\frac{d(\ )}{dt}$$

$$i_{xz} = \frac{8I_{xz}}{\rho Sb^3} \qquad \mu = \frac{2m}{\rho Sb}$$

$$i_z = \frac{8I_z}{\rho Sb^3}$$

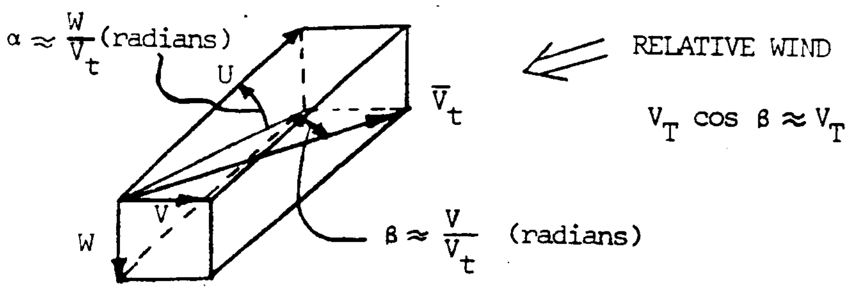
3. STABILITY AXIS



A. Fixed to aircraft at t = 0.

B. Assume $\beta = 0$ (no sideslip) at t = 0.

4. DEFINITION OF β (Sideslip)

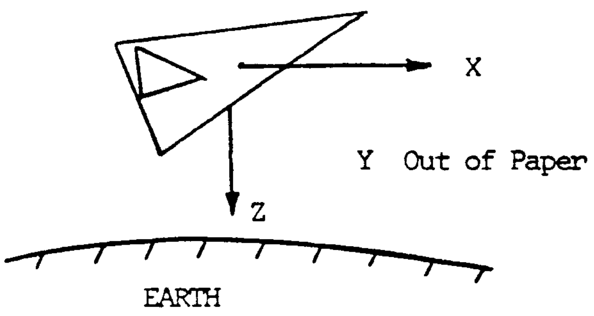


A. Note positive β is wind in the right ear.

EQUATIONS OF MOTION

(SUPPLEMENT)

1. MOVING EARTH AXIS

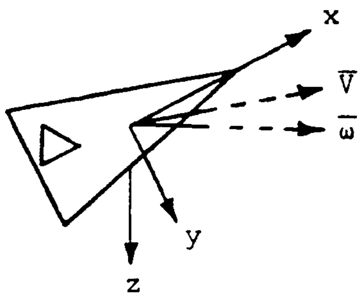


A. X and Y axis always remain horizontal.

B. Z axis always points to center of earth.

C. Assumed as an inertial reference.

2. BODY AXIS



$$\bar{\omega} = P\bar{i} + Q\bar{j} + R\bar{k}$$
$$\bar{V} = U\bar{i} + V\bar{j} + W\bar{k}$$

A. Fixed to aircraft, origin at CG.

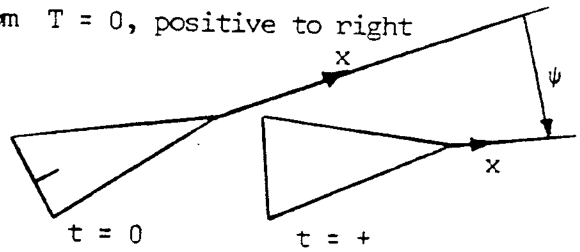
B. x out nose, y out RT wing, z out bottom.

C. Aircraft and body axis rotate at $\bar{\omega}$ with respect to moving earth axis.

5. EULER ANGLES

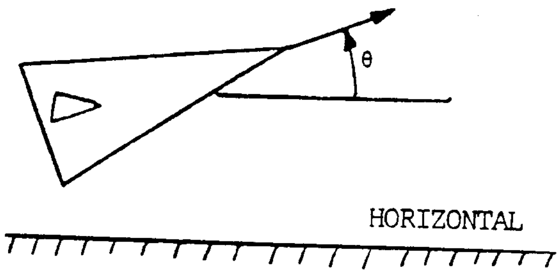
A. ψ change in heading from T = 0, positive to right

TOP VIEW

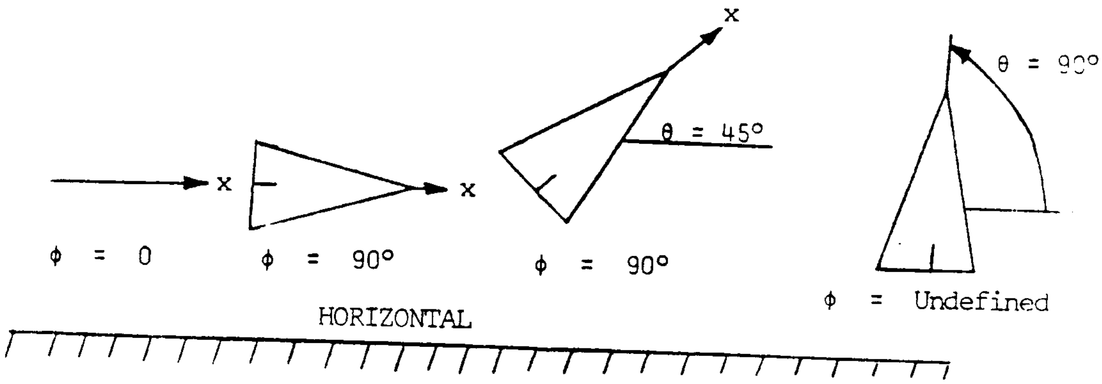


B. θ angle between x axis and horizontal, positive nose up.

SIDE VIEW



C. ϕ Bank Angle: For a given ψ and θ, bank angle is a measure of the rotation about the x axis to put the aircraft in the desired position from a wings horizontal condition. Positive right wing down.



6. Relation between P, Q, R and ψ, θ, ϕ

$$P = \dot{\phi} - \dot{\psi} \sin \theta$$

$$Q = \dot{\psi} \cos \theta \sin \phi + \dot{\theta} \cos \phi$$

$$R = \dot{\psi} \cos \theta \cos \phi - \dot{\theta} \sin \phi$$

7. MAJOR ASSUMPTIONS

A. Rigid Body.

B. Earth and atmosphere assumed fixed.

C. Constant mass.

D. xz - plane is a plane of symmetry.

8. IN THE BEGINNING.

$$\overline{F} = m\overline{a} \rightarrow m \left[\frac{d\overline{V}}{dt} + \overline{\omega} \times \overline{V}\right]$$

$$\overline{G} = \frac{d\overline{H}}{dt} \rightarrow \left[\frac{d\overline{H}}{dt} + \overline{\omega} \times \overline{H}\right]$$

$$\overline{H} = \Sigma m_1 [\overline{R}_1 \times \overline{\omega} \times \overline{R}_1]$$

9. RIGHT-HAND SIDE (RHS) OF LINEAR FORCE EQ.

$$\bar{F} = m \frac{d\bar{V}}{dt} + \bar{\omega} \times \bar{V}$$

	$\bar{i}$	$\bar{j}$	$\bar{k}$
$\bar{F} =$	F_x	F_y	F_z
$\frac{d\bar{V}}{dt} =$	$\dot{U}$	$\dot{V}$	$\dot{W}$
$\bar{\omega} =$	P	Q	R
$\bar{V} =$	U	V	W
$\bar{\omega} \times \bar{V} =$	$QW - RV$	$RU - PW$	$PV - QU$

$$F_x = m [\dot{U} + QW - RV]$$

$$F_y = m [\dot{V} + RU - PW]$$

$$F_z = m [\dot{W} + PV - QU]$$

10. WE NEED $\bar{H}$

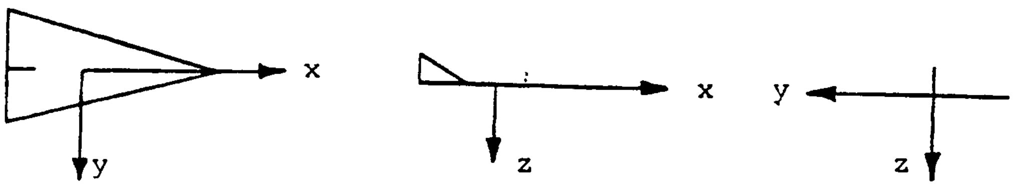
$$\bar{H} = m_1 (\bar{R}_1 \times \bar{\omega} \bar{R}_1)$$

GIVES

$$H_x = P I_x - Q I_{xy} - R I_{xz}$$

$$H_y = Q I_y - R I_{yz} - P I_{xy}$$

$$H_z = R I_z - P I_{xz} - Q I_{yz}$$



I_z LARGEST

I_y SMALLER

I_x SMALLEST

$I_{xy} = 0$

$I_{xz} \neq 0$

$I_{yz} = 0$

11. RHS OF MOMENT EQ.

$$\bar{G} = \frac{d\bar{H}}{dt} + \omega \times H$$

	$\bar{i}$	$\bar{j}$	$\bar{k}$
$\bar{G} =$	G_x	G_y	G_z
$\dot{\bar{H}} =$	$\dot{P} I_x - \dot{R} I_{xz}$	$\dot{Q} I_y$	$\dot{R} I_z - \dot{P} I_{xz}$
$\bar{\omega} =$	P	Q	R

$$G_x = \dot{P} I_x + QR (I_z - I_y) - (\dot{R} + PQ) I_{xz}$$

$$G_y = \dot{Q} I_y - PR (I_z - I_x) + (P^2 - R^2) I_{xz}$$

$$G_z = \dot{R} I_z + PQ (I_y - I_x) + (QR - \dot{P}) I_{xz}$$

12. CONVENIENT GROUPINGS

A. LONGITUDINAL SET

"DRAG" $F_x = m[\dot{U} + QW - RV]$

"LIFT" $F_z = m[\dot{W} + PV - QU]$

"PITCH" $G_y = \dot{Q} I_y - PR(I_z - I_x) + (P^2 - R^2) I_{xz}$

B. LATERAL-DIRECTIONAL SET

"SIDEFORCE" $F_y = m[\dot{V} + RU - PW]$

"ROLL" $G_x = \dot{P} I_x + QR(I_z - I_y) - (\dot{R} + PQ) I_{xz}$

"YAW" $G_z = \dot{R} I_z + PQ(I_y - I_x) + (QR - \dot{P}) I_{xz}$

13. DEFINITIONS: LEFT-HAND SIDE (LHS)

			AERO		THRUST		GRAVITY
"DRAG"	F_x	=	$-D$	+	T_x	−	D_g
"LIFT"	F_z	=	$-L$	+	T_z	+	L_g
"PITCH"	G_y	=	$\mathcal{M}$	+	M_T		
			$U, \alpha, \dot{\alpha}, Q, \delta_e$		U, δ_{RPM}		θ
"SIDEFORCE"	F_y	=	Y	+	Y_T	+	Y_g
"ROLL"	G_x	=	$\mathcal{L}$	+	L_T		
"YAW"	G_z	=	$\mathcal{N}$	+	N_T		
			$\beta, P, R, \delta_a, \delta_r$		δ_{RPM}		ϕ

14. SMALL PERTURBATION THEORY

A. Let $U = U_o + \Delta U = U_o + u$

$\alpha = \alpha_o + \Delta\alpha = \alpha_o + \alpha$

$Q = Q_o + \Delta Q = Q_o + q$

etc.

B. THEN BY TAYLOR SERIES APPROXIMATION

$$D = D_o + \frac{\partial D}{\partial u} u + \frac{\partial D}{\partial \alpha} \alpha + \frac{\partial D}{\partial \dot{\alpha}} \dot{\alpha} + \frac{\partial D}{\partial q} q + \frac{\partial D}{\partial \delta_e} \delta_e$$

$$T_x = T_{x_o} + \frac{\partial T_x}{\partial u} u + \frac{\partial T_x}{\partial \delta_{RPM}} \delta_{RPM}$$

etc.

15. THE COMPLETE DRAG EQUATION

$$-\left[D_o + \frac{\partial D}{\partial u} u + \frac{\partial D}{\partial \alpha} \alpha + \frac{\partial D}{\partial \dot{\alpha}} \dot{\alpha} + \frac{\partial D}{\partial q} q + \frac{\partial D}{\partial \delta_e} \delta_e\right]$$

$$+ T_{x_o} + \frac{\partial T_x}{\partial u} u + \frac{\partial T_x}{\partial \delta_{RPM}} \delta_{RPM} - D_{g_o} - \frac{\partial D_g}{\partial \theta} \theta$$

$$= m [\dot{U} + QW - RV]$$

16. DRAG EQUATION IN STABILITY PARAMETER NOTATION

A. FIRST DIVIDE BY m "FORCE/UNIT MASS"

B. DEFINE $\frac{1}{m}\frac{\partial D}{\partial u} u = D_u u$

$$-\frac{D_o}{m} + \frac{T_{x_o}}{m} - \frac{D_{g_o}}{m} - \left[D_u u + D_\alpha \alpha + D_{\dot{\alpha}} \dot{\alpha} + D_q q + D_{\delta_e} \delta_e \right]$$

$$+ T_{x_u} u + T_{x_{\delta_{RPM}}} \delta_{RPM} - D_{g_\theta} \theta = \dot{U} + QW - RV$$

C. FOR STABILITY AND CONTROL PURPOSES

$$\dot{u} + D_\alpha \alpha + D_u u + D_\theta \theta = 0$$

17. LONGITUDINAL EQ. (STABILITY PARAMETERS)

A. ASSUMPTIONS APPROPRIATE FOR SIMPLIFICATION YIELD:

"DRAG" $\dot{u} + D_u u + D_\alpha \alpha + D_\theta \theta = 0$

"LIFT" $-L_u u - (1 + L_{\dot{\alpha}})\dot{\alpha} - L_\alpha \alpha + (1 - L_q)\dot{\theta} = L_{\delta_e} \delta_e$

"PITCH" $-\mathcal{M}_u u - \mathcal{M}_{\dot{\alpha}} \dot{\alpha} - \mathcal{M}_\alpha \alpha + \ddot{\theta} - \mathcal{M}_q \dot{\theta} = \mathcal{M}_{\delta_e} \delta_e$

18. LATERAL-DIRECTIONAL EQ (STABILITY PARAMETERS)

A. ASSUMPTIONS APPROPRIATE FOR SIMPLIFICATION YIELD:

"SIDE FORCE" $\dot{\beta} - Y_\beta \beta - Y_p \dot{\phi} - Y_\phi \phi + (1 - Y_r) r = Y_{\delta_r} \delta_r$

"ROLL" $-\mathcal{L}_\beta \beta + \ddot{\phi} - \mathcal{L}_p \dot{\phi} - \frac{I_{xz}}{I_x} \dot{r} - \mathcal{L}_r r = \mathcal{L}_{\delta_a} \delta_a + \mathcal{L}_{\delta_r} \delta_r$

"YAW" $-\mathcal{N}_\beta \beta - \frac{I_{xz}}{I_z} \ddot{\phi} - \mathcal{N}_p \dot{\phi} + \dot{r} - \mathcal{N}_r r = \mathcal{N}_{\delta_a} \delta_a + \mathcal{N}_{\delta_r} \delta_r$

19. LONGITUDINAL EQ (NON-DIMENSIONAL)

A. STABILITY PARAMETERS REWRITTEN IN TERMS OF STABILITY DERIVATIVES

"DRAG" $(2\mu\nabla + C_{D_u} + 2C_{D_o}) \hat{u} + C_{D_\alpha} \alpha + C_{L_o} \theta = 0$

"LIFT" $-(C_{L_u} + 2C_{L_o}) \hat{u} - (2\mu\nabla + C_{L_{\dot{\alpha}}} \nabla + C_{L_\alpha}) \alpha + (2\mu\nabla - C_{L_q} \nabla) \theta = C_{L_{\delta_e}} \delta_e$

"PITCH" $-C_{m_u} \hat{u} - (C_{m_{\dot{\alpha}}} \nabla + C_{m_\alpha}) \alpha + (i_y \nabla^2 - C_{m_q} \nabla) \theta = C_{m_{\delta_e}} \delta_e$

Where

$i_y = \frac{8 I_y}{\rho S c^3}$ $\mu = \frac{2m}{\rho S c}$

$\nabla = \frac{c}{2U_o} \frac{d(\;)}{dt}$

20. LATERAL-DIRECTIONAL EQ. (NON-DIMENSIONAL)

"SIDE FORCE" $(2\mu\nabla - C_{y_\beta})\beta - (C_{Y_p}\nabla + C_{L_o})\phi + (2\mu - C_{Y_r})\hat{r} = C_{Y_{\delta_r}}\delta_r$

"ROLL" $-C_{\ell_\beta}\beta + (i_x\nabla^2 - C_{\ell_p}\nabla)\phi - (i_{xz}\nabla + C_{\ell_r})\hat{r} = C_{\ell_{\delta_a}}\delta_a + C_{\ell_{\delta_r}}\delta_r$

"YAW" $-C_{n_\beta}\beta - (i_{xz}\nabla^2 + C_{n_p}\nabla)\phi + (i_z\nabla - C_{n_r})\hat{r} = C_{n_{\delta_a}}\delta_a + C_{n_{\delta_r}}\delta_r$

Where

$$i_x = \frac{8I_x}{\rho S b^3} \qquad \mu = \frac{b}{2U_o}\frac{d(\ )}{dt}$$

$$i_{xz} = \frac{8I_{xz}}{\rho S b^3} \qquad = \frac{2m}{\rho S b}$$

$$i_z = \frac{8I_z}{\rho S b^3}$$

LONGITUDINAL STATIC STABILITY

3.1 DEFINITION OF LONGITUDINAL STATIC STABILITY

Static stability is the reaction of a body to a disturbance from equilibrium. To determine the static stability of a body, the body must be initially disturbed from its equilibrium state. If when disturbed from equilibrium, the body returns to its original equilibrium position, the body displays positive static stability or is stable. If the body remains in the disturbed position, the body is said to be neutrally stable. However, should the body, when disturbed, continue to displace from equilibrium, the body has negative static stability or is unstable.

Longitudinal static stability or "gust stability" of an aircraft is determined similarly. If an aircraft in equilibrium is momentarily disturbed by a vertical gust, the resulting change in angle of attack causes changes in lift coefficients on the aircraft. (Velocity is constant for this time period.) The changes in lift coefficients produce additional aerodynamic forces and moments in this disturbed position. If the aerodynamic forces and moments created tend to return the aircraft to its original undisturbed condition, the aircraft possessses positive static stability or is stable. Should the aircraft remain in the disturbed position, it possesses neutral stability. If the forces and moments cause the aircraft to diverge further from equilibrium, the aircraft possesses negative longitudinal static stability or is unstable.

3.2 ANALYSIS OF LONGITUDINAL STATIC STABILITY

Longitudinal static stability is only a special case for the total equations of motion of an aircraft. Of the six equations of motion, longitudinal static stability is concerned with only one, the pitch equation, that equation describing the aircraft's motion about the y - axis.

$$G_y = \dot{q}I_y - pr(I_z - I_x) + (p^2 - r^2)I_{xz} \quad (3.1)$$

The fact that theory pertains to an aircraft in straight, steady, symmetrical flight with no unbalance of forces or moments permits longitudinal static stability motion to be independent of the lateral and directional equations of motion. This is not an oversimplification since most aircraft spend much of the flight under symmetric equilibrium conditions. Furthermore the disturbance required for stability determination and the measure of the aircraft's response takes place about the y - axis or in the longitudinal plane.

Since longitudinal static stability is concerned with resultant aircraft pitching moments caused by momentary changes in angle of attack and lift coefficients, the primary stability derivatives become C_{m_α} and $C_{m_{C_L}}$. The value of either derivative is a direct indication of the longitudinal static stability of the particular aircraft.

To determine an expression for the derivative, $C_{m_{C_L}}$, an aircraft in stabilized, equilibrium flight with horizontal stabilizer control surface fixed will be analyzed. A moment equation will be determined from the forces and moments acting on the aircraft. Once this equation is nondimensionalized, in moment coefficient form, the derivative with respect to C_L will be taken. This differential equation will be an expression for Cm_{C_L} and will relate directly to the aircraft's stability. Individual term contribution to stability will in turn be analyzed. A flight test relationship for determining the stability of an aircraft will be developed followed by analysis of the aircraft with a free control surface.

3.3 THE STABILITY EQUATION

To derive the longitudinal pitching moment equation, refer to the aircraft in figure 3.1. Writing the moment equation using the sign convention of pitchup being a positive moment and assuming a small angle of attack α so that cos α ≈ 1 and sin α ≈ α;

If an order of magnitude check is made, some of the terms can be logically eliminated **because** of their relative size. C_T can be omitted since

$$C_T = \frac{C_w}{10} = \frac{N_w}{100}$$

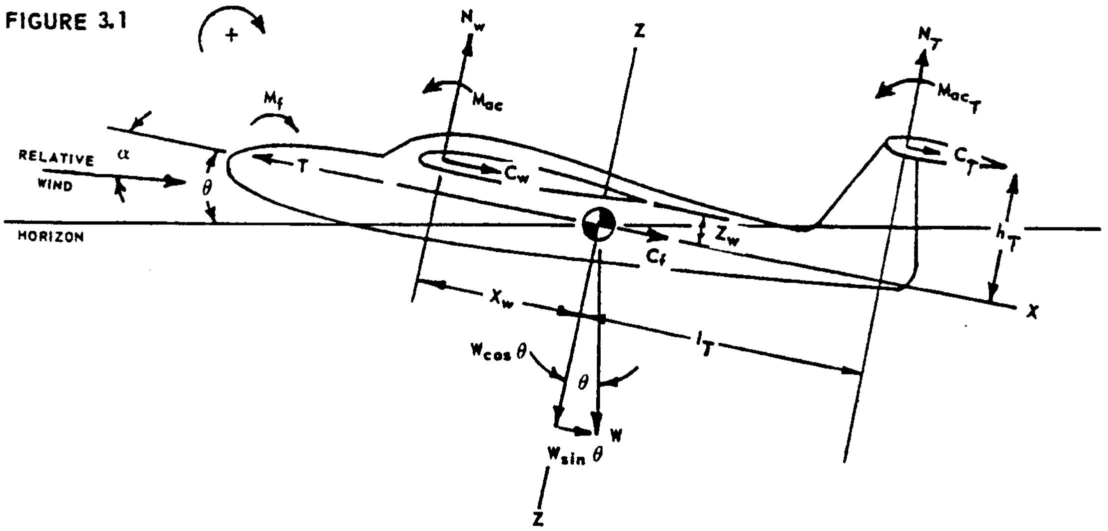
M_{ac_T} is zero for a symmetrical airfoil horizontal stabilizer section. Rewriting the simplified equation:

$$\overset{+}{\curvearrowright} \quad M_{CG} = N_w X_w + C_w Z_w - M_{ac} + M_f - N_T \ell_T \quad (3.3)$$

It is convenient to express equation 3.3 in nondimensional coefficient form by dividing both sides of the equation by $q_w S_w c_w$

$$\frac{M_{CG}}{q_w S_w c_w} = \frac{N_w X_w}{q_w S_w c_w} + \frac{C_w Z_w}{q_w S_w c_w} - \frac{M_{ac}}{q_w S_w c_w} + \frac{M_f}{q_w S_w c_w} - \frac{N_T \ell_T}{q_w S_w c_w} \quad (3.4)$$

FIGURE 3.1



$$\overset{+}{\curvearrowright} \quad M_{CG} = N_w X_w + C_w Z_w - M_{ac} + M_f - N_T \ell_T + C_T h_T - M_{ac_T} \quad (3.2)$$

Substituting the following coefficients in equation 3.4:

$C_{m_{CG}} = \dfrac{M_{CG}}{q_w S_w c_w}$	total pitching moment coefficient
$C_{m_{ac}} = \dfrac{M_{ac}}{q_w S_w c_w}$	wing aerodynamic pitching moment coefficient
$C_{m_f} = \dfrac{M_f}{q_w S_w c_w}$	fuselage aerodynamic pitching moment coefficient
$C_N = \dfrac{N_w}{q_w S_w}$	wing aerodynamic normal force coefficient
$C_{N_T} = \dfrac{N_T}{q_T S_T}$	tail aerodynamic normal force coefficient
$C_C = \dfrac{C_w}{q_w S_w}$	wing aerodynamic chordwise force coefficient

Equation 3.4 may now be written:

$$C_{m_{CG}} = C_N \frac{X_w}{c} + C_C \frac{Z_w}{c} - C_{m_{ac}} + C_{m_f} - \frac{N_T \ell_T}{q_w S_w c_w} \qquad (3.5)$$

where c and c_w are used interchangeably to represent the mean aerodynamic chord of the wing. To have the tail term in terms of a coefficient, multiply and divide the term by $q_T S_T$

$$\frac{N_T \ell_T}{q_w S_w c_w} \cdot \frac{q_T S_T}{q_T S_T}$$

Substituting tail efficiency factor $\eta_T = q_T/q_w$ and designating tail volume coefficient $V_H = \ell_T S_T / c_w S_w$, Equation (3.5) becomes:

$$C_{m_{CG}} = C_N \frac{X_w}{c} + C_C \frac{Z_w}{c} - C_{m_{ac}} + C_{m_f} - C_{N_T} V_H \eta_T \qquad (3.6)$$

Equation 3.6 is referred to as the equilibrium equation in pitch. If the magnitudes of the individual terms in the above equation are adjusted to the proper value, the aircraft may be placed in equilibrium flight where $C_{m_{CG}} = 0$.

Taking the derivative of equation 3.6 with respect to C_L and assuming that X_w, Z_w, V_H, and η_T do not vary with C_L,

$$\frac{dC_{m_{CG}}}{dC_L} = \underbrace{\frac{dC_N}{dC_L}\frac{X_w}{c} + \frac{dC_C}{dC_L}\frac{Z_w}{c} - \frac{dC_{m_{ac}}}{dC_L}}_{\text{WING}} + \underbrace{\frac{dC_{m_f}}{dC_L}}_{\text{FUSELAGE}} - \underbrace{\frac{dC_{N_T}}{dC_L} V_H \eta_T}_{\text{TAIL}} \qquad (3.7)$$

Equation 3.7 is the stability equation and is related to the stability derivative C_{m_α} by the slope of the lift curve, a.

$$\frac{dC_m}{d\alpha} = a \frac{dC_m}{dC_L} \qquad (3.8)$$

Equation 3.6 and equation 3.7 determine the two criteria necessary for longitudinal stability:

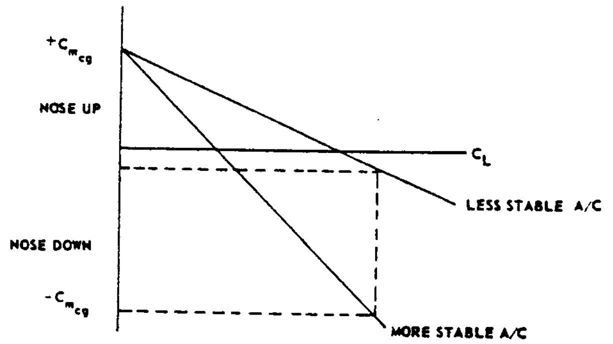
1. The aircraft is balanced.

2. The aircraft is stable.

The first condition is satisfied if the pitching moment equation may be forced to $C_{m_{CG}} = 0$ for all useful positive values of C_L. This condition is achieved by trimming the aircraft so that moments about the center of gravity are zero (i.e., $M_{CG} = 0$).

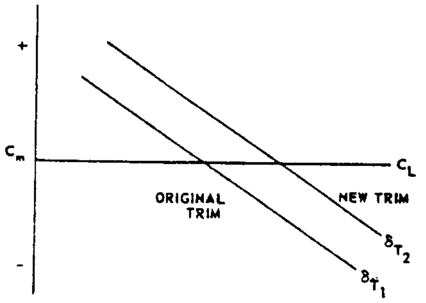
The second condition is satisfied if equation 3.7 or $dC_{m_{CG}}/dC_L$ has a negative value. From figure 3.2 a negative value for equation 3.7 is necessary if the aircraft is to be stable. Should a gust cause an angle of attack increase and a corresponding increase in C_L, a negative $C_{m_{CG}}$ should be produced to return the aircraft to equilibrium, or $C_{M_{CG}} = 0$. The greater the slope or the negative value, the more restoring moment is generated for an increase in C_L. The slope or dC_m/dC_L is a direct measure of the "gust stability" of the aircraft.

FIGURE 3.2



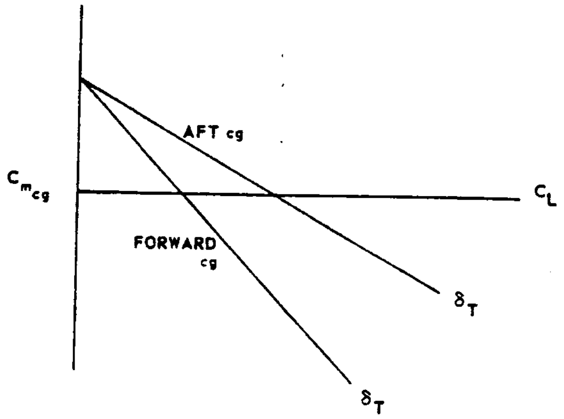
If the aircraft is retrimmed from one angle of attack to another, the basic stability of the aircraft or slope dC_m/dC_L does not change. Note figure 3.3.

FIGURE 3.3



However, if the cg is changed or values of X_W, Z_W, and V_H are changed, the slope or stability of the aircraft is changed. See equation 3.7. For no change in trim tab setting, the stability curve may shift as in figure 3.4.

FIGURE 3.4

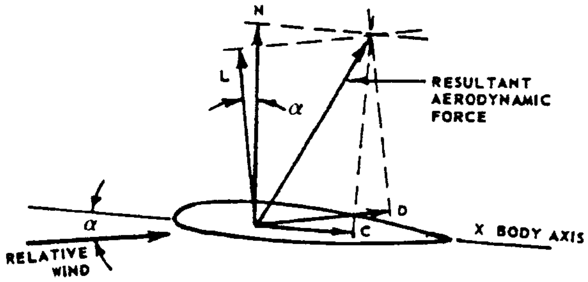


3.4 EXAMINATION OF THE WING, FUSELAGE, AND TAIL CONTRIBUTION TO THE STABILITY EQUATION

The Wing Contribution to Stability:

The lift and drag are by definition always perpendicular and parallel to the relative wind. It is therefore inconvenient to use these forces to obtain moments, for their arms to the center of gravity vary with angle of attack. For this reason, all forces are resolved into normal and chordwise forces whose axes remain fixed with the aircraft and whose arms are therefore constant:

FIGURE 3.5



Assuming the wing lift to be the airplane lift and the angle of attack of the wing to be the airplane's angle of attack, the following relationship exists between the normal and lift forces (figure 3.5):

$$N = L\cos\alpha + D\sin\alpha \qquad (3.9)$$

$$C = D\cos\alpha - L\sin\alpha \qquad (3.10)$$

Therefore, the coefficients are similarly related:

$$C_N = C_L\cos\alpha + C_D\sin\alpha \qquad (3.11)$$

$$C_C = C_D\cos\alpha - C_L\sin\alpha \qquad (3.12)$$

The stability contributions, dC_N/dC_L and dC_C/dC_L, are obtained:

$$\frac{dC_N}{dC_L} = \frac{dC_L}{dC_L}\cos\alpha - C_L\frac{d\alpha}{dC_L}\sin\alpha + \frac{dC_D}{dC_L}\sin\alpha + C_D\frac{d\alpha}{dC_L}\cos\alpha \qquad (3.13)$$

$$\frac{dC_C}{dC_L} = \frac{dC_D}{dC_L}\cos\alpha - C_D\frac{d\alpha}{dC_L}\sin\alpha - \frac{dC_L}{dC_L}\sin\alpha - C_L\frac{d\alpha}{dC_L}\cos\alpha \qquad (3.14)$$

Making an additional assumption that:

$$C_D = C_{D_{parasite}} + \frac{C_L^2}{\pi \mathit{AR}\, e}$$ and if

$C_{D_{parasite}}$ is constant with change in C_L:

Then $$\frac{dC_D}{dC_L} = \frac{2C_L}{\pi \mathit{AR}\, e}$$

If the angles of attack are small such that $\cos\alpha = 1.0$ and $\sin\alpha \approx \alpha$, equations 3.13 and 3.14 become:

$$\frac{dC_N}{dC_L} = 1 + C_L\alpha\left(\frac{2}{\pi \mathit{AR}\, e} - \frac{d\alpha}{dC_L}\right) + C_D\frac{d\alpha}{dC_L} \qquad (3.15)$$

$$\frac{dC_C}{dC_L} = \frac{2}{\pi \mathit{AR}\, e}C_L - C_D\frac{d\alpha}{dC_L}\alpha - \alpha - C_L\frac{d\alpha}{dC_L} \qquad (3.16)$$

Examining the above equations for relative magnitude,

C_D	is on the order of 0.03
C_L	usually ranges from .2 to 2.0
α	is small, $\approx$ .2 radians
$\frac{d\alpha}{dC_L}$	is nearly constant at .2 radians
$\frac{2}{\pi \mathit{AR}\, e}$	is on the order of .1

Making these substitutions, equations 3.15 and 3.16 become

$$\frac{dC_N}{dC_L} = 1 - .04 + .06$$

$$= 1.02 \approx 1.0 \qquad (3.17)$$

$$\frac{dC_C}{dC_L} = .1\, C_L - .012 - .2 - .2C_L$$

$$= -\,(.2 + .1\, C_L) \qquad (3.18)$$

By definition the coefficient of moment about the aerodynamic center is invariant with respect to angle of attack. Therefore,

$$\frac{dC_{m_{ac}}}{dC_L} = 0$$

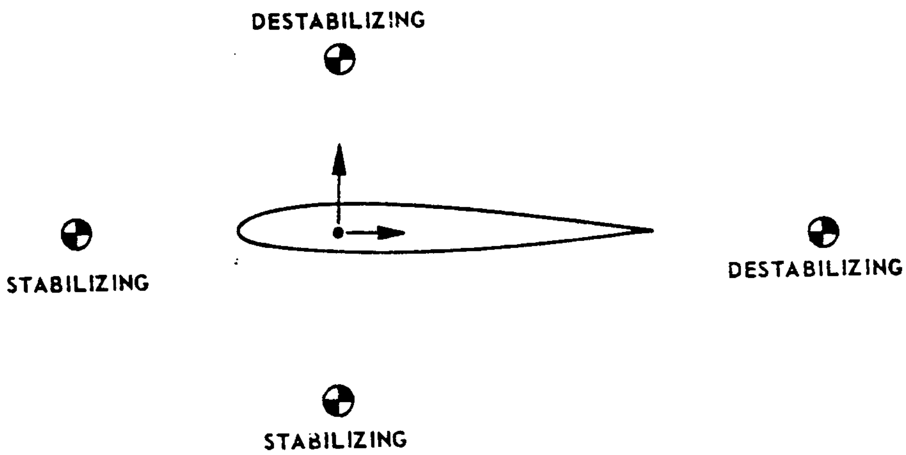
Rewriting the wing contribution of the stability equation 3.7,

$$\frac{dC_m}{dC_{L_{WING}}} = \frac{X_w}{c} - (.2 + .1\, C_L)\, \frac{Z_w}{c} \qquad (3.19)$$

From figure 3.5 when α increases, the normal force increases and the chordwise force decreases. Equation 3.19 shows the relative magnitude of these changes. The position of the cg above or below the aerodynamic center (a.c.) has a much smaller effect on stability than does the position of the cg ahead or behind the a.c. With the cg ahead of the a.c., the normal force is stabilizing. From equation 3.19, the more forward the cg location, the more stable the aircraft. With the cg below the a.c., the chordwise force is stabilizing since this force decreases as the angle of attack increases. The further the cg is located below the a.c., the more stable the aircraft or the more negative the value of dC_m/dC_L. The wing contribution to stability depends on the cg and a.c. relationship shown in figure 3.6.

FIGURE 3.6

WING CONTRIBUTION TO STABILITY



For a stable wing contribution to stability, the aircraft should be designed with a high wing aft of the center of gravity.

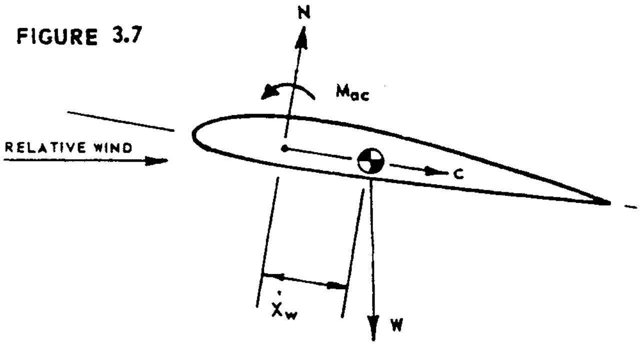
Fighter type aircraft and most low wing, large aircraft have cg's very close to the top of the mean aerodynamic chord. Z_w is on the order of .03. For these aircraft, the chordwise force contribution to stability can be neglected. The wing contribution then becomes:

$$\frac{dC_m}{dC_{L_{WING}}} = \frac{X_w}{c} \qquad (3.20)$$

The Flying Wing.

In order for a flying wing to be a usable aircraft, it must be balanced (fly in equilibrium at a useful positive C_L) and be stable. The problem may be analyzed as follows:

FIGURE 3.7



For the wing in figure 3.7, assuming that the chordwise force acts through the cg, the equilibrium equation in pitch may be written:

$$M_{CG} = NX_w - M_{ac} \qquad (3.21)$$

Writing the equation in coefficient form,

$$C_{m_{CG}} = C_N \frac{X_w}{c} - C_{m_{ac}} \qquad (3.22)$$

For controls fixed, the stability equation becomes,

$$\frac{dC_{m_{CG}}}{dC_L} = \frac{dC_N}{dC_L} \frac{X_w}{c} \qquad (3.23)$$

Equations 3.22 and 3.23 show that the wing in figure 3.7, is balanced and unstable. To make the wing stable, or dC_m/dC_L negative, the center of gravity must be ahead of the wing aerodynamic center. Making this cg change, however, now changes the signs in equation 3.21. The equilibrium and stability equations become:

$$C_{m_{CG}} = -C_N \frac{X_w}{c} - C_{m_{ac}} \qquad (3.24)$$

$$\frac{dC_{m_{CG}}}{dC_L} = -\frac{dC_N}{dC_L} \frac{X_w}{c} \qquad (3.25)$$

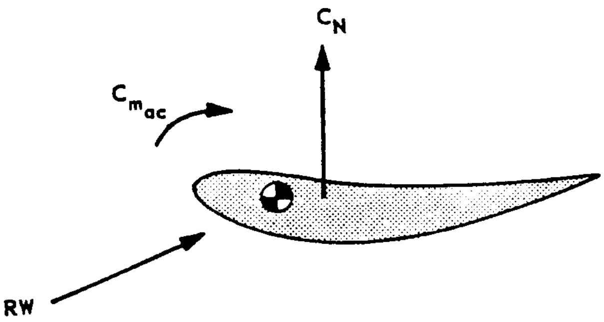
The wing is now stable but unbalanced. The balanced condition is possible with a positive $C_{m_{ac}}$.

Three methods of obtaining a positive $C_{m_{ac}}$ are:

1. Use a negative camber airfoil section. The positive $C_{m_{ac}}$ will give a flying wing that is stable and balanced (figure 3.8).

FIGURE 3.8

NEGATIVE CAMBERED FLYING WING



This type of wing is not realistic because of unsatisfactory dynamic characteristics, small cg range, and extremely low C_L capability.

2. A reflexed airfoil section reduces the effect of camber by creating a download near the trailing edge. Similar results are possible with an upward deflected flap on a symmetrical airfoil.

3. A symmetrical airfoil section in combination with sweep and wing tip washout (reduction in angle of incidence at the tip) will produce a positive $C_{m_{ac}}$ by virtue of the aerodynamic couple produced between the down loaded tips and the normal lifting force.

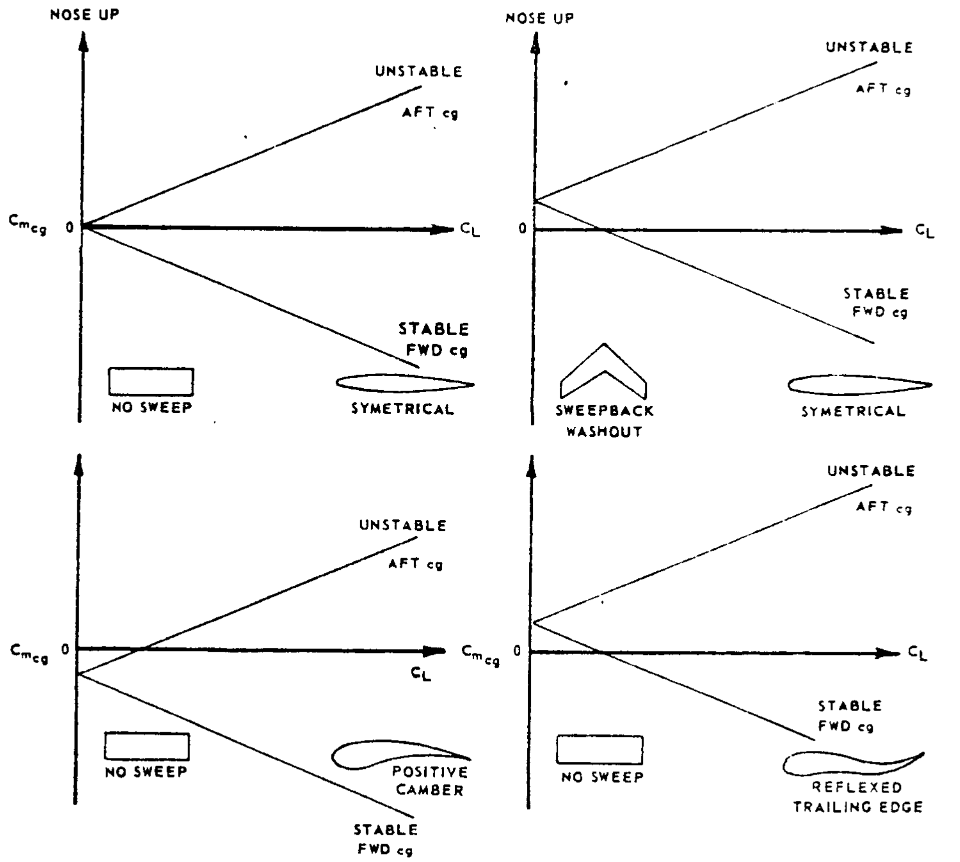
Figure 3.9 shows idealized $C_{m_{cg}}$ versus C_L for various wings in a control fixed position. Only two of the wings are capable of sustained flight.

The Fuselage Contribution to Stability:

The fuselage contribution is difficult to separate from the wing terms because it is strongly influenced by interference from the wing flow field. Wind tunnel tests of the wing body combination are used by airplane designers to obtain information about the fuselage influence on stability.

A fuselage by itself is almost always destabilizing because the center of pressure is usually

FIGURE 3.9



ahead of the center of gravity. The magnitude of the destabilizing effects of the fuselage requires their consideration in the equilibrium and stability equations.

$$\frac{dC_m}{dC_{L_{Fus}}} = \text{Positive quantity}$$

The Tail Contribution to Stability:

From equation 3.7, the tail contribution to stability was found to be:

$$\frac{dC_m}{dC_{L_{Tail}}} = -\frac{dC_{N_T}}{dC_L} V_H \eta_T \qquad (3.26)$$

For small angles of attack, the lift curve slope of the tail is very nearly the same as the slope of the normal force curve.

$$a_T = \frac{dC_L}{d\alpha_{Tail}} \approx \frac{dC_N}{d\alpha_{Tail}} \qquad (3.27)$$

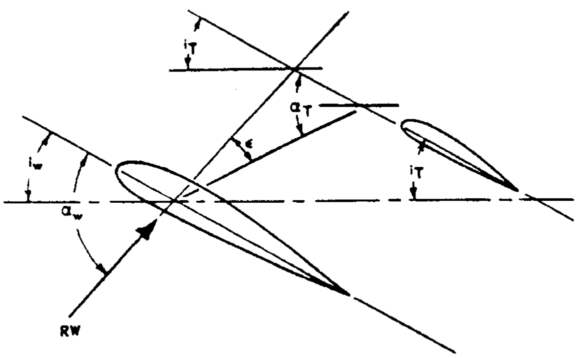
Therefore:

$$C_{N_T} = a_T \alpha_T \qquad (3.28)$$

An expression for α_T in terms of C_L is required before solving for dC_{N_T}/dC_L.

FIGURE 3.10

TAIL ANGLE OF ATTACK



From figure 3.10,

$$\alpha_T = \alpha_w - i_w + i_T - \epsilon \qquad (3.29)$$

Substituting equation 3.29 into 3.28 and taking the derivative with respect to C_L, where $a_w = dC_L/d\alpha$

$$\frac{dC_{N_T}}{dC_L} = a_T\left(\frac{d\alpha_w}{dC_L} - \frac{d\epsilon}{dC_L}\right)$$

$$= a_T\left(\frac{1}{a_w} - \frac{d\epsilon}{d\alpha}\frac{1}{a_w}\right) \qquad (3.30)$$

upon factoring out $1/a_w$,

$$\frac{dC_{N_T}}{dC_L} = \frac{a_T}{a_w}\left(1 - \frac{d\epsilon}{d\alpha}\right) \qquad (3.31)$$

Substituting equation 3.31 into 3.26, the expression for the tail contribution becomes,

$$\frac{dC_m}{dC_{L_{Tail}}} = -\frac{a_T}{a_w}\left(1 - \frac{d\epsilon}{d\alpha}\right) V_H \eta_T \qquad (3.32)$$

The value of a_T/a_w is very nearly constant. These values are usually obtained from experimental data.

The tail volume coefficient, V_H, is a term determined by the geometry of the aircraft. To vary this term is to redesign the aircraft.

$$V_H = \frac{\ell_T S_T}{cS} \qquad (3.33)$$

The further the tail is located aft of the cg (increase ℓ_T) or the greater the tail surface area (S_T), the greater the tail volume coefficient V_H which increases the tail contribution to stability.

The expression, η_T, is the ratio of the tail dynamic pressure to the wing dynamic pressure and η_T is greater than unity for a prop aircraft and less than unity for a jet. For power-off considerations, $\eta_T \approx 1.0$.

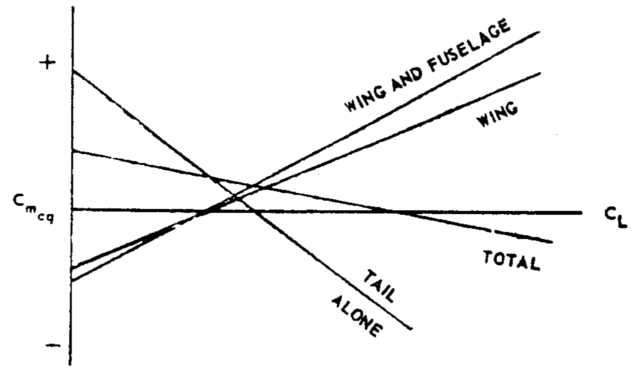
The term $(1 - d\varepsilon/d\alpha)$ is an important factor in the stability contribution of the tail. Large positive values of $d\varepsilon/d\alpha$ produce destabilizing effects by reversing the sign of the term $(1 - d\varepsilon/d\alpha)$ and consequently, the sign of $dC_m/dC_{L_{Tail}}$.

For example at high angles of attack the F-104 experiences a sudden increase in $d\varepsilon/d\alpha$. The term $(1 - d\varepsilon/d\alpha)$ goes negative causing the entire tail contribution to be positive or destabilizing, causing aircraft pitchup. The stability of an aircraft is definitely influenced by the wing vortex system. For this reason the downwash variation with angle of attack should be evaluated in the wind tunnel.

The horizontal stabilizer provides the necessary positive stability contribution (negative dC_m/dC_L) to offset the negative stability of the wing and fuselage combination and to make the entire aircraft stable and balanced (figure 3.11).

FIGURE 3.11

CONTRIBUTIONS TO STABILITY



The stability equation 3.7 may now be written as,

$$\frac{dC_m}{dC_L} = \frac{X_w}{c} + \frac{dC_m}{dC_{L_{Fus}}} - \frac{a_T}{a_w} V_H \eta_T \left(1 - \frac{d\varepsilon}{d\alpha}\right) \qquad (3.33a)$$

The Power Contribution to Stability:

The addition of a power plant to the aircraft may have a decided effect on the equilibrium as well as the stability equations. The overall effect may be quite complicated. This section will be a qualitative discussion of the power effects. The actual end result as to the power effects on trim and stability should come from large scale wind tunnel models or actual flight test.

The power effects on a propeller-driven aircraft which influence the static longitudinal stability of the aircraft are:

1. Thrust force effect - effect on stability from the thrust force acting along the propeller axis.
2. Normal force effect - effect on stability from a force normal to the thrust line and in the plane of the propeller.
3. Indirect effects - power plant effects on the stability contribution of other parts of the aircraft.

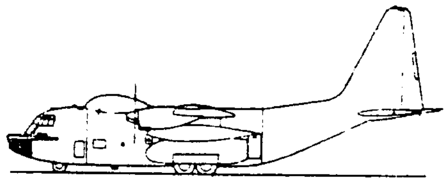
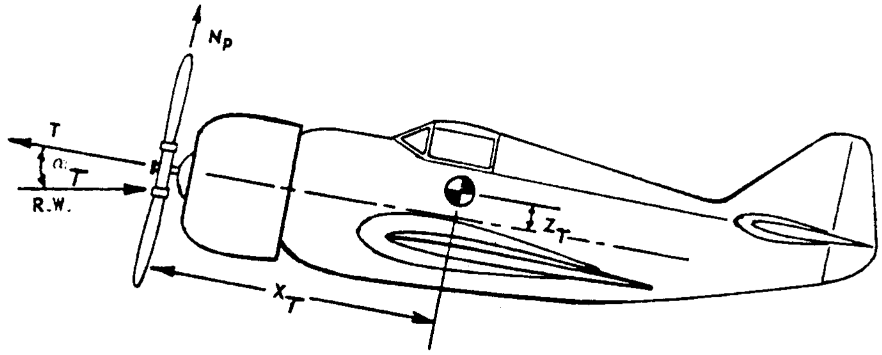


FIGURE 3.12

PROPELLER THRUST AND NORMAL FORCE



Writing the moment equation for the power terms as:

$$\overset{+}{\curvearrowright} \quad M_{CG} = TZ_T + N_p X_T \qquad (3.34)$$

In coefficient form,

$$C_{m_{CG}} = C_T \frac{Z_T}{c} + C_{N_p} \frac{X_T}{c} \qquad (3.35)$$

The direct power effect on the aircraft's stability equation is then:

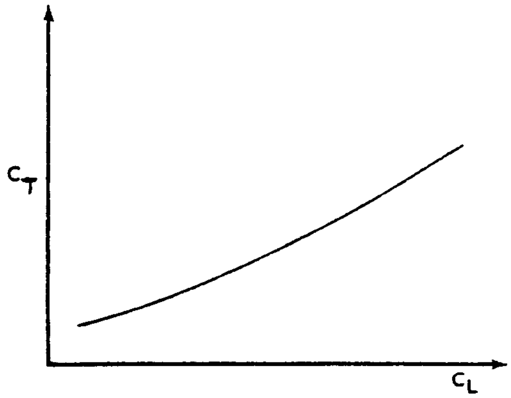
$$\frac{dC_m}{dC_{L_{power}}} = \frac{dC_T}{dC_L}\frac{Z_T}{c} + \frac{dC_{N_p}}{dC_L}\frac{X_T}{c} \qquad (3.36)$$

The sign of $dC_m/dC_{L_{power}}$ then depends on the sign of the derivatives dC_{N_p}/dC_L and dC_T/dC_L.

We shall first consider the dC_T/dC_L derivative. If speed varies at different flight conditions with throttle position held constant, then C_T varies in a manner that can be represented by dC_T/dC_L. The coefficient of thrust for a reciprocating power plant varies with C_L and propeller efficiency. Propeller efficiency which is available from propeller performance estimates in manufacturer's data, decreases with increase in velocity. Coefficient of thrust variation with C_L is nonlinear with the derivative large at low speeds. The combination of these two variations approximately linearize C_T versus C_L (figure 3.13). The sign of dC_T/dC_L is positive.

FIGURE 3.13

COEFFICIENT OF THRUST CURVE RECIPROCATING POWER PLANT WITH PROPELLER



The derivative, dC_{N_p}/dC_L, is positive since the normal propeller force increases linearly with the local angle of attack of the propeller axis, α_T.

The direct power effects are then destabilizing if the cg is as shown in figure 3.12, or where the power plant is ahead and below the cg.

The indirect power effects must also be considered in evaluating the overall stability contribution of the propeller power plant. No attempt will be made to determine their quantitative magnitudes; however, their general influence on the aircraft's stability and trim condition can be great.

1. Increase of Angle of Downwash, ε:

Since the normal force on the propeller increases with angle of attack under powered flight, the slipstream is deflected downward netting an increase downwash at the tail. The downwash in the slipstream will increase more rapidly with angle of attack than the downwash outside the slipstream. The derivative $d\varepsilon/d\alpha$ has a positive increase with power. The term $(1 - d\varepsilon/d\alpha)$ in equation 3.32 is reduced causing the tail trim contribution to be less negative or less stable than the power-off situation.

2. Increase of $\eta_T = (q_T/q_W)$:

The dynamic pressure, q_T, of the tail is increased by the slipstream and η_T is greater than unity. From equation 3.32, the increase of η_T with addition of a power plant increases the tail contribution to stability. However, if the tail is carrying a download at trim and if it should move into a high velocity region of the slipstream at higher C_L, more of a noseup moment would be present as C_L increased, causing an obvious destablizing effect.

Both slipstream effects mentioned above may be reduced by locating the horizontal stabilizer high on the tail and out of the slipstream at operating angles of attack.

Power Effects on Jet Aircraft.

The magnitude of the power effects on jet-powered aircraft are generally smaller than on propeller-driven aircraft. By assuming that jet engine thrust does not change with velocity or angle of attack, and by assuming constant power settings, smaller power effects would be expected than with a similar reciprocating engine aircraft.

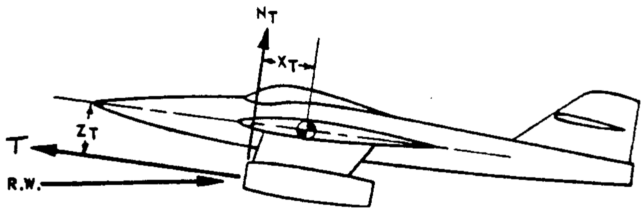
There are three major contributions of a jet engine to the equilibrium static longitudinal stability of the aircraft. These are the direct thrust effects, the normal force effects at the air duct inlet, and the indirect effect of the induced flow at the tail.

The thrust and normal force contribution may be determined from figure 3.13a.

FIGURE 3.13a

TEST THRUST AND NORMAL FORCE

JET THRUST AND NORMAL FORCE



Writing the equation,

$$\overset{+\curvearrowright}{M_{CG}} = TZ_T + N_T X_T \qquad (3.38)$$

or

$$C_{m_{CG}} = \frac{T}{qSc} Z_T + C_{N_T} \frac{X_T}{c} \qquad (3.39)$$

With the aircraft in unaccelerated flight, the dynamic pressure is a function of lift coefficient.

$$q = \frac{W}{C_L S} \qquad (3.40)$$

Therefore,

$$C_{m_{CG}} = \frac{T}{W}\frac{Z_T}{c} C_L + C_{N_T}\frac{X_T}{c} \qquad (3.41)$$

If thrust is considered independent of speed,* then

$$\frac{dC_m}{dC_L} = \frac{T}{W}\frac{Z_T}{c} + \frac{dC_{N_T}}{dC_L}\frac{X_T}{c} \qquad (3.42)$$

The thrust contribution to stability then depends on whether the thrust line is above or below the cg. Locating the engine below the cg causes a destablizing influence, and above the cg a stabilizing influence.

The normal force contribution depends on the sign of the derivative dC_{N_T}/dC_L. The normal force N_T is created at the air-duct inlet to the turbojet unit. This force is created as a result of the momentum change of the free stream which bends to flow along the duct axis. The magnitude of the force is a function of the mass airflow rate, W_a, and the angle α_T between the local flow at the duct entrance and the duct axis.

$$N_T = \frac{Wa}{g} V \quad \alpha_T \qquad (3.43)$$

With an increase in α_T, N_T will increase, causing dC_{N_T}/dC_L to be positive. The normal force contribution will be destabilizing if the inlet duct is ahead of the center of gravity. The magnitude of the destabilizing moment will depend on the distance the inlet duct is ahead of the center of gravity.

For a jet engine to definitely contribute to positive longitudinal stability, (dC_m/dC_L negative), the jet engine would be located above and behind the center of gravity.

The indirect contribution of the jet unit to longitudinal stability is the effect of the jet induced downwash at the horizontal tail. This applies to the situation where the jet exhaust passes under or over the horizontal tail surface. The jet exhaust as it discharges from the tail pipe spreads outward. Turbulent mixing causes outer air to be drawn in towards the exhaust area. Downwash at the tail is directly affected. With the exhaust below the tail surface, the downwash is increased, causing the tail term to be less stabilizing.

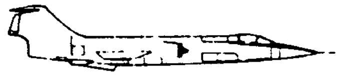
From the above discussion it can be seen that several factors are important in deciding the power effect on stability. Each aircraft must be examined individually. This is the reason that aircraft are tested for stability in several configurations and at different power settings.

3.5 THE NEUTRAL POINT

The stick-fixed neutral point is defined as the center of gravity position at which the aircraft displays neutral stability or where $dC_m/dC_L = 0$.

The symbol h is used for center of gravity position where,

$$h = \frac{X_{CG}}{c} \qquad (3.44)$$



* For aircraft which have large thrust variation with airspeed, the pitching moment coefficient must be calculated for different values of the aircraft's lift coefficient.

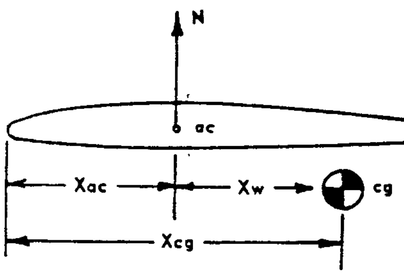
The stability equation for the powerless aircraft is:

$$\frac{dC_m}{dC_L} = \frac{X_w}{c} + \frac{dC_m}{dC_{L_{Fus}}} - \frac{a_T}{a_w} V_H \eta_T \left(1 - \frac{d\varepsilon}{d\alpha}\right) \quad (3.33a)$$

Looking at the relationship between cg and a.c. in figure 3.14,

FIGURE 3.14

cg AND ac RELATIONSHIP



$$\frac{X_w}{c} = h - \frac{X_{ac}}{c} \quad (3.46)$$

Substituting equation 3.46 into equation 3.45 and setting dC_m/dC_L equal to zero,

$$\frac{dC_m}{dC_L} = 0 = h - \frac{X_{ac}}{c} + \frac{dC_m}{dC_{L_{Fus}}} - \frac{a_T}{a_w} V_H \eta_T \left(1 - \frac{d\varepsilon}{d\alpha}\right) \quad (3.47)$$

Solving for h which is h_n,

$$h_n = \frac{X_{ac}}{c} - \frac{dC_m}{dC_{L_{Fus}}} + \frac{a_T}{a_w} V_H \eta_T \left(1 - \frac{d\varepsilon}{d\alpha}\right) \quad (3.48)$$

Substituting equation 3.48 back into equation 3.47, the stick-fixed stability derivative in terms of cg positions becomes,

$$\frac{dC_m}{dC_L} = h - h_n \quad (3.49)$$

The stick-fixed static stability is equal to the distance between the cg position and the neutral point in percent of the mean aerodynamic chord. "Static Margin" refers to the same distance but is positive in sign for a stable aircraft.

$$\text{"Static Margin"} = h_n - h \quad (3.50)$$

It is the test pilot's responsibility to evaluate the aircraft's handling qualities and to determine the acceptable static margin for the aircraft.

● 3.6 ELEVATOR POWER

As previously mentioned, for an aircraft to be a usable flying machine, it must possess stability and must be capable of being placed in equilibrium ($C_{m_{cg}} = 0$) throughout the useful C_L range (balanced).

For trimmed or equilibrium flight, $C_{m_{cg}}$ must be zero. Some means must be available for balancing the various terms in the moment coefficient of equation 3.51,

$$C_{m_{CG}} = C_N \frac{X_w}{c} + C_C \frac{Z_w}{c} - C_{m_{ac}} + C_{m_f} - a_T \alpha_T V_H \eta_T \quad (3.51)$$

Eq 3.51 is obtained by substituting Eq 3.28 into 3.6.

Several possibilities are available. The center of gravity could be moved fore and aft or up and down thus changing X_w/C or Z_w/C. However, this would not only affect the equilibrium lift coefficient but would also change the stability dC_m/dC_L in the stability equation 3.52. This is undesirable.

$$\frac{dC_m}{dC_L} = \frac{dC_N}{dC_L}\frac{X_w}{c} + \frac{dC_C}{dC_L}\frac{Z_w}{c} + \frac{dC_m}{dC_{L_{Fus}}}$$

$$- \frac{a_T}{a_w} V_H \eta_T \left(1 - \frac{d\varepsilon}{d\alpha}\right) \qquad (3.52)$$

Eq 3.52 is obtained by substituting Eq 3.31 into 3.7.

The pitching moment coefficient about the aerodynamic center could be changed by effectively changing the camber of the wing by using trailing edge flaps as is done in flying wing vehicles. On the conventional tail-to-the-rear aircraft, trailing edge wing flaps are ineffective in trimming the pitching moment coefficient to zero.

The remaining solution is to change the angle of attack of the horizontal stabilizer to achieve a $C_{m_{cg}} = 0$ without a change to the basic aircraft stability. The control means is either an elevator on the stabilizer or an all moving stabilizer (called a slab). The slab is used in most high speed aircraft and is the most powerful means of longitudinal control.

Movement of the slab or elevator changes the effective angle of attack of the horizontal stabilizer and, consequently, the lift on the horizontal tail. This in turn changes the moment about the center of gravity due to the horizontal tail. It is of interest to know the amount of pitching moment change associated with a degree of elevator deflection. This may be determined by differentiating equation 3.51 with respect to δ_e.

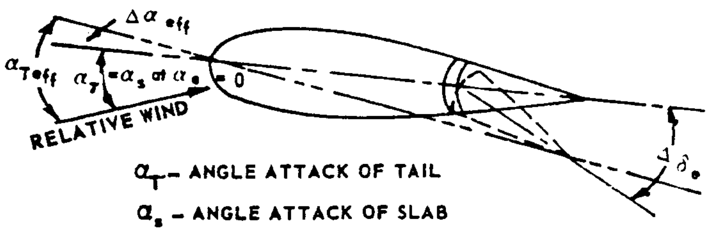
$$\frac{dC_m}{d\delta_e} = C_{m_{\delta_e}} = - a_T V_H \eta_T \frac{d\alpha_T}{d\delta_e} \qquad (3.53)$$

This change in pitching moment coefficient with respect to elevator deflection $C_{m_{\delta_e}}$ is referred to as "elevator power." It indicates the capability of the elevator in producing moments about the center of gravity.

The term $d\alpha_T/d\delta_e$ in equation 3.53 is termed elevator effectiveness and is given the shorthand notation τ. The elevator effectiveness may be considered as the equivalent change in effective tail plane angle of attack per unit change in elevator deflection. The relationship between elevator effectiveness τ and the effective angle of attack of the stabilizer is seen in figure 3.15.

FIGURE 3.15

CHANGE IN EFFECTIVE ANGLE OF ATTACK WITH ELEVATOR DEFLECTION



As seen, elevator effectiveness is a design parameter and is determined from wind tunnel tests. Elevator effectiveness is a negative number for all tail to the rear aircraft. The values range from zero to the limiting case of the all moving stabilizer (slab) where τ equals (-1). The tail angle of attack would change plus one degree for every minus degree the slab moves. For the elevator stabilizer combination, the elevator effectiveness is a function of the ratio of overall elevator area to the entire horizontal tail area.

● 3.7 STABILITY CURVES

Figure 3.16 is a wind tunnel plot of C_m versus C_L for an aircraft tested under two cg positions and two elevator positions.

$$\frac{dC_m}{dC_L} = \frac{dC_N}{dC_L}\frac{X_w}{c} + \frac{dC_C}{dC_L}\frac{Z_w}{c} + \frac{dC_m}{dC_{L_{Fus}}} - \frac{a_T}{a_w} V_H n_T \left(1 - \frac{d\varepsilon}{d\alpha}\right) \quad (3.52)$$

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$$\frac{dC_m}{d\delta_e} = C_{m_{\delta_e}} = -a_T V_H n_T \frac{d\alpha_T}{d\delta_e} \quad (3.53)$$

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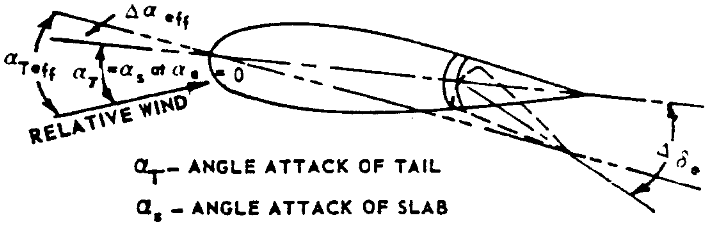
CHANGE IN EFFECTIVE ANGLE OF ATTACK WITH ELEVATOR DEFLECTION



As seen, elevator effectiveness is a design parameter and is determined from wind tunnel tests. Elevator effectiveness is a negative number for all tail to the rear aircraft. The values range from zero to the limiting case of the all moving stabilizer (slab) where τ equals (-1). The tail angle of attack would change plus one degree for every minus degree the slab moves. For the elevator stabilizer combination, the elevator effectiveness is a function of the ratio of overall elevator area to the entire horizontal tail area.

3.7 STABILITY CURVES

Figure 3.16 is a wind tunnel plot of C_m versus C_L for an aircraft tested under two cg positions and two elevator positions.

FIGURE 3.16

cg AND δ_e VARIATION ON STABILITY

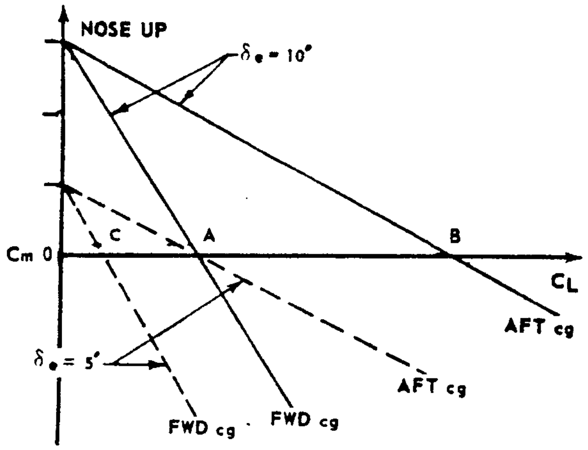
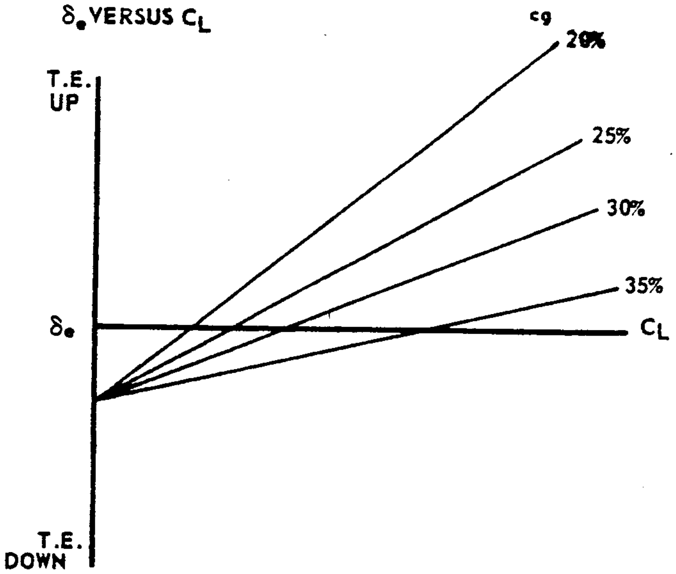


FIGURE 3.17

δ_e VERSUS C_L



Assuming the elevator effectiveness and the elevator power to be constant, then equal elevator deflections produce equal moments about the cg. Points A and B represent the same elevator deflection corresponding to the $C_{m_{cg}}$ needed to maintain equilibrium. The pilot selects elevator deflection of 10 degrees. In the aft cg condition, the aircraft will fly in equilibrium at point B. If the cg is moved forward with no change to the elevator deflection, the equilibrium point is now at A or at a new C_L. Note the increase in the stability of the aircraft (greater negative slope dC_m/dC_L).

If the pilot desires to fly at a lower C_L or at A and not change the cg, he does so by deflecting the elevator to 5 degrees. The stability level of the aircraft has not changed (same slope).

A cross plot of figure 3.16 is elevator deflection versus C_L for $C_m = 0$. This is shown in figure 3.17. The slopes of the cg curves are indicative of the aircraft's stability.

● 3.8 FLIGHT TEST RELATIONSHIP

The stability equation 3.52 derived previously pertains to theoretical applications and text book solutions. The equation has no use in flight testing. There is no aircraft instrumentation which will measure the change in pitching moment coefficient with change in lift coefficient or angle of attack. Therefore an expression involving parameters easily measurable in flight is required. This expression should relate directly to the stick-fixed longitudinal static stability dC_m/dC_L of the aircraft.

The external moment acting longitudinally on an aircraft is:

$$M = f(\alpha, \dot{\alpha}, q, V, \delta_e) \tag{3.54}$$

Assuming further that the aircraft is in equilibrium and in unaccelerated flight, then

$$M = f(\alpha, \delta_e) \tag{3.55}$$

Therefore,

$$\Delta M = \frac{\partial M}{\partial \alpha} \Delta\alpha + \frac{\partial M}{\partial \delta_e} \Delta\delta_e \qquad (3.56)$$

and

$$C_m = C_{m_\alpha}\alpha + C_{m_{\delta_e}}\delta_e = 0 \qquad (3.57)$$

where $\Delta\alpha = \alpha - \alpha_o = \alpha$

$\Delta\delta_e = \delta_e - \delta_{e_o} = \delta_e$

assuming $\alpha_o = 0$

$\delta_{e_o} = 0$

The elevator deflection required to maintain equilibrium is,

$$\delta_e = -\frac{C_{m_\alpha}\alpha}{C_{m_{\delta_e}}} \qquad (3.58)$$

Taking the derivative of δ_e with respect to C_L,

$$\frac{d\delta_e}{dC_L} = -\frac{\frac{dC_m}{d\alpha}\frac{d\alpha}{dC_L}}{C_{m_{\delta_e}}} = -\frac{\frac{dC_m}{dC_L}}{C_{m_{\delta_e}}} \qquad (3.59)$$

In terms of the static margin, the flight test relationship is,

$$\frac{d\delta_e}{dC_L} = \frac{h_n - h}{C_{m_{\delta_e}}} \qquad (3.60)$$

The amount of elevator required to fly at equilibrium varies directly as the amount of static stick-fixed stability and inversely as the amount of elevator power.

3.9 LIMITATION TO DEGREE OF STABILITY

The degree of stability tolerable in an aircraft is determined by the physical limits of the longitudinal control. The elevator power and amount of elevator deflection is fixed once the aircraft has been designed. If the relationship between δ_e required to maintain the aircraft in equilibrium flight and C_L is linear, then the elevator deflection required to reach any C_L is,

$$\delta_e = \delta_{e_{Zero\ Lift}} + \frac{d\delta_e}{dC_L} C_L \qquad (3.61)$$

The elevator stop determines the absolute limit of the elevator deflection available. Similarly, the elevator must be capable of bringing the aircraft into equilibrium at $C_{L_{Max}}$.

Recalling Equation 3.59

$$\frac{d\delta_e}{dC_L} = \frac{-\frac{dC_m}{dC_L}}{C_{m_{\delta_e}}} \qquad (3.59)$$

Substituting equation 3.59 into 3.61 and solving for $dC_m/dC_{L_{Max}}$ corresponding to $C_{L_{Max}}$

$$\frac{dC_m}{dC_{L_{Max}}} = \frac{\left(\delta_{e_{Zero\ Lift}} - \delta_{e_{Limit}}\right)}{C_{L_{Max}}} C_{m_{\delta_e}} \qquad (3.63)$$

Given a maximum C_L required for landing approach, equation 3.63 represents the maximum stability possible, or defines the most forward cg movement. A cg forward of this point prevents obtaining maximum C_L with limit elevator.

If a pilot were to maintain the $C_{L_{Max}}$ for the approach, the value of dC_m/dC_L corresponding to this $C_{L_{Max}}$ would be satisfactory. However, as is the case, the pilot desires additional C_L to maneuver as in flaring the aircraft. Additional elevator is required. This requirement then dictates a $dC_m/dC_{L_{Max}}$ less than the value required for $C_{L_{Max}}$ only.

In addition to maneuvering the aircraft in the landing flare, the pilot must adjust for ground effect. The ground imposes a boundary condition which affects the downwash associated with the lifting action of the wing. This ground interference places the horizontal stabilizer at a reduced angle of attack. The equilibrium condition at the desired C_L is disturbed. To maintain the desired C_L, the pilot must increase δ_e to obtain the original tail angle of attack. The maximum stability dC_m/dC_L must be further reduced to obtain additional δ_e to counteract the reduction in downwash.

The three conditions that limit the amount of static longitudinal stability or most forward cg position are:

a. The ability to land at high C_L in ground effect.

b. The ability to maneuver at landing C_L (flare capability).

c. The total elevator deflection available.

Figure 3.17A illustrates the limitations in $dC_m/dC_{L_{Max}}$.

3.10 STICK-FREE STABILITY

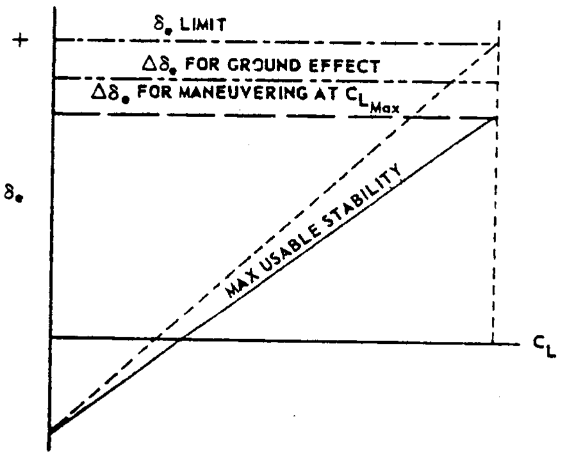
The name stick-free stability comes from the era of reversible control systems and is that variation related to the longitudinal stability which an aircraft would possess if the longitudinal control surface were left free to float in the slip stream. The control **force** variation with a change in airspeed is a measure of this stability.

If an airplane had an elevator that would float in the slip stream when the controls were free, then the change in the dynamic pressure pattern of the stabilizer would cause a change in the stability level of the airplane. The change in the stability contribution of the tail would be manifested by the floating characteristics of the elevator. Thus, the stick-free stability would depend upon the elevator hinge moments, control friction, or any device that would affect the moment of the elevator.

An airplane with an irreversible control system has very little tendency for its elevator to float. Yet the control forces presented to the pilot during flight, even though artificially produced, appear to be the effects of having a free elevator. If the control feel system can be altered artifically, then the pilot will see only good handling qualities and be able to fly

FIGURE 3.17a

LIMITATIONS ON $dC_m/dC_{L_{Max}}$



what would normally be an unsatisfactory flying machine.

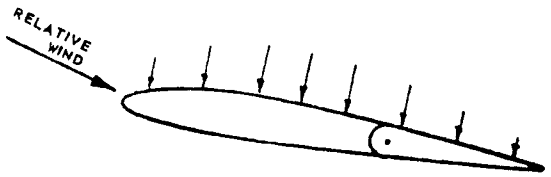
Stick-free stability can be analyzed by considering the effect of freeing the elevator of a tail-to-the-rear aircraft with a reversible control system. In this case the stick free stability would be indicated by the stick forces required to maintain the airplane in equilibrium at some speed other than trim.

The change in stability due to freeing the elevator, is a function of the floating characteristics of the elevator. The floating characteristics depend upon the elevator hinge moments which depend upon the change in pressure distribution over the elevator associated with changes in elevator deflection and tail angle of attack.

The analysis will look at the effect that pressure distribution has on the elevator hinge moments, the floating characteristics of the elevator, and then the effects of freeing the elevator.

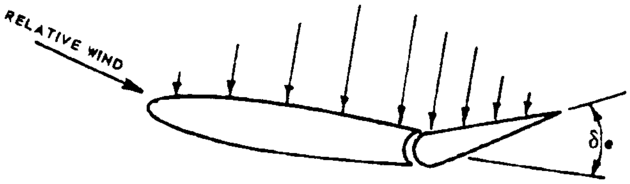
For a standard stable tail to the rear airplane, the pressure distribution would produce a downward load on the tail.

FIGURE 3.18



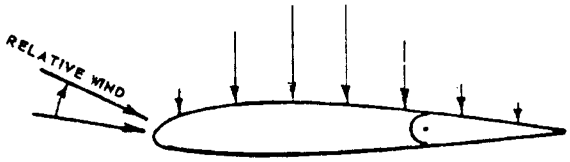
When the elevator is deflected the pressure distribution is changed.

FIGURE 3.19



When the stabilizer angle of attack is changed the pressure distribution is also changed.

FIGURE 3.20

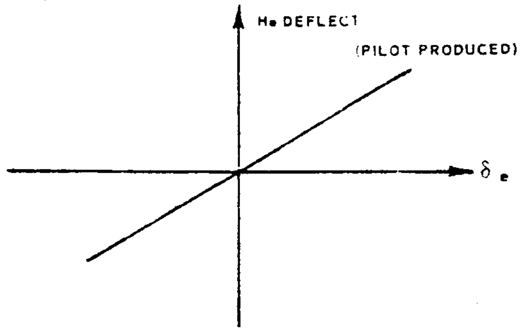


When the pressure distribution is changed, the hinge moments are changed. In order to deflect the elevator, the pilot had to apply a force to the stick and create a moment about the elevator hinge. The elevator hinge moment the pilot applied is now balanced by a moment caused by the pressure distribution on the control surface, and the elevator remains in the deflected position.

The pilot normally pulls back on the stick in order to produce a pitchup moment on the airplane. The hinge moment produced tends to move the control such that a positive moment on the airplane results. Therefore, the hinge moment is called positive. The pilot applies a positive moment to move the elevator. The pressure distribution produces a negative moment that opposes that of the pilot.

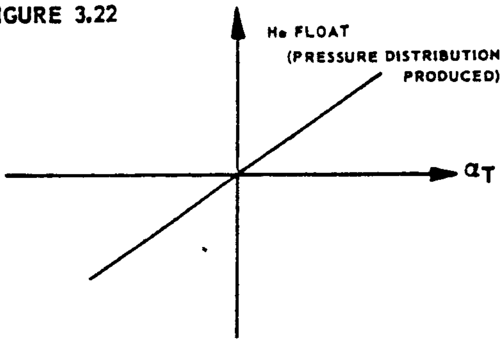
A plot of the pilot's hinge moment to deflect the elevator would be:

FIGURE 3.21



The hinge moment produced by the pressure distribution would be as shown in figure 3.24.

FIGURE 3.22

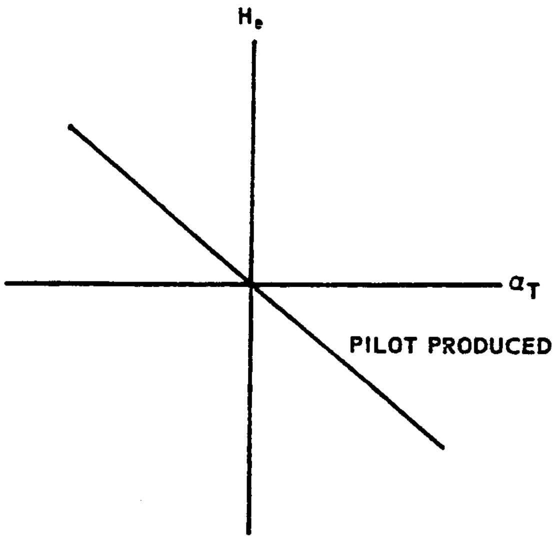


When the stabilizer angle of attack (α_T) is changed, the pilot must produce a control force in order to keep the elevator from floating in the slip stream.

Normally as the angle of attack is increased the elevator would tend to float up and the pilot would have to apply a negative push force in order to keep the stick from moving.

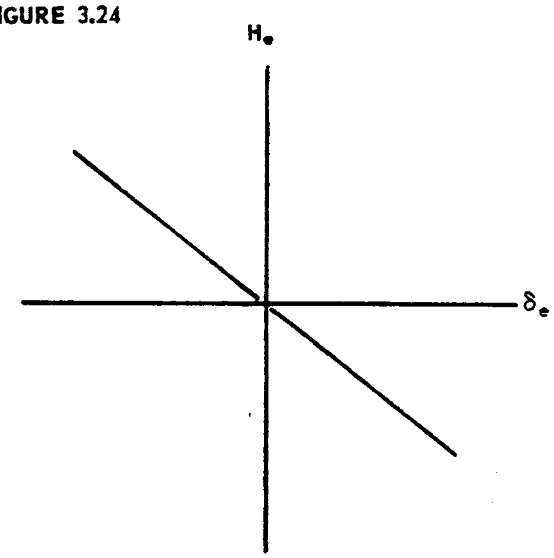
The hinge moment produced by the pilot to maintain trim deflection would be:

FIGURE 3.23



The hinge moment produced by the pressure distribution to float the elevator would be as shown in figure 3.22 .

FIGURE 3.24



If we consider the moments produced by the pressure distribution on the elevator only, then we could analyze the floating characteristics of the elevator.

The hinge moments can be put in coefficient form in much the same manner as the airplane's aerodynamic moments. The H_e Restore Slope due to elevator deflection in coefficient form would be:

$$\frac{\partial C_h}{\partial \delta_e} \text{ Restore } = C_{h_\delta} \qquad (3.64)$$

The He_{Float} slope due to angle of attack change in coefficient form would be:

$$\frac{\partial C_h}{\partial \alpha_e} \text{ Float } = C_{h_\alpha} \qquad (3.65)$$

Examining a floating elevator, it is seen that the total hinge moment is a function of elevator deflection, angle of attack, and mass distribution.

$$H_e = f(\delta_e, \alpha_T, W) \qquad (3.66)$$

If the elevator is held at zero elevator deflection and zero angle of attack there may be some residual aerodynamic hinge moment Ch_o. The total hinge moment where W = weight of the elevator would be:

$$C_h = C_{h_o} + C_{h_\alpha}\alpha_T + C_{h_\delta}\delta_e + \frac{W}{qS}\frac{X}{c} \qquad (3.67)$$

The weight effect is usually eliminated by mass balancing the elevator. Proper design of a symmetrical airfoil will cause Ch_o to be negligible.

When the elevator assumes its equilibrium position the total hinge moments will be zero and solving for the elevator deflection at this floating position.

$$\delta_{e_{Float}} = -\frac{C_{h_\alpha}}{C_{h_\delta}}\alpha_T \qquad (3.68)$$

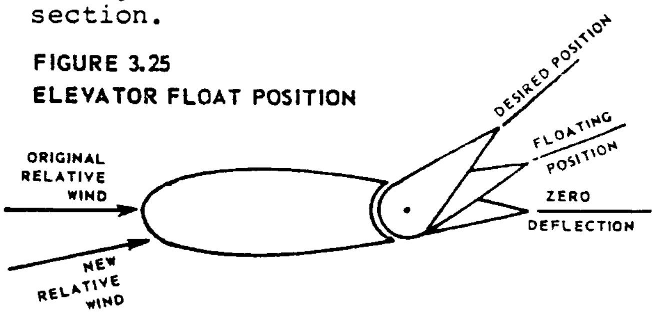
The stability of the aircraft with the elevator free is going to be affected by this floating position.

If the pilot desires to hold a new angle of attack from trim, he will have to deflect the elevator from this floating position to the position desired.

The floating position will greatly affect the forces the pilot is required to use. If the ratio Ch_α/Ch_δ can be adjusted, then the forces the pilot is required to use can be controlled.

If Ch_α/Ch_δ is small, then the elevator will not float very far and the stick-free stability characteristics will be much the same as those with the stick-fixed. But Ch_δ must be small or the stick forces required to hold deflection will be unreasonable. The values of Ch_α and Ch_δ can be controlled by aerodynamic balance. Types of aerodynamic balancing will be covered in a later section.

FIGURE 3.25
ELEVATOR FLOAT POSITION



3.11 STICK-FREE STABILITY EQUATIONS

Stick free stability may be considered the summation of the stick-fixed stability and the contribution to stability of freeing the elevator.

$$\frac{dC_m}{dC_L}\bigg|_{Stick\text{-}Free} = \frac{dC_m}{dC_L}\bigg|_{Stick\text{-}Fixed} + \frac{dC_m}{dC_L}\bigg|_{Freeing\ Elevator} \qquad (3.69)$$

Solving first for the effect to stability of freeing the elevator,

$$\frac{dC_m}{dC_L}\bigg|_{Free\ Elev.} = \frac{dC_m}{d\delta_e}\frac{d\delta_e}{dC_L} = C_{m_{\delta_e}}\frac{d\delta_e}{dC_L} \qquad (3.70)$$

The stability contribution of the free elevator depends upon the elevator floating position. Equation 3.68 relates this position.

$$\delta_{e_{Float}} = -\frac{C_{h_\alpha}}{C_{h_\delta}}\alpha_T \qquad (3.68)$$

Substituting for α_T from equation 3.29,

$$\delta_{e_{Float}} = -\frac{C_{h_\alpha}}{C_{h_\delta}} (\alpha_w - i_w + i_T - \varepsilon) \qquad (3.72)$$

Taking the derivative of equation 3.72 with respect to C_L,

$$\frac{d\delta_e}{dC_L} = -\frac{C_{h_\alpha}}{C_{h_\delta}} \frac{\left(1 - \frac{d\varepsilon}{d\alpha}\right)}{a_w} \qquad (3.73)$$

Substituting the expression for elevator power, Eq 3.74 and Eq 3.73 above into Eq 3.70 gives Eq 3.75.

$$C_{m_{\delta_e}} = -a_T \tau V_H \eta_T \qquad (3.74)$$

$$\frac{dC_m}{dC_{L_{Free}}}_{\text{Elevator}} = -\frac{a_T}{a_w} V_H \eta_T \left(1 - \frac{d\varepsilon}{d\alpha}\right)\left(-\tau \frac{C_{h_\alpha}}{C_{h_\delta}}\right) \qquad (3.75)$$

Substituting equation 3.75 and equation 3.33a ($dC_m/dC_{L_{Fixed}}$) into equation 3.69, the stick-free stability becomes

$$\frac{dC_m}{dC_{L_{Stick\ Free}}} = \frac{X_w}{c} + \frac{dC_m}{dC_{L_{Fus}}} - \frac{a_T}{a_w} V_H \eta_T \left(1 - \frac{d\varepsilon}{d\alpha}\right)\left(1 - \tau \frac{C_{h_\alpha}}{C_{h_\delta}}\right) \qquad (3.76)$$

The difference between stick-fixed and stick-free stability is the multiplier in equation 3.76, $(1 - \tau\, C_{h_\alpha}/C_{h_\delta})$, called the "free elevator factor" and which is designated F. The magnitude and sign of F depends on the relative magnitudes of τ and the ratio of $C_{h_\alpha}/C_{h_\delta}$. An elevator with only slight floating tendency has a small $C_{h_\alpha}/C_{h_\delta}$ giving a value of F around unity. The stick fixed and stick free stability are practically the same. If the elevator has a large floating tendency (ratio of $C_{h_\alpha}/C_{h_\delta}$ large), the stability contribution of the horizontal tail is reduced materially ($dC_m/dC_{L_{Free}}$ is less negative).

For instance, a ratio of $C_{h_\alpha}/C_{h_\delta} = -2$ and a τ of $-.5$, the floating elevator can obviate the whole tail contribution to stability. Generally, freeing the elevator causes a destablizing effect. With elevator free to float, the aircraft is less stable.

The stick-free neutral point, h_n', is that cg position at which $dC_m/dC_{L_{Free}}$ is zero. Continuing as in the stick-fixed case, the stick-free neutral point is,

$$h_n' = \frac{X_{ac}}{c} - \frac{dC_m}{dC_{L_{Fus}}} + \frac{a_T}{a_w} V_H \eta_T \left(1 - \frac{d\varepsilon}{d\alpha}\right) F \qquad (3.77)$$

and

$$\frac{dC_m}{dC_{L_{Free}}} = h - h_n' \qquad (3.78)$$

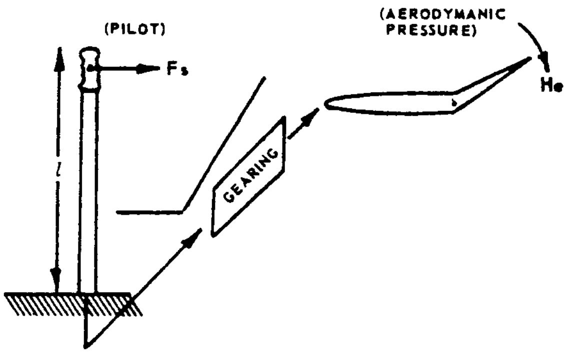
The stick-free static margin is defined as,

$$\text{Static Margin} = h_n' - h \qquad (3.79)$$

3.12 STICK-FREE FLIGHT TEST RELATIONSHIP

As was done for stick-fixed stability, a flight test relationship is required that will relate measureable flight test parameters with the stick-free stability of the aircraft $dC_m/dC_{L_{Free}}$. This relationship may be developed with reference to figure 3.26.

FIGURE 3.26
ELEVATOR-STICK GEARING



The pilot holds a stick deflected with a stick force F_s. The control system transmits the moment from the pilot through the gearing to the elevator. The elevator deflects and the aerodynamic pressure produces a hinge moment at the elevator that exactly balances the moment produced by the pilot with force F_s.

$$F_s \ell = - G' H_e \qquad (3.80)$$

If the length ℓ is included with the gearing, the stick force becomes,

$$F_s = {}' - GH_e \qquad (3.81)$$

The hinge moment H_e may be written,

$$H_e = C_h q S_e c_e \qquad (3.82)$$

Equation 3.81 then becomes,

$$F_s = - GC_h q S_e c_e \qquad (3.83)$$

Substituting

$$C_h = C_{h_o} + C_{h_{\alpha_T}} \alpha_T + C_{h_{\delta_e}} \delta_e + C_{h_{\delta_T}} \delta_T \qquad (3.84)$$

where $C_{h_{\delta_T}} \delta_T$ represents the tab contribution for an elevator with tab

where

$$\delta_e = \delta_{e_{zero\ lift}} + \frac{d\delta_e}{dC_L} C_L \qquad (3.61)$$

$$\alpha_T = \alpha_w - i_w + i_T - \varepsilon \qquad (3.29)$$

Equation 3.83 may be written,

$$F_s = \overbrace{- GS_e c_e q}^{A} \Bigg(\overbrace{C_{h_o} + C_{h_\alpha} (\alpha_{OL} - i_w + i_T) + \left(C_{h_{\delta_e}} \delta_e\right)_{Zero\ Lift}}^{B} + C_{h_{\delta_T}} \delta_T - \frac{C_L C_{h_{\delta_e}}}{C_{m_{\delta_e}}} \frac{dC_m}{dC_{L_{Free}}} \Bigg) \qquad (3.87)$$

Rewriting equation 3.87 with the above substitutions,

$$F_s = Aq \left(B + C_{h_{\delta_T}} \delta_T - \frac{C_L C_{h_{\delta_e}}}{C_{m_{\delta_e}}} \frac{dC_m}{dC_{L_{Free}}}\right) \qquad (3.88)$$

Writing equation 3.88 as a function of airspeed and substituting for unaccelerated flight, $C_L q = W/S$ and using equivalent airspeed, V_e,

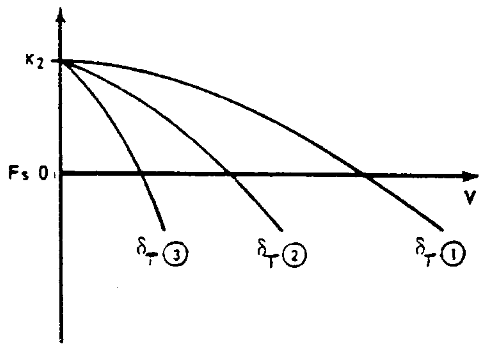
$$F_s = 1/2\, \rho_o V_e^2 A(B + C_{h_{\delta_T}} \delta_T) - A\frac{W}{S} \frac{C_{h_{\delta_e}}}{C_{m_{\delta_e}}} \frac{dC_m}{dC_{L_{Free}}} \qquad (3.89)$$

Simplifying equation 3.89 by combining constant terms,

$$F_s = K_1 V_e^2 + K_2 \qquad (3.90)$$

K_1 contains δ_T which determines trim speed. K_2 contains $dC_m/dC_{L_{Free}}$. Equation 3.90 gives a relationship between an inflight measurement of stick force gradient and stick free stability. The equation is plotted in figure 3.27.

FIGURE 3.27
STICK FORCE VERSUS AIRSPEED



The plot is made up of a constant force springing from the stability term plus a variable force proportional to the velocity squared, introduced through some constants and the tab term $C_{h_{\delta_T}}\delta_T$. Equation 3.90 introduces an interesting fact that the stick force variation with airspeed is apparently dependent on the first term only and independent in general of the stability level. That is, the slope of the curve F_s versus V is not a direct function of $dC_m/dC_{L_{Free}}$. If the derivative of equation 3.89 is taken with respect to V, the second term containing the stability drops out.

$$\frac{dF_s}{dV} = \rho_o V_e A(B + C_{h_{\delta_T}} \delta_T) \qquad (3.91)$$

However, dF_s/dV may be made a function of the stability term using another approach. The tab setting δ_T in equation 3.89 should be adjusted to obtain $F_s = 0$. **This** is δ_T for trim velocity, i.e.,

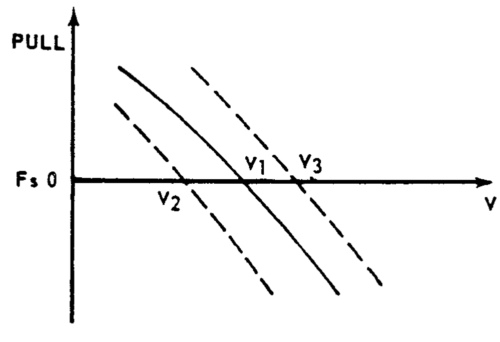
$$\delta_{T_{F_s}} = 0 = f(V_{Trim}, \frac{dC_m}{dC_{L_{Free}}}) \qquad (3.92)$$

This value of $\delta_{T_{F_s}} = 0$ is then substituted into equation 3.91 so that,

$$\frac{dF_s}{dV_{Trim}} = f\,(V_{Trim}, \frac{dC_m}{dC_{L_{Free}}}) \qquad (3.93)$$

Thus it appears that if an aircraft is flown at least two cg locations and dF_s/dV_{Trim} through the same trim speed each time is determined, then one could extrapolate or interpolate to determine the stick-free neutral point h_n'. Unfortunately, if there is a significant amount of friction in the control system, it is impossible to precisely determine this trim speed. In order to investigate briefly the effects of friction on the longitudinal control system, suppose that the aircraft represented in figure 3.28 is perfectly trimmed at V_1 (i.e., $\delta_e = \delta_{e1}$ and $\delta_T = \delta_{T_1}$). If the airspeed is decreased or increased with no change to the trim setting, the friction in the control system will

FIGURE 3.28
CONTROL SYSTEM FRICTION



prevent the elevator from returning all the way back to δ_{e_1} when the controls are released. The aircraft will return only to V_2 or V_3. With the trim tab at δ_{T_1}, the aircraft is content to fly at any speed between V_2 and V_3. The more friction that exists in the system, the wider this speed range becomes.

Therefore, if there is a significant amount of friction in the control system, it becomes impossible to say that there is one exact speed for which the aircraft is trimmed. Equation 3.93 then, is something less than perfect for predicting the stick-free neutral point of an aircraft. To reduce the undesirable effect of friction in the control system, a different approach is made to equation 3.88.

If equation 3.88 is divided by the dynamic pressure q, then,

$$F_s/q = A(B + C_{h_{\delta_T}} \delta_T) - \frac{AC_L C_{h_{\delta_e}}}{C_{m_{\delta_e}}} \frac{dC_m}{dC_{L_{Free}}} \qquad (3.94)$$

Differentiating with respect to C_L,

$$\frac{dF_s/q}{dC_L} = - \frac{AC_{h_{\delta_e}}}{C_{m_{\delta_e}}} \frac{dC_m}{dC_{L_{Free}}} \qquad (3.95)$$

or

$$\frac{dF_s/q}{dC_L} = f\left(\frac{dC_m}{dC_{L_{Free}}}\right) \qquad (3.96)$$

Trim velocity is now eliminated from consideration, and the prediction of stick-free neutral point h_n' is more exact. A plot of $dF_s/q/dC_L$ versus cg position may be extrapolated to obtain h_n'.

3.13 APPARENT STICK-FREE STABILITY

Speed stability or stick force gradient dF_s/dV in most cases does not reflect the actual stick-free stability $dC_m/dC_{L_{Free}}$ of an aircraft. In fact this apparent stability dF_s/dV may be quite different from the actual stability of the aircraft. Where the actual stability of the aircraft may be marginal ($dC_m/dC_{L_{Free}}$ small), or even unstable ($dC_m/dC_{L_{Free}}$ positive), the apparent stability dF_s/dV may be such as to make the aircraft quite acceptable. In flight, the test pilot feels and evaluates the apparent stability of the aircraft and not the actual stability $dC_m/dC_{L_{Free}}$.

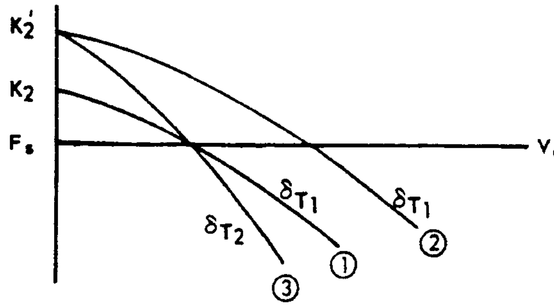
The apparent stability dF_s/dV is affected by:

1. Changes in $dC_m/dC_{L_{Free}}$
2. Aerodynamic balancing
3. Downsprings and/or bob weights.

The apparent stability or the stick force gradient through a given trim speed increases if $dC_m/dC_{L_{Free}}$ is made more negative. The constant K_2 of equation 3.90 is made more positive and in order for the stick force curve to continue to pass through the desired trim speed, a more positive tab selection is required. An aircraft operating at a certain cg with a tab setting δ_{T_1} is shown on figure 3.29, line 1.

FIGURE 3.29

EFFECT ON APPARENT STABILITY



If $dC_m/dC_{L_{Free}}$ is increased by moving the cg forward, K_2, which is a function of $dC_m/dC_{L_{Free}}$ in equation 3.89 becomes more positive or increases. The new equation becomes,

$$F_s = K_1 V_e^2 + K_2' \tag{3.97}$$

This equation plots as line 2 in figure 5.7. The aircraft with no change in tab setting δ_{T_1} operates on line 2 and is trimmed to V_2. Stick forces at all airspeeds have increased. At this juncture, although the actual stability $dC_m/dC_{L_{Free}}$ has increased, there has been no effect on the stick force gradient or apparent stability. (The slopes of line 1 and line 2 being the same.) So as to retrim to the original trim airspeed V_1, the pilot applies additional nose up tab to δ_{T_2}. The aircraft is now operating in line 3. The stick force gradient through V_1 has increased because of an increase in the K_1 term in equation 3.89. The apparent stability dF_s/dV has increased.

The same effect on apparent stability as cg movement may be obtained by means of aerodynamic balancing. This is a design means of controlling the hinge moment coefficients, Ch_α and Ch_δ. The primary reason for aerodynamic balancing is to increase or reduce the hinge moments and, in turn, the control stick forces. Changing Ch_δ, affects the stick forces as seen in equation 3.89. In addition to the influence on hinge moments, aerodynamic balancing may very well affect the real and apparent stability of the aircraft. Assuming that the restoring hinge moment coefficient Ch_δ is increased by an appropriate aerodynamic balanced control surface, the ratio of Ch_α/Ch_δ in stability equation 3.76 is decreased.

$$\frac{dC_m}{dC_{L_{Free}}} = \frac{X_w}{c} + \frac{dC_m}{dC_{L_{Fus}}} - \frac{a_T}{a_w} V_H \eta_T \left(1 - \frac{d\varepsilon}{d\alpha}\right)\left(1 - \tau \frac{C_{h_\alpha}}{C_{h_\delta}}\right) \tag{3.76}$$

The combined increase in $dC_m/dC_{L_{Free}}$ and Ch_δ, increases the K_2 term in equation 3.90 since

$$K_2 = -A \frac{W}{S} \frac{C_{h_\delta}}{C_{m_{\delta_e}}} \frac{dC_m}{dC_{L_{Free}}} \tag{3.99}$$

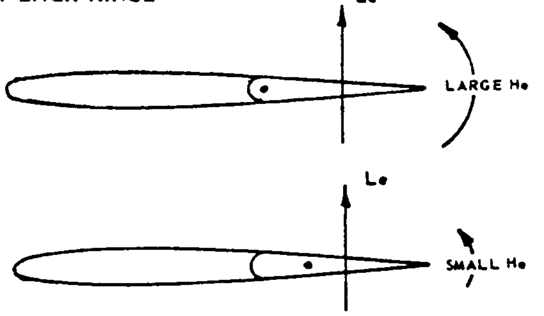
Figure 3.29 shows the effect of increased K_2. The apparent stability is not affected by the increase in K_2 while the aircraft retrims at V_2. However, once the aircraft is retrimmed to the original airspeed V_1 by increasing the tab setting to δ_{T_2}, the apparent stability is increased.

Types of aerodynamic balancing used to control the hinge moment coefficients are as follows:

<u>Set-Back-Hinge:</u>

Perhaps the simplest method of reducing the aerodynamic hinge moments is simply to move the hinge line rearward. Thus the hinge moment is reduced because of the moment arm between the elevator lift and the elevator hinge line is reduced. (One may arrive at the same conclusion by arguing that part of the elevator lift acting behind the hinge line has been reduced, while that in front of the hinge line has been increased.) The net result is a reduction in the absolute value of both Ch_α and Ch_δ. In fact if the hinge line is set back far enough, the sign of both derivatives can be changed.

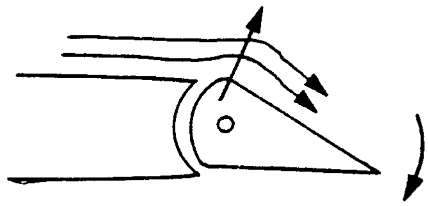
FIGURE 3.30
SET-BACK-HINGE



Overhang Balance:

This method is simply a special case of set-back hinge in which the elevator is designed so that when the leading edge protrudes into the airstream, the local velocity is increased significantly; causing an increase in negative pressure at that point. This negative pressure peak creates a hinge moment which opposes the normal restoring hinge moment, reducing Ch_δ. Figure 3.31 shows an elevator with an overhang balance.

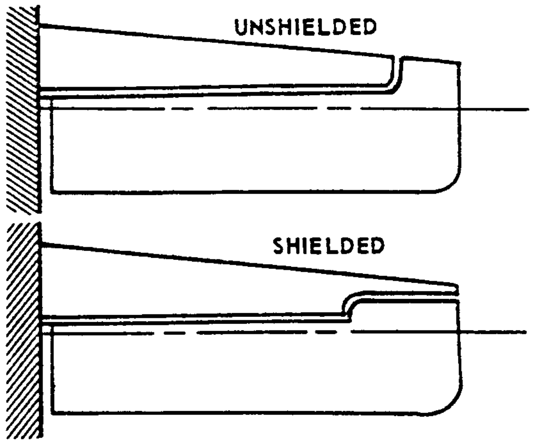
FIGURE 3.31
OVERHANG BALANCE



Horn Balance:

The horn balance works on the same principle as the set-back hinge, i.e., to reduce hinge moments by increasing the area forward of the hinge line. The horn balance, especially the unshielded horn, is very effective in reducing Ch_α and Ch_δ. This arrangement shown in figure 3.32, is also a handy way of improving the mass balance of the control surface.

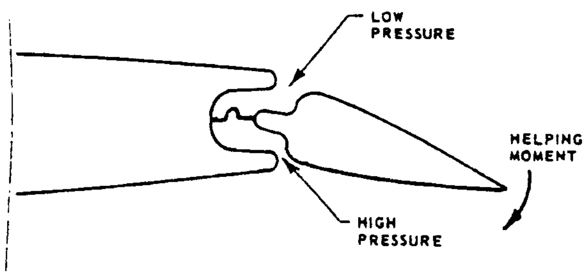
FIGURE 3.32
HORN BALANCE



Internal Balance or Internal Seal:

The internal seal allows the negative pressure on the upper surface and the positive pressure on the lower surface to act on an internal sealed surface forward of the hinge line in such a way that a helping moment is created, again opposing the normal hinge moments. As a result, the absolute values of Ch_α and Ch_δ are both reduced. This method is good at high indicated airspeeds but is sometimes troublesome at high Mach numbers.

FIGURE 3.33
INTERNAL SEAL



Elevator Modifications:

Bevel Angle on Top or on Bottom of the Stabilizer.

This device, which causes flow separation on one side but not on the other, reduced the absolute values of Ch_α and Ch_δ.

Trailing Edge Strips.

This device, found on the B-57, increases both Ch_α and Ch_δ. In combination with a balance tab, trailing edge strips produce a very high positive Ch_α, but still a low Ch_δ. This results in what is called a favorable "Response Effect," i.e., it takes a lower control force to hold a deflection than was originally required to produce it.

Convex Trailing Edge.

This type surface produces a more negative Ch_δ, but tends as well to produce a dangerous short-period oscillation.

Tabs:

A tab is simply a small flap which has been placed on the trailing edge of a larger one. The tab greatly modifies the flap hinge moments but has only a small effect on the lift of the elevator or the entire airfoil. Tabs in general are designed to modify stick-forces and therefore Ch_δ but will not affect Ch_α. A positive tab deflection is one which will tend to move the elevator in a positive direction.

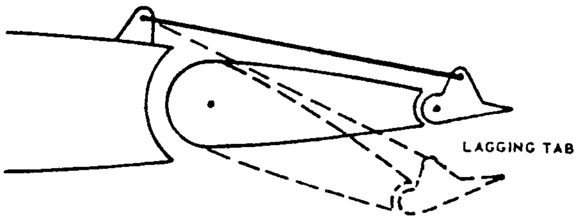
Trim Tab.

A tab which is controlled by a switch or control separate from the normal cockpit pilot control is called a trim tab. The purpose of the trim tab is to reduce the elevator hinge moment and, therefore, the stick force to zero for a given flight condition. A satisfactory trim tab should be able to accomplish this throughout the aircraft flight envelope. Ordinarily, a trim tab will not significantly vary Ch_α or Ch_δ. The functions of the spring and trim or balance and trim tabs may be combined in a single tab. Another method of trimming an aircraft is the use of an adjustable horizontal stabilizer. Normally the trim tab or horizontal stabilizer setting will have a small effect on stability.

Balance Tab.

A balance tab is a simple tab which is mechanically geared to the elevator so that a certain elevator deflection produces a given tab deflection. If the tab is geared to move in the same direction as the surface, it is called a leading tab. If it moves in the opposite direction, it is called a lagging tab. The purpose of the balance tab is usually to reduce the hinge moments and stick forces (lagging tab) at the price of a certain loss in control effectiveness. Sometimes, however, a leading tab is used to increase control effectiveness at the price of increased stick forces. The leading tab may also be used for the express purpose of increasing control forces. Thus Ch_δ may be increased or decreased, while Ch_α remains unaffected. If the linkage shown in figure 3.34 is made so that the length may be varied by the pilot, then the tab may also serve as a trimming device.

FIGURE 3.34
BALANCE TAB



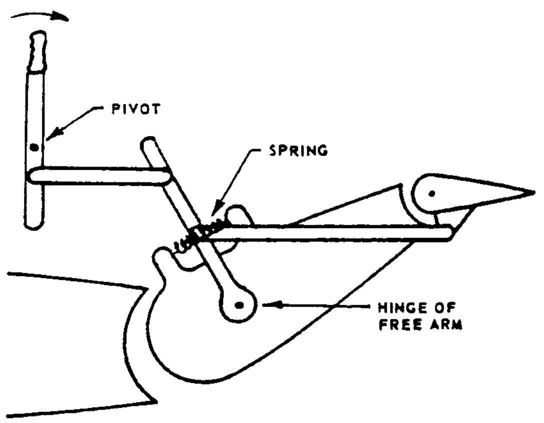
Servo or Control Tab.

The servo tab is linked directly to the aircraft control system in such a manner that the pilot moves the tab and the tab moves the elevator, which is free to float. The summation of elevator hinge moments, therefore, always equals zero since the elevator will float until the hinge moment due to elevator deflection just balances out the hinge moments due to α_s and δ_t. The stick forces are now a function of the tab hinge moment or Ch_{δ_T}. Again Ch_α is not affected.

Spring Tab.

A spring tab is a lagging balance tab which is geared in such a way that the pilot receives the most help from the tab at high speeds where he needs it the most, i.e., the gearing is a function of dynamic pressure. The basic principles of its operation are:

FIGURE 3.35
SPRING TAB



1. An increase in dynamic pressure causes an increase in hinge moment and stick force for a given control deflection.

2. The increased stick force causes an increased spring deflection and, therefore, an increased tab deflection.

3. The increased tab deflection causes a decrease in stick force. Thus an increased proportion of the hinge moment is taken by the tab.

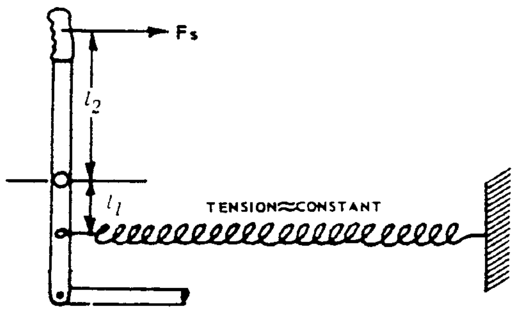
4. Therefore, the spring tab is a geared balance tab where the gearing is a function of dynamic pressure.

5. Thus the stick forces are more nearly constant over the speed range of the aircraft. That is, the stick force required to produce a given deflection at 300 knots is still greater than at 150 knots, but not by as much as before.

6. After full spring or tab deflection is reached the balancing feature is lost and the pilot must supply the full force necessary for further deflection. (This acts as a safety feature.)

Because of the very low force gradients in most modern aircraft at the aft center of gravity ($dC_m/dC_{L_{Free}}$ less negative), improvements in the stick-free longitudinal stability are obtained by devices which produce a constant pull force on the stick independent of airspeed which allows a more noseup tab setting and steeper stick force gradients. Two such gadgets for improving the stick force gradients are the downspring and bobweight. Both effectively increase the apparent stability of the aircraft.

FIGURE 3.36
DOWNSPRING



Downspring:

A virtually constant stick force may be demanded of the pilot by incorporating a downspring or bungee into the control system which tends to pull the top of the stick forward. From figure 3.36, the force required to counteract the spring is,

$$F_{s_{Downspring}} = T\frac{\ell_1}{\ell_2} = K_3 \qquad (3.100)$$

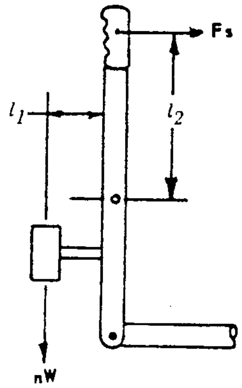
If the spring is a long one, the tension in it will be increased only slightly as the top moves rearward and can be considered to be constant.

The equation with the downspring in the control system becomes,

$$F_s = K_1 V_e^2 + K_2 + K_{3_{Downspring}} \qquad (3.101)$$

As shown in figure 3.29, the apparent stability will increase when the aircraft is once again retrimmed to the original trim airspeed by increasing the tab setting. Note that the downspring increases apparent stability but does not affect the actual stability $dC_m/dC_{L_{Free}}$ (no change to K_2) of the aircraft.

FIGURE 3.37
BOBWEIGHT



Bobweight:

Another method of introducing a nearly constant stick force is by placing a bobweight somewhere in the control system which causes a constant moment which must be overcome by the pilot. The force which the pilot must apply to counteract the bobweight is,

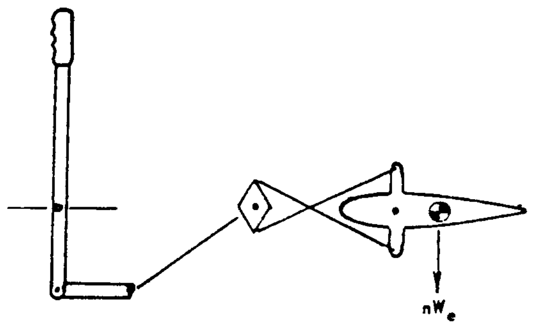
$$F_{s_{Bobweight}} = nW\frac{\ell_1}{\ell_2} = K_3 \qquad (3.102)$$

Like the downspring the bobweight increases the stick force throughout the airspeed range and, at increased tab settings, the apparent stability or stick force gradient. The bobweight has no effect on the actual $dC_m/dC_{L_{Free}}$ of the aircraft.

Elevator Unbalance:

There are other devices which increase the stick force gradient through trim or apparent stability. The unbalance in the control system resulting from the center of gravity of the elevator falling aft of the hinge line is shown in figure 3.38.

FIGURE 3.38
ELEVATOR UNBALANCE



From the figure it can be seen that an elevator cg behind the hinge line will tend to rotate the top of the stick forward. This must be counteracted by a positive pull stick force.

As the elevator is moved from the horizontal, the hinge moment is reduced by the cosine of the deflection angle; this moment remains virtually constant. Thus a forward hinge line which usually produces a destabilizing (positive) Ch_α will also produce a "stabilizing" elevator unbalance.

Comment:

Since h_n' is usually found by equation 3.96, it would be worthwhile to examine the effect of the stick force gradient dF_s/dV on this equation. Rewriting equation 3.88, with a downspring used as the control system gadget,

$$F_s = Aq(B + C_{h_{\delta_T}} \delta_T) - AC_L q \frac{C_{h_\delta}}{C_{m_{\delta_e}}} \left.\frac{dC_m}{dC_L}\right|_{\substack{Free \\ No\ Gadget}} + K_3'{}_{Gadget} \qquad (3.103)$$

$$F_s/q = A(B + C_{h_{\delta_T}} \delta_T) - AC_L \frac{C_{h_\delta}}{C_{m_{\delta_e}}} \left.\frac{dC_m}{dC_L}\right|_{\substack{Free \\ No\ Gadget}} + \frac{K_3' C_L}{W/S} \qquad (3.104)$$

$$\frac{dF_s/q}{dC_L} = K_2' \left.\frac{dC_m}{dC_L}\right|_{\substack{Free \\ No\ Gadget}} + \frac{K_3'}{W/S} \qquad (3.105)$$

Obviously the cg location at which $dF_s/q/dC_L$ goes to zero will not be the true h_n'. However, the only reason that the term $dC_m/dC_{L_{Free}}$ was of interest in the first place was because it was proportional to the stick force gradient. The pilot is more interested in the apparent stability for the same reason. The fact that the addition to the stick-free stability caused by this gadgetry is "artificial" rather than genuine is only of academic interest.

3.14 HIGH SPEED LONGITUDINAL STATIC STABILITY

The effects of high speeds (transonic and supersonic) on longitudinal static stability can be analyzed in the same manner as that done for subsonic speeds. The assumptions that were made for the incompressible flow are no longer valid.

Compressibility associated with the transonic and supersonic speed regime has noticeable effect upon both the gust stability (longitudinal static stability $C_{m_{C_L}}$) and speed stability (F_s/V). The gust stability depends mainly on the contributions to stability of the wing, fuselage, and tail in the stability equation below during transonic and supersonic flight.

$$\frac{dC_m}{dC_L} = \frac{X_w}{c} + \left.\frac{dC_m}{dC_L}\right|_{Fus} - \frac{a_T}{a_w} V_H \eta_T \left(1 - \frac{d\varepsilon}{d\alpha}\right) \qquad (3.106)$$

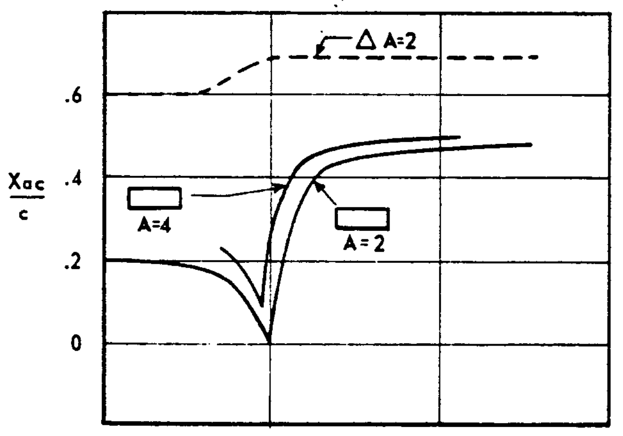
The terms in the stability equation will be examined in turn.

The Wing Contribution:

In subsonic flow or low Mach flight, the aerodynamic center is at the quarter chord. As subsonic Mach approaches unity or the transonic speed is approached, flow separation occurs behind the shock formations causing the aerodynamic center to move forward of the quarter chord position. The immediate effect is a reduction in stability since X_w/c increases. Following the flow separation behind the shocks at positions of sonic speed, the flow pattern on the airfoil eventually transitions to supersonic flow. The shocks move off the surface and the wing recovers lift. The aerodynamic center now moves aft towards the 50-percent chord position. There is a sudden increase in the wing's contribution to stability since X_w/c is reduced (figure 3.1).

The extent of the aerodynamic center shift forward and rearward depends greatly on the aspect ratio of the aircraft. The shift is least for low aspect ratio aircraft. Among the plan forms, the rectangular wing has the largest shift for a given aspect ratio whereas the triangular wing has the least (figures 3.39 and 3.14).

FIGURE 3.39
ac VARIATION WITH MACH



The Fuselage Contribution:

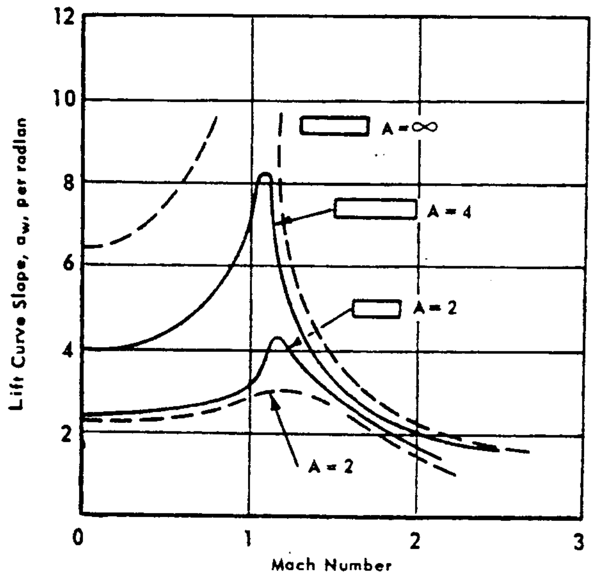
In supersonic flow the fuselage center of pressure moves forward causing a positive increase in the fuselage dC_m/dC_L or a destabilizing influence on the stability equation. The fuselage term variation with Mach number will be ignored.

The Tail Contribution:

The tail contribution to stability depends on the variation of lift curve slopes, a_W and a_T, plus downwash ε with Mach during transonic and supersonic flight. It is expressed as: $-a_T/a_W \; V_H \eta_T \; (1 - d\varepsilon/d\alpha)$

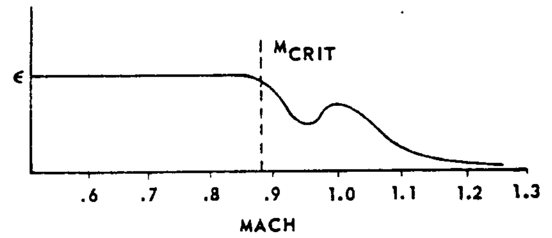
During subsonic flight a_T/a_W remains approximately constant. The slope of the lift curve, a_W varies as shown in figure 3.40. This variation of a_W in the transonic speed range is a function of geometry (i.e., aspect ratio, thickness, camber, and sweep). Limiting further discussion to aircraft designed for transonic flight or aircraft which employ airfoil shapes with small thickness to chord ratios, then a_W increases slightly in the transonic regime. For all airfoil shapes the values of a_W and a_T decrease as the airspeed increases supersonically. The overall a_T/a_W contribution is generally destabilizing in the transonic regime and stabilizing in the supersonic regime.

FIGURE 3.40
LIFT CURVE SLOPE VARIATION WITH MACH



The tail contribution is further affected by the variation in downwash, ε, with Mach increase. The downwash at the tail is a result of the vortex system associated with the lifting wing. It is recognized that the tail location will have considerable influence as to the degree of variation of δ_e with $\Delta\varepsilon$. An aircraft such as the F-100 has a great deal more variation of δ_e due to downwash effects than the F-104. Since downwash is a direct function of wing lift, a sudden loss of downwash occurs transonically with a resulting increase in tail angle of attack. The effect is to require the pilot to apply additional up elevator with increasing airspeed to maintain altitude. This additional up elevator contributes to speed instability. (Speed stability will be covered more thoroughly later.) Downwash variation with Mach is seen in figure 3.41.

FIGURE 3.41
DOWNWASH VARIATION WITH MACH

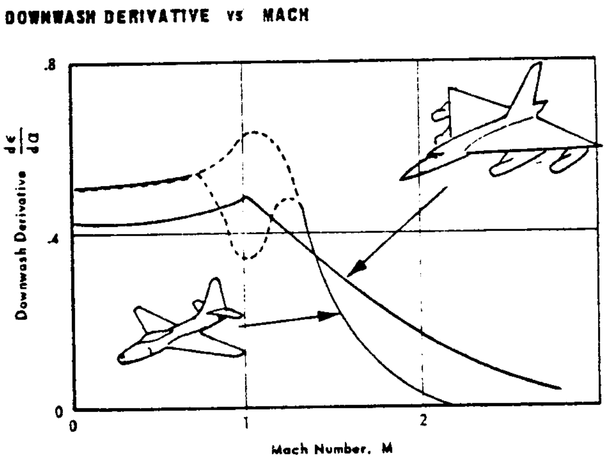


The variation of $d\varepsilon/d\alpha$ with Mach number greatly influences the aircraft's gust stability dC_m/dC_L. Recalling,

$$\varepsilon = \frac{114.6 C_L}{\pi A\!\!R} \qquad \text{where} \qquad \frac{d\varepsilon}{d\alpha} = \frac{114.6 a_w}{\pi A\!\!R} \tag{3.107}$$

Since the downwash angle behind the wing is directly proportional to the lift coefficient of the wing, it is apparent that the value of the derivative $d\varepsilon/d\alpha$ is a function of a_w. The general trend of $d\varepsilon/d\alpha$ is an initial increase with Mach starting at subsonic speeds. This increase follows a trend similar to but at a lesser slope than the increase of the lift curve slope, a_w, of the wing. Above Mach 1.0, $d\varepsilon/d\alpha$ decreases and approaches zero. This variation depends on the particular wing geometry of the aircraft. As shown in figure 3.42, $d\varepsilon/d\alpha$ may dip for thicker wing sections where considerable flow separation occurs. Again, $d\varepsilon/d\alpha$ is very much dependent on a_w.

FIGURE 3.42
DOWNWASH DERIVATIVE vs MACH



For an aircraft designed for high speed flight, the variation of $d\varepsilon/d\alpha$ with increasing Mach number results in a slight destabilizing effect in the transonic regime and contributes to increased stability in the supersonic speed regime.

As the wing surface becomes a less efficient lifting surface, a loss of stabilator effectiveness is experienced in supersonic flight. The elevator power, $C_{m\delta_e}$, increases as airspeed approaches Mach 1.0. Beyond Mach 1.0, elevator effectiveness decreases. Consequently, increase of elevator power causes a positive $\Delta\delta_e$ contribution or again an indication of speed instability as Mach 1.0 is approached. With decrease in elevator power, a negative $\Delta\delta_e$ contribution once again produces speed stability. For the F-104 the relative order of magnitude of these values cause an initial increase in gust stability in the transonic regime followed by a steadily decreasing stability influence as $C_{m_{\delta_e}}$ approaches zero.

$$\frac{dC_m}{dC_{L_{Tail}}} = \frac{C_{m_{\delta_e}}}{a_w \tau}\left(1 - \frac{d\epsilon}{d\alpha}\right) \qquad (3.108)$$

The overall effect of transonic and supersonic flight on gust stability or dC_m/dC_L is shown in figure 3.43. Static longitudinal stability increases transonically and then decreases supersonically. The speed stability of the aircraft is affected as well. Recalling the pitching moment coefficient equation,

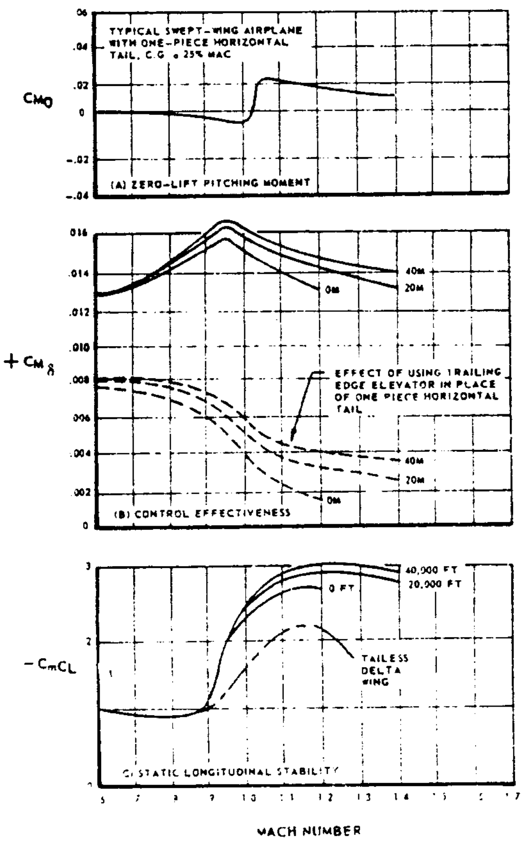
$$\Delta C_m = C_{m_o} + C_{m_\alpha}\Delta\alpha + C_{m_{\dot{\alpha}}}\Delta\dot{\alpha} + C_{m_{\delta_e}}\Delta\delta_e$$
$$+ C_{m_V}\Delta V + C_{m_q}\Delta q \qquad (3.109)$$

and since $C_{m_{C_L}} = \frac{1}{a} C_{m_\alpha}$, then: Assuming no change in speed or pitch rate, and since under compressibility C_{m_o} is not zero, the elevator required to maintain steady flight is:

$$\Delta\delta_e = -\frac{C_{m_o}}{C_{m_{\delta_e}}} - \frac{C_{m_{C_L}}}{C_{m_{\delta_e}}}\Delta C_L \qquad (3.110)$$

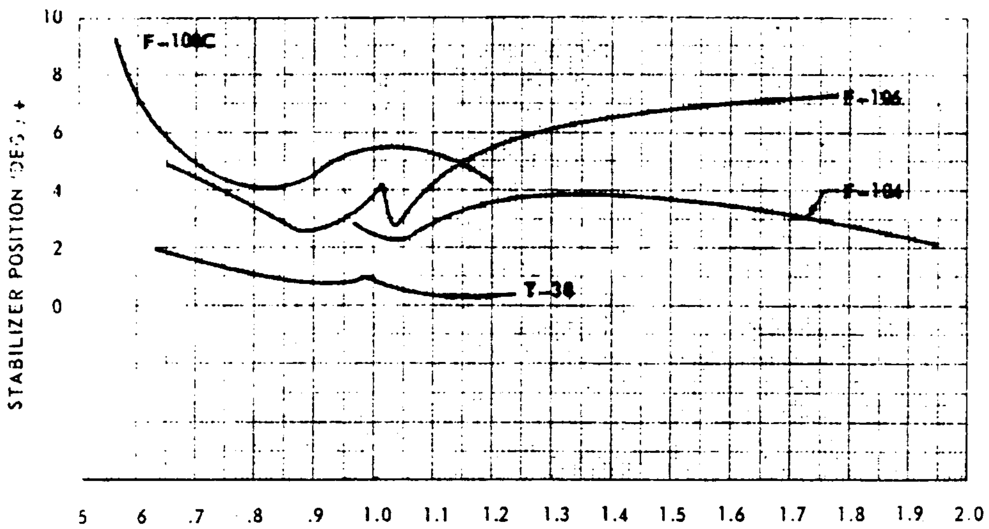
Speed stability depends on the variations of δ_e with transonic and supersonic speeds and according to equation 3.110, depends on how C_{m_o}, $C_{m_{\delta_e}}$, and $C_{m_{C_L}}$ vary.

FIGURE 3.43
MACH VARIATIONS ON C_{M_O}, $C_{m_{\delta_e}}$ AND $C_{m_{C_L}}$



If equation 3.110 is analyzed using the plots in figure 3.43, speed instability during transonic flight becomes obvious. The value of $-C_{m_o}/C_{m_{\delta_e}}$ increases from approximately zero in the subsonic range to some positive value as the aircraft passes through Mach 1.0. The value of $C_{m_{C_L}}/C_{m_{\delta_e}}$ increases to a very large number in comparison to $C_{m_o}/C_{m_{\delta_e}}$ through this same range. The result is a positive $\Delta\delta_e$ or a reversal of elevator deflection with increasing airspeed. This manifests itself as a relaxation of forward pressure or even a pull force to maintain attitude or prevent a nose down tendency. As the aircraft speed increases to supersonic speed, $\Delta\delta_e$ again becomes negative and the pilot regains speed stability or decreasing δ_e with increasing airspeed. The actual results of some aircraft flown in this range are shown in figure 3.44.

FIGURE 3.44
δ_e vs MACH



Whether the speed instability or reversal in elevator deflections and stick forces are objectionable, depends on many factors such as magnitude of variation, length of time required to transverse the region of instability, control system characteristics, and conditions of flight. It is impossible for the engineer to determine from data plots if the degree of instability is acceptable. The pilot is the only one capable of evaluating these effects.

In the F-100C, a pull force of 14 pounds was required when accelerating from Mach 0.87 to Mach 1.0. The test pilot described this trim change as disconcerting while attempting to maneuver the aircraft in this region and recommended that a "q" or Mach sensing device be installed to eliminate this characteristic. Consequently, a mechanism was incorporated to automatically change the artificial feel gradient as the aircraft accelerates through the transonic range. Also, the longitudinal trim is automatically changed in this region by the use of a "Mach Trimmer."

In the F-104, the test pilot stated that transonic trim changes required an aft stick movement with increasing speed and a forward stick movement when decreasing speed, but described this speed instability as acceptable.

In the F-106 the pilot stated that the 1.0 to 1.1 Mach region is characterized by a moderate trim change necessitating pilot technique to avoid large variations in altitude during accelerations. Minor trim changes are encountered up to

Mach 1.35. His report concluded that the speed instabilities were not unsatisfactory.

In the T-38 which embodies the latest design concept, a departure is noted from the low tail configuration difficulties where the pilot described the transonic trim change as being hardly perceptible.

Aircraft design considerations are, of course, influenced by the stability aspects of high speed flight. It is desirable to design an aircraft where trim changes through transonic speeds are small. A flat wing without camber, twist, or incidence or a low aspect ratio wing and tail provide values of X_W/c, a_W, a_T, and $d\varepsilon/d\alpha$ which vary minimally over the Mach number range. An all-moving tail (slab) for control gives negligible variation of τ with Mach and maximum control effectiveness. A full power, irreversible control system is necessary to counteract the erratic changes in pressure distribution which affect Ch_α and Ch_{δ_e}.

In the transonic speed regime the meaning or importance of "neutral point" is reduced. At transonic speeds the variation of control angle and trim force with speed, although important, is not affected by cg position. Instead of relating trim gradients to a cg margin, it is more useful to view variation of control for trim as a function of compressibility and ignore cg position.

CHAPTER 4 MANEUVERABILITY

REVISED FEBRUARY 1977

4.1 MANEUVERING FLIGHT

The method used to analyze maneuvering flight will be to determine stick-fixed maneuver points (h_m) and stick-free maneuver points (h'_m). It will be seen that these are analogous to their counterparts in static stability, the stick-fixed and stick-free neutral points. The maneuver points will also be defined in terms of the neutral points and the theory will help to predict which of these points will be critical as regards the aft center of gravity location. It will also be shown how the forward center of gravity is affected by the parameters that define the maneuver points.

Maneuvering flight will be analyzed much in the same manner used in determining a flight test relationship in longitudinal stability. For stick-fixed longitudinal stability, the flight test relationship was determined to be

$$\frac{d\delta_e}{dC_L} = -\frac{dC_m/dC_L}{C_{m_{\delta_e}}} \tag{4.1}$$

This equation gave the static longitudinal stability of the aircraft in terms that could easily be measured in flight test.

In maneuvering flight, a similar stick-fixed equation relating to easily measurable flight test quantities is desirable. Where in longitudinal stability, the elevator deflection was related to lift coefficient or angle of attack, one may surmise that in maneuvering flight elevator deflection will relate to load factor n.

To determine this expression, one must refer to the aircraft's basic equations of motion. As in longitudinal stability, the six equations of motion are the basis for all analysis of aircraft stability and control. In maneuvering an aircraft the same equations will hold true, but one additional derivative will have to be added to the analysis. Recalling the pitching moment equation

$$M = \dot{q}I_y + pr\,(I_x - I_z) + (p^2 - r^2)\,I_{xz} \tag{4.2}$$

and the fact that in static stability analysis we have no roll rate, yaw rate, or pitch acceleration, equation 4.2 reduces to:

$$M = \dot{q}\,I_y = 0 \tag{4.3}$$

The variables that cause external pitching moments on an aircraft are infinite, i.e., speed brakes, canopy, elevator, etc. There are, however, five primary variables that we can consider.

$$M = f(V,\ \alpha,\ \dot{\alpha},\ \delta_e,\ q) \tag{4.4}$$

If any or all of these variables change, there will be a change of total pitching moment that will equal the sum of the partial changes of all the variables. This is written as

$$\Delta M = \frac{\partial M}{\partial \alpha}\Delta\alpha + \frac{\partial M}{\partial \dot{\alpha}}\Delta\dot{\alpha} + \frac{\partial M}{\partial \delta_e}\Delta\delta_e + \frac{\partial M}{\partial V}\Delta V + \frac{\partial M}{\partial q}\Delta q \tag{4.5}$$

Since in maneuvering flight, ΔV and $\Delta\dot{\alpha}$ are zero, equation 4.5 becomes:

$$\Delta M = \frac{\partial M}{\partial \alpha}\Delta\alpha + \frac{\partial M}{\partial \delta_e}\Delta\delta_e + \frac{\partial M}{\partial q}\Delta q = 0 \qquad (4.6)$$

and since $M = qSc\,C_m$, then

$$\frac{\partial M}{\partial \alpha} = q\,Sc\,\frac{\partial C_m}{\partial \alpha} = q\,Sc\,C_{m_\alpha} \qquad (4.7)$$

$$\frac{\partial M}{\partial \delta_e} = q\,Sc\,\frac{\partial C_m}{\partial \delta_e} = q\,Sc\,C_{m_{\delta_e}} \qquad (4.8)$$

$$\frac{\partial M}{\partial q} = q\,Sc\,\frac{\partial C_m}{\partial q} \qquad (4.9)$$

Substituting these values into equation 4.6, and multiplying by $1/q\,Sc$,

$$C_{m_\alpha}\Delta\alpha + C_{m_{\delta_e}}\Delta\delta_e + \frac{\partial C_m}{\partial q}\Delta q = 0 \qquad (4.10)$$

The derivative $\partial C_m/\partial q$ is carried instead of C_{m_q} since the compensating factor $c/2V$ is not used at this time.

Solving for the change in elevator deflection $\Delta\delta_e$,

$$\Delta\delta_e = \frac{-C_{m_\alpha}\Delta\alpha - \partial C_m/\partial q\,\Delta q}{C_{m_{\delta_e}}} \qquad (4.11)$$

The analysis of equation 4.11 may be continued by substituting in values for $\Delta\alpha$ and Δq. The final equation obtained should be in the form of some flight test relationship. Since maneuvering is related to load factor, the elevator deflection required to obtain different load factors will define the stick-fixed maneuver point. The immediate goal then is to determine the change in angle of attack, $\Delta\alpha$, and change in pitch rate, Δq, in terms of load factor n.

4.2 THE PULL UP MANEUVER

In the pull up maneuver, the change in angle of attack of the aircraft, $\Delta\alpha$, may be related to the lift coefficient of the aircraft. In the pull up with constant velocity, the angle of attack of the whole aircraft will be changed since the aircraft has to fly at a higher C_L to obtain the load factor required. The change in C_L required to maneuver at high load factors at a constant velocity comes from two sources: (1) load factor increase, (2) elevator deflection. Although often ignored because of its small value when compared to total C_L, the change in lift with elevator deflection $C_{L_{\delta_e}}\Delta\delta_e$ will be carried along for a more general analysis.

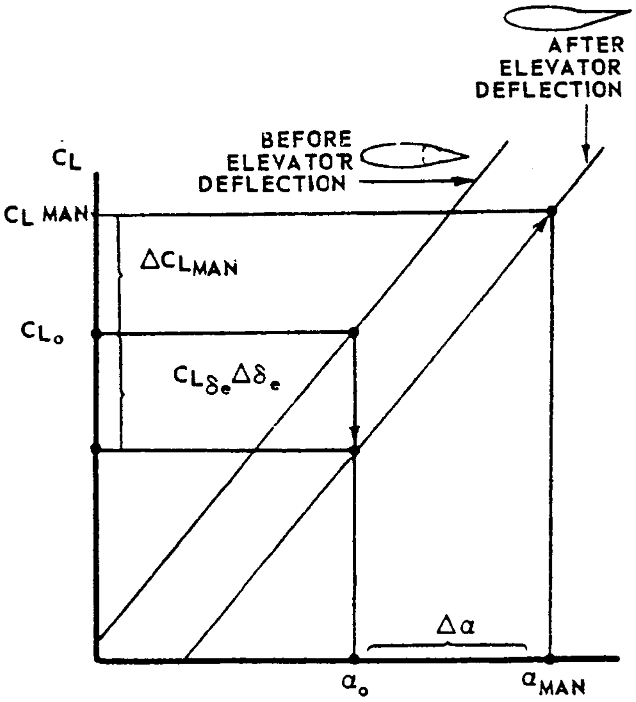


Figure 4.1 Lift Coefficient Versus Angle of Attack

Referring to figure 4.1, the aircraft is in equilibrum at some C_{L_o} corresponding to some α_o before the elevator is deflected to initiate the pull up. If the elevator is considered as a flap, its deflection will affect the lift curve as follows. When the elevator is deflected upward, the lift curve shifts downward and does not change slope. This says that a certain amount of lift is initially lost when the elevator is deflected upward. The loss in lift because of elevator deflection is designated $C_{L_{\delta_e}}\Delta\delta_e$. The increase in downloading on the tail or increase in negative lift on the horizontal stabilizer causes a moment on the aircraft which creates a nose up pitch rate. The aircraft continues to pitch upward and increase its angle of attack until it reaches a new C_L and an equilibrium load factor. In other words a pitch rate is initiated and α increases until a maneuvering lift coefficient $C_{L_{MAN}}$ is reached for the deflected elevator δ_e. The change in angle of attack is $\Delta\alpha$. The change in C_L has come partially from the deflected elevator and mainly from the pitching maneuver. The change in C_L due to the maneuver is from C_{L_o} to $C_{L_{MAN}}$. Since it did not change the slope of the lift curve, if the change in lift caused by elevator deflection is included, the expression for $\Delta\alpha$ becomes:

$$C_L = a\alpha \qquad (4.12)$$

$$\Delta C_L = a\Delta\alpha \qquad (4.12a)$$

$$\Delta C_L = \Delta C_{L_{MAN}} - C_{L_{\delta_e}}\Delta\delta_e = a\Delta\alpha \qquad (4.13)$$

$$\Delta\alpha = \frac{1}{a}\left[\Delta C_{L_{MAN}} - C_{L_{\delta_e}}\Delta\delta_e\right] \qquad (4.14)$$

To put equation 4.14 in terms of load factor, $\Delta C_{L_{MAN}}$ must be defined. This is the change in lift from the initial condition to the final maneuvering condition. This change can occur from one g flight to some other load factor or it can start at 2 or 3 g's and progress to some new load factor. If C_L is at one g then

$$C_L = \frac{W}{qS} \qquad (4.15)$$

$$C_{L_o} = \frac{n_o W}{qS} \quad n_o \text{ is initial load factor} \qquad (4.16)$$

$$C_{L_{MAN}} = \frac{nW}{qS} \quad n \text{ is final load factor} \qquad (4.17)$$

$$\Delta C_{L_{MAN}} = C_{L_{MAN}} - C_{L_o}$$

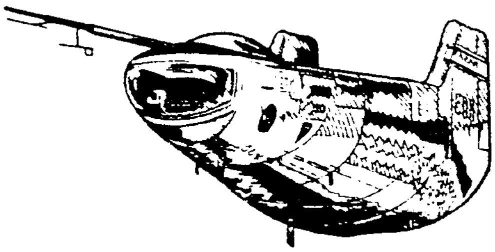
$$= \frac{nW}{qS} - \frac{n_o W}{qS} = C_L(n-n_o) \qquad (4.18)$$

Finally substituting $\Delta C_{L_{MAN}}$ into equation 4.14,

$$\Delta\alpha = \frac{1}{a}\left[C_L\,(n-n_o) - C_{L_{\delta_e}}\Delta\delta_e\right] \qquad (4.19)$$

Equation 4.12 is now ready for substitution into equation 4.11.

An expression for Δq in equation 4.11 will be derived using the pull up maneuver analysis.



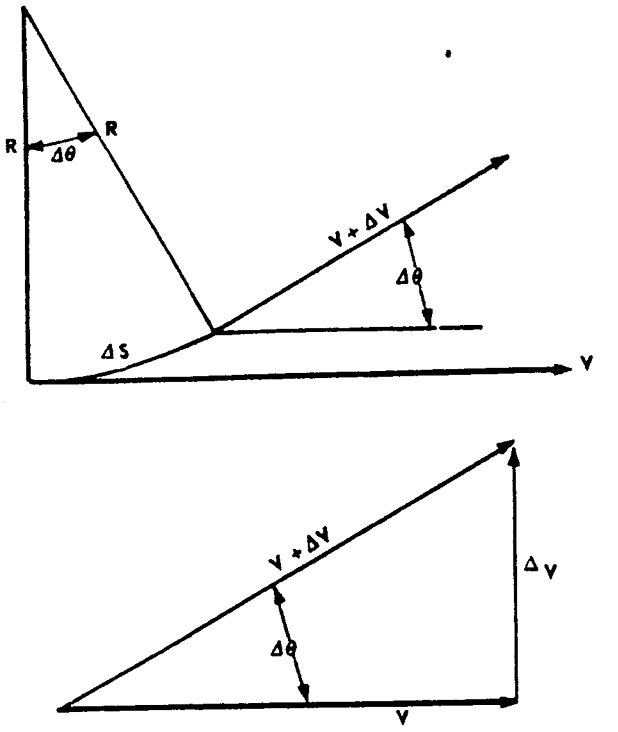


Figure 4.2 Curvilinear Motion

Referring to figure 4.2

$$\Delta\theta = \frac{\Delta S}{R} \qquad (4.20)$$

$$\frac{d\theta}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta S}{\Delta t}\,\frac{1}{R} \qquad (4.21)$$

$$\frac{d\theta}{dt} = \frac{V}{R} = q \qquad (4.22)$$

From figure 4.2

$$\frac{\Delta V}{V} = \Delta\theta \quad \text{(small angles)} \qquad (4.23)$$

$$\frac{d\theta}{dt} = \lim_{\Delta t \to 0} \frac{\Delta V}{\Delta t}\,\frac{1}{V} = \frac{1}{V}\,\frac{dV}{dt} \qquad (4.24)$$

combining equations 4.24 and 4.22,

$$\frac{dV}{dt} = \frac{V^2}{R} \qquad (4.25)$$

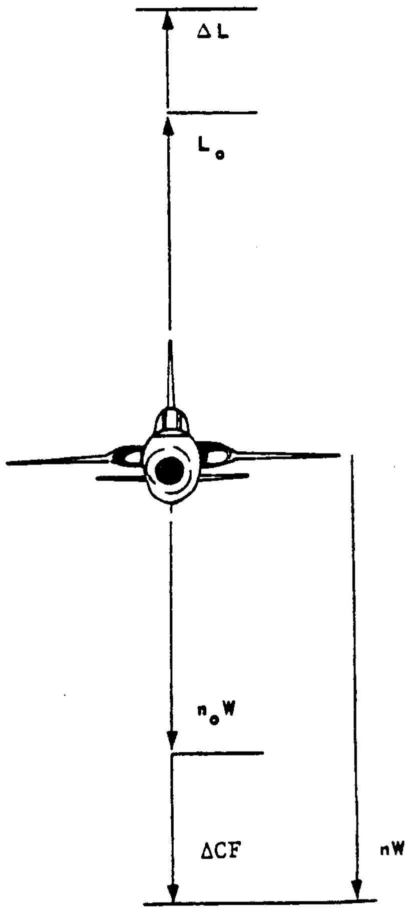


Figure 4.3 Wings Level Pull Up

From figure 4.3,

$$\Delta L = \Delta CF = nW - n_0 W \qquad (4.26)$$

Again the factor now indicates that the change may take place from any original load factor and is not limited to the straight and level flight condition. The centrifugal force that holds the aircraft in equilibrium can be expressed as:

$$CF = \frac{W}{g}\,a = \frac{W}{g}\,\frac{V^2}{R} \qquad (4.27)$$

Therefore:

$$\Delta L = W(n - n_o) = \frac{WV}{g}\left(\frac{V}{R} - \frac{V}{R_o}\right) = \Delta CF \qquad (4.28)$$

$$\frac{g}{V}(n - n_o) = \frac{V}{R} - \frac{V}{R_o} = q - q_o = \Delta q \qquad (4.29)$$

$$\Delta q = \frac{g}{V}(n - n_o) \qquad (4.30)$$

Now equations 4.30 and 4.19 may be substituted into equation 4.11.

$$\Delta \delta_e = \frac{-C_{m_\alpha} \frac{1}{a}\left[C_L (n - n_o) - C_{L_{\delta_e}} \Delta \delta_e\right]}{C_{m_{\delta_e}}}$$

$$\frac{-\frac{\partial C_m}{\partial q}\frac{g}{V}(n - n_o)}{C_{m_{\delta_e}}} \qquad (4.31)$$

From longitudinal stability,

$$C_{m_\alpha} = \frac{\partial C_m}{\partial C_L}\frac{\partial C_L}{\partial \alpha} = a(h - h_n) \qquad (4.32)$$

Also to help further in reducing the equation to its simplest terms,

$$V^2 = \frac{2W}{\rho S C_L} \qquad (4.33)$$

and

$$\frac{\partial C_m}{\partial q} = \frac{c}{2V} C_{m_q} \qquad (4.34)$$

Substituting equations 4.34, 4.33 and 4.22 into equation 4.31 and turning the algebra crank, results in,

$$\frac{\Delta \delta_e}{n - n_o} = \frac{a C_L}{C_{m_\alpha} C_{L_{\delta_e}} - C_{m_{\delta_e}} a}\left[h - h_n + \rho \frac{Sc}{4m} C_{m_q}\right] \qquad (4.35)$$

Equation 4.35 is now in the form that will define the stick-fixed maneuver point for the pull up. The definition of the maneuver point (h_m) is the cg position at which the elevator deflection per g goes to zero. Taking the limit of equation 4.35, where Δn is defined as $(n - n_o)$,

$$\lim_{\Delta n \to o} \frac{\Delta \delta_e}{\Delta n} = \frac{d\delta_e}{dn} \qquad (4.36)$$

or

$$\frac{d\delta_e}{dn} = \frac{a C_L}{C_{m_\alpha} C_{L_{\delta_e}} - C_{m_{\delta_e}} a}\left[h - h_n + \rho \frac{Sc}{4m} C_{m_q}\right] \qquad (4.37)$$

Setting equation 4.37 equal to zero will give the cg position (h) as the maneuver point (h_m).

$$h_m = h_n - \rho \frac{Sc}{4m} C_{m_q} \qquad (4.38)$$

Solving equation 4.38 for h_n and substituting into equation 4.37,

$$\frac{d\delta_e}{dn} = \frac{a C_L}{C_{m_\alpha} C_{L_{\delta_e}} - C_{m_{\delta_e}} a}(h - h_m) \qquad (4.39)$$

where $(h_m - h)$ is defined as the stick-fixed maneuver margin.

The significant points to be made about equation 4.39 are:

1. The derivative $d\delta_e/dn$ varies with the maneuver margin. The more forward the cg, the more elevator will be required to obtain the limit load factor. That is, as the cg moves forward, more elevator deflection is necessary to obtain a given load factor.

2. The higher the C_L, the more elevator will be required to obtain the limit load factor. That is, at low speeds (high

C_L) more elevator deflection is necessary to obtain a given load factor than is required to obtain the same load factor at a higher speed (lower C_L).

3. The derivative $d\delta_e/dn$ should be linear with respect to cg at a constant C_L.

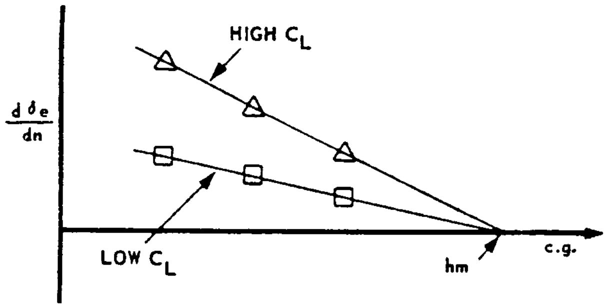


Figure 4.4 $\frac{d\delta_e}{dn}$ vs cg

Another approach to solving for the maneuver point (h_m) is to return to the original stability equation.

$$\frac{dC_m}{dC_L} = h - \frac{X_{ac}}{c} + \frac{dC_m}{dC_{L_{Fus.}}} - \frac{a_T}{a_w} V_H \eta_T \left(1 - \frac{d\epsilon}{d\alpha}\right) \quad (4.40)$$

The effect of pitch damping on the aircraft stability will be determined and added to equation 4.40. Recalling the relationship:

$$\frac{dC_m}{dq} = \frac{c}{2V} C_{m_q} \quad (4.41)$$

or

$$\Delta C_m = \frac{c}{2V} C_{m_q} \Delta q \quad (4.42)$$

Substituting the value obtained for Δq from equation 4.30,

$$\Delta C_m = \frac{cg}{2V^2} C_{m_q} (n - n_o) \quad (4.43)$$

Substituting the appropriate C_L expression for load factor,

$$\Delta C_m = \rho \frac{Sc}{4m} C_{m_q} (C_{L_{MAN}} - C_{L_o}) \quad (4.44)$$

if

$$\Delta C_L = C_{L_{MAN}} - C_{L_o}, \quad \text{then}$$

$$\lim_{\Delta C_L \to 0} \frac{\Delta C_m}{\Delta C_L} = \frac{dC_m}{dC_L}_{\text{Pitch Damping}} = \rho \frac{Sc}{4m} C_{m_q} \quad (4.45)$$

This term may now be added to equation 4.45. If the sign of C_{m_q} is negative, then the term is a stabilizing contribution to the stability equation. C_{m_q} will be analyzed further.

$$\frac{dC_m}{dC_L} = h - \frac{X_{ac}}{c} + \frac{dC_m}{dC_{L_{Fus}}} - \frac{a_T}{a_w} V_H \eta_T \left(1 - \frac{d\epsilon}{d\alpha}\right) + \rho \frac{Sc}{4m} C_{m_q} \quad (4.46)$$

The maneuver point is found by setting dC_m/dC_L equal to zero and solving for the cg position where this occurs.

$$h_m = \frac{X_{ac}}{c} - \frac{dC_m}{dC_{L_{Fus}}} + \frac{a_T}{a_w} V_H \eta_T \left(1 - \frac{d\epsilon}{d\alpha}\right) - \rho \frac{Sc}{4m} C_{m_q} \quad (4.47)$$

The first three terms on the right side of equation 4.47 may be identified as the expression for the neutral point h_n. If this substitution is made in equation 4.47, equation 4.38 is again obtained.

$$h_m = h_n - \rho \frac{Sc}{4m} C_{m_q} \quad (4.48)$$

The derivative C_{m_q} found in equation 4.37, 4.38, and 4.46 needs to be examined before proceeding with further discussion.

The damping that comes from the pitch rate established in a pull up, comes from the wing, tail, and fuselage components. The tail is the largest contributor to the pitch damping because of the long moment arm. For this reason it is usually used to derive the value of C_{m_q}. Sometimes an empirical value is added to account for the rest of the aircraft but more often than not, the value for the tail alone is used to estimate the derivative. The effect of the tail may be calculated in the following manner:

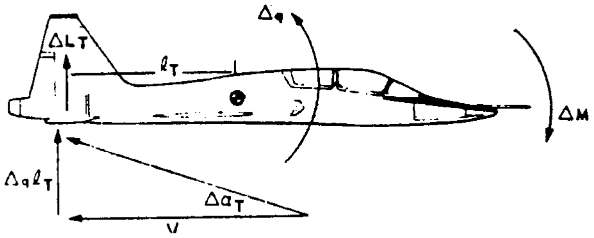


Figure 4.5 Pitch Damping

The pitching moment effect on the aircraft from the downward moving horizontal stabilizer is:

$$\Delta M = -\ell_T \Delta L_T = q_w S_w c_w \Delta C_m \quad (4.49)$$

where

$$\Delta L_T = q_T S_T \Delta C_{L_T} \quad (4.50)$$

Solving for ΔC_m,

$$\Delta C_m = -\frac{q_T \ell_T S_T}{q_w c_w S_w} \Delta C_{L_T} \quad (4.51)$$

The combination $\ell_T S_T / c_w S_w$ can be recognized as the tail volume coefficient V_H. The term q_T/q_w is referred to as the tail efficiency factor η_T.

Equation 4.51 may then be written:

$$\Delta C_m = -V_H \eta_T \Delta C_{L_T} \quad (4.52)$$

which can be further refined to:

$$\Delta C_m = -V_H \eta_T a_T \Delta \alpha_T \quad (4.53)$$

From figure 4.5, the change in angle of attack at the tail caused by the pitch rate will be:

$$\Delta \alpha_T = \tan^{-1} \frac{\Delta q \ell_T}{V} \approx \Delta q \frac{\ell_T}{V} \quad (4.54)$$

Substituting equation 4.54 into 4.53

$$\Delta C_m = -a_T V_H \eta_T \frac{\ell_T}{V} \Delta q \quad (4.55)$$

Taking the limit of equation 4.55 gives

$$\frac{\partial C_m}{\partial q} = -a_T V_H \eta_T \frac{\ell_T}{V} \quad (4.56)$$

Equation 4.56 shows that the damping expression $\partial C_m / \partial q$ is a function of airspeed, i.e., this term is greater at lower speeds.

Solving for C_{m_q},

$$C_{m_q} = \frac{2V}{c} \frac{\partial C_m}{\partial q} = -2 a_T V_H \eta_T \frac{\ell_T}{c} \quad (4.57)$$

The damping derivative is not a function of airspeed but rather a value determined by design considerations only (subsonic flight). The damping in pitch derivative may be increased by increasing S_T or ℓ_T.

When this value for C_{m_q} is substituted into equation 4.48,

$$h_m = h_n + \rho \frac{S\, a_T\, \eta_T\, \ell_T}{2m} V_H \qquad (4.58)$$

The following conclusions are apparent from equation 4.58.

1. The maneuver point should always be behind the neutral point. This is verified since the addition of a pitch rate increases the stability (C_{m_q} is negative in equation 4.46) of the aircraft. Therefore, the stability margin should increase.

2. Aircraft geometry is influential in locating the maneuver point aft of the neutral point.

3. As altitude increases, the distance between the neutral point and maneuver point decreases.

4. As weight decreases at any given altitude, the maneuver point moves further behind the neutral point and the maneuver stability margin increases.

5. The largest variation between maneuver point and neutral point occurs with a light aircraft flying at sea level.

■ 4.3 AIRCRAFT BENDING

Before the pull up analysis is completed, one more subject should be covered. One of the assumptions made early in stability was that the aircraft was a rigid body. It is a well known fact that all aircraft bend when a load is applied. The bigger the aircraft, the more they bend. The effect on the aircraft bending is shown in figures 4.6a and 4.6b.

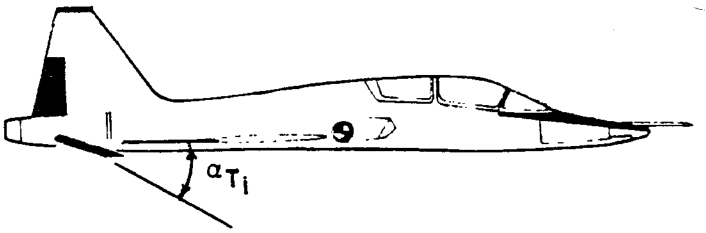


Figure 4.6a Rigid Aircraft Under High Load Factor

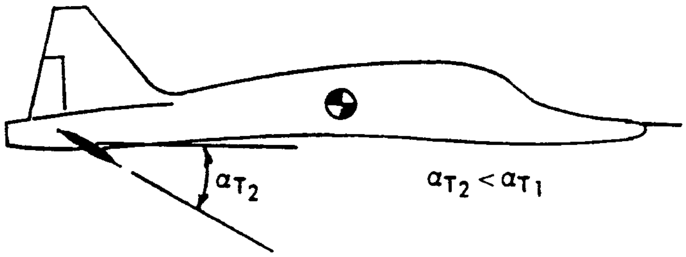


Figure 4.6b Non-Rigid Aircraft under High Load Factor

The angle of attack of the tail is approximately the same as the angle of attack of the wing with the exception of downwash, incidence, etc., for a particular elevator deflection. As the non-rigid aircraft bends, the angle of attack α_T of the horizontal stabilizer decreases. In order to keep the aircraft at the same overall angle of attack, the original angle of attack of the tail must be reestablished. This requires an increase in the elevator (slab) deflection or a $\Delta\delta_e$ to reestablish the necessary α_T and to maintain the required maneuvering C_L. This additional elevator requirement under aircraft bending gives an apparent increase in the maneuvering stability of the aircraft or an additional $\Delta\delta_e$ per load factor.

■ 4.4 THE TURN MANEUVERING

The subject of maneuvering in pull ups has been thoroughly discussed and while it is the easiest method for a test pilot to perform, it is also the most time consuming.

Therefore, most maneuvering data is collected by turning. There are several methods used to collect data in a turn and these are discussed in Chapter 5, Vol I, Stability and Control Flight Test Techniques.

In order to analyze the maneuvering turn, equation 4.11 is recalled:

$$\Delta \delta_e = - \frac{C_{m_\alpha} \Delta \alpha - \partial C_m / \partial q \ \Delta q}{C_{m_{\delta_e}}} \qquad (4.59)$$

The expression for $\Delta\alpha$ in equation 3.19, derived for the pullup maneuver, is also applicable to the turning maneuver.

$$\Delta \alpha = \frac{1}{a} \left[C_L (n - n_o) - C_{L_{\delta_e}} \Delta \delta_e \right] \qquad (4.60)$$

Such is not the case for the Δq expression in equation 4.59. Another expression for Δq pertaining to the turn maneuver must be developed.

Referring to figure 4.7, the lift vector will be statically balanced by the weight and centrifugal force. One component ($L \cos \phi$) balances the weight and the other ($L \sin \phi$) balances the centrifugal force.

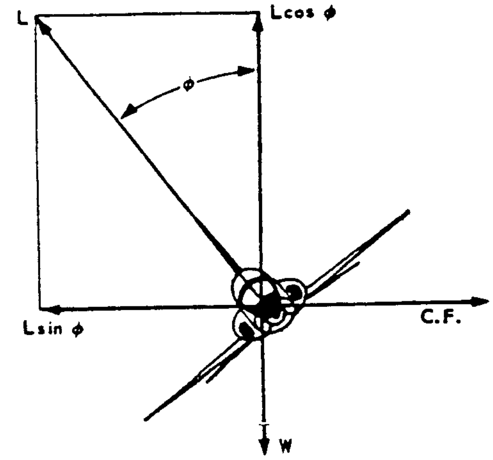


Figure 4.7a

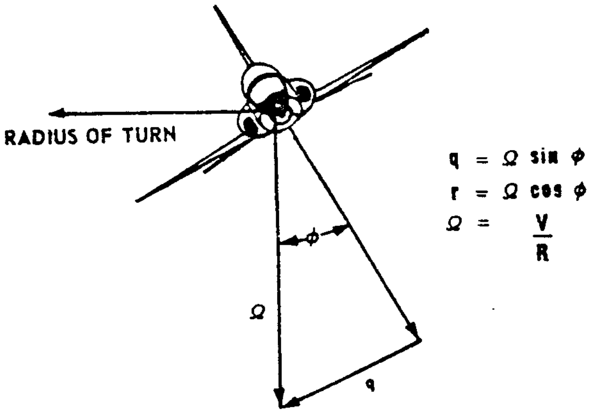


Figure 4.7b Aircraft in the Turn Maneuver

$$L \sin \phi = CF = \frac{W}{g} \frac{V^2}{R} \qquad (4.61)$$

$$L \cos \phi = W \qquad (4.62)$$

$$n = L/W = \frac{1}{\cos \phi} \qquad (4.63)$$

Now dividing 4.61 by 4.62 and rearranging terms:

$$\frac{V}{R} = \frac{g}{V} \frac{\sin \phi}{\cos \phi} \qquad (4.64)$$

Referring to figure 4.7 where pitch rate is represented by a vector along the wings and yaw rate a vector vertically through the center of gravity, the following relationships can be derived.

$$\Omega = \frac{V}{R} \qquad (4.65)$$

$$q = \Omega \sin \phi \qquad (4.66)$$

$$q = \frac{V}{R} \sin \phi \qquad (4.67)$$

Therefore, most maneuvering data is collected by turning. There are several methods used to collect data in a turn and these are discussed in Chapter 5, Vol I, Stability and Control Flight Test Techniques.

In order to analyze the maneuvering turn, equation 4.11 is recalled:

$$\Delta \delta_e = - \frac{C_{m_\alpha} \Delta \alpha - \partial C_m / \partial q \; \Delta q}{C_{m_{\delta_e}}} \qquad (4.59)$$

The expression for $\Delta\alpha$ in equation 3.19, derived for the pullup maneuver, is also applicable to the turning maneuver.

$$\Delta \alpha = \frac{1}{a} \left[C_L \; (n - n_o) - C_{L_{\delta_e}} \; \Delta \delta_e \right] \qquad (4.60)$$

Such is not the case for the Δq expression in equation 4.59. Another expression for Δq pertaining to the turn maneuver must be developed.

Referring to figure 4.7, the lift vector will be statically balanced by the weight and centrifugal force. One component ($L \cos \phi$) balances the weight and the other ($L \sin \phi$) balances the centrifugal force.

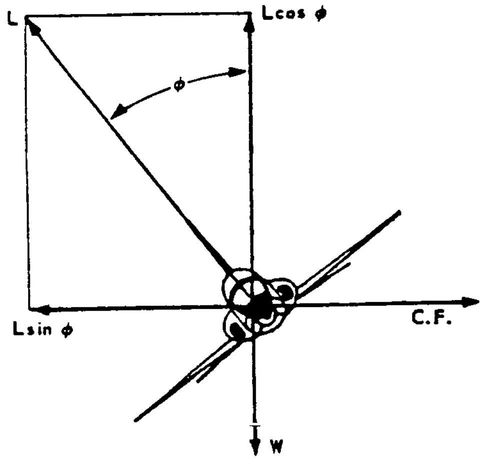


Figure 4.7a

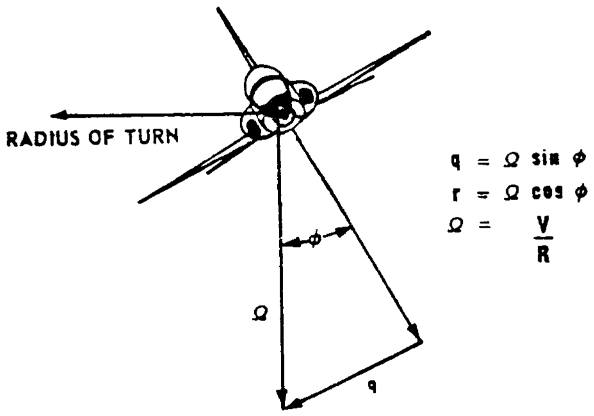


Figure 4.7b Aircraft in the Turn Maneuver

$$L \sin \phi = CF = \frac{W}{g} \frac{V^2}{R} \qquad (4.61)$$

$$L \cos \phi = W \qquad (4.62)$$

$$n = L/W = \frac{1}{\cos \phi} \qquad (4.63)$$

Now dividing 4.61 by 4.62 and rearranging terms:

$$\frac{V}{R} = \frac{g}{V} \frac{\sin \phi}{\cos \phi} \qquad (4.64)$$

Referring to figure 4.7 where pitch rate is represented by a vector along the wings and yaw rate a vector vertically through the center of gravity, the following relationships can be derived.

$$\Omega = \frac{V}{R} \qquad (4.65)$$

$$q = \Omega \sin \phi \qquad (4.66)$$

$$q = \frac{V}{R} \sin \phi \qquad (4.67)$$

Substituting 4.64 into 4.67,

$$q = \frac{g}{V}\ \frac{\sin^2 \phi}{\cos \phi} \qquad (4.68)$$

$$q = \frac{g}{V}\ \frac{1 - \cos^2\phi}{\cos \phi} \qquad (4.69)$$

$$q = \frac{g}{V}\left(\frac{1}{\cos \phi} - \cos \phi\right) \qquad (4.70)$$

$$q = \frac{g}{V}\left(n - \frac{1}{n}\right) \qquad (4.71)$$

When maneuvering from initial conditions of n_o to n, the Δq equation becomes,

$$\Delta q = q - q_o = \frac{g}{V}\left(n - \frac{1}{n}\right) - \frac{g}{V}\left(n_o - \frac{1}{n_o}\right) \qquad (4.72)$$

$$\Delta q = \frac{g}{V}(n - n_o)\left(1 + \frac{1}{nn_o}\right) \qquad (4.73)$$

The general expression for Δq in equation 4.73 and the value for ... in equation 4.60 may now be substituted into equation 4.59 to determine $\Delta\delta_e$

$$\Delta\delta_e = \frac{-C_{m_\alpha}\frac{1}{a}\left[C_L(n - n_o) - C_{L_{\delta_e}}\Delta\delta_e\right]}{C_{m_{\delta_e}}} \qquad (4.74)$$

$$-\frac{\partial C_m}{\partial q}\ \frac{g}{V}(n - n_o)\left(1 + \frac{1}{nn_o}\right)$$

Substituting

$$\frac{\partial C_m}{\partial q} = \frac{c}{2V}\ C_{m_q}$$

$$\left(C_{m_{\delta_e}} - \frac{C_{m_\alpha} C_{L_{\delta_e}}}{a}\right)\Delta\delta_e =$$

$$-\frac{C_{m_\alpha} C_L}{a}(n - n_o)$$

$$- C_{m_q}\frac{cg}{2V^2}(n - n_o)\left(1 + \frac{1}{nn_o}\right) \qquad (4.75)$$

$$\Delta\delta_e = \frac{C_{m_\alpha} C_L(n - n_o) + C_{m_q}\, a\, \frac{cg}{2V^2}(n - n_o)\left(1 + \frac{1}{nn_o}\right)}{C_{m_\alpha} C_{L_{\delta_e}} - C_{m_{\delta_e}}\, a} \qquad (4.76)$$

Now

$$C_{m_\alpha} = a\,(h - h_n) \text{ and } V^2 = \frac{2W}{C_L \rho S}$$

$$\frac{\Delta\delta_e}{(n - n_o)} = \frac{aC_L}{C_{m_\alpha} C_{L_{\delta_e}} - C_{m_{\delta_e}}\, a}\cdot\left[(h - h_n) + C_{m_q}\frac{\rho Sc}{4m}\left(1 + \frac{1}{nn_o}\right)\right] \qquad (4.77)$$

Taking the limit of $\Delta\delta_e/\Delta n$ in equation 4.77 and,

$$\frac{d\delta_e}{dn} = \frac{aC_L}{C_{m_\alpha}C_{L_{\delta_e}} - C_{m_{\delta_e}}\, a}\cdot\left[h - h_n + \rho\frac{Sc}{4m}C_{m_q}\left(1 + \frac{1}{n^2}\right)\right] \qquad (4.78)$$

The maneuver point is determined by setting $d\delta_e/dn$ equal to zero and solving for the cg position at this point.

$$h_m = h_n - \rho\frac{Sc}{4m}C_{m_q}\left(1 + \frac{1}{n^2}\right) \qquad (4.79)$$

The maneuver point in a turn differs from the pullup by the factor $(1 + 1/n^2)$. This means that at high load factors the turn and pullup maneuver points will be very nearly the same. If equation (4.79) is solved for h_n and substituted back into equation 4.78, the result is:

$$\frac{d\delta_e}{dn} = \frac{aC_L}{C_{m_\alpha}C_{L_{\delta_e}} - C_{m_{\delta_e}}\, a}(h - h_m) \qquad (4.80)$$

The conclusion that $d\delta_e/dn$ is the same for both pullup and turn would be untrue since h_m in equation 4.80 for turns (includes the factor $(1 + 1/n^2)$ is different from the h_m found for the pullup maneuver. The same conclusions reached for 4.39 and 4.58 apply to 4.80 and 4.81 as well.

$$h_m = h_n + \frac{\rho S a_T V_H \eta_T \ell_T}{2m}(1 + 1/n^2) \qquad (4.81)$$

4.5 RECAPITULATION

Before looking further into the stick-free maneuverability case, it would be well to review the development in the preceding paragraphs and relate it to the results of chapter 3.

The basic approach to longitudinal stability was centered around finding a value for dC_m/dC_L. It was found that a negative value for this derivative meant that the aircraft was statically stable. The derivative was analyzed for the stick-fixed case first and then the stick-free case. The cg position where this derivative was zero, was defined as the neutral point. Static margin was defined as the difference between the neutral point and the cg location. The stick-free case was determined by:

$$\left.\frac{dC_m}{dC_L}\right|_{\text{Stick-Free Aircraft}} = \left.\frac{dC_m}{dC_L}\right|_{\text{Stick-Fixed Aircraft}} + \Delta \left.\frac{dC_m}{dC_L}\right|_{\text{Effect of Free Elevator}} \qquad (4.82)$$

The free elevator case was merely the basic stability of the aircraft with the effect of freeing the elevator added to it.

When the maneuvering case was introduced, it was shown that there was a new derivative to be discussed but the basic stability of the aircraft would not change - only the effect of pitch rate was added to it.

$$\left.\frac{dC_m}{dC_L}\right|_{\text{Stick-Fixed Aircraft Pitching}} = \left.\frac{dC_m}{dC_L}\right|_{\text{Stick-Fixed Aircraft}} + \Delta \left.\frac{dC_m}{dC_L}\right|_{\text{Effect of The Pitch Rate}} \qquad (4.83)$$

For the stick-free case, the following must be true,

$$\left.\frac{dC_m}{dC_L}\right|_{\text{Stick-Free Aircraft Pitching}} = \left.\frac{dC_m}{dC_L}\right|_{\text{Stick-Fixed Aircraft}} + \Delta \left.\frac{dC_m}{dC_L}\right|_{\text{Effect of Free Elevator}} + \Delta \left.\frac{dC_m}{dC_L}\right|_{\text{Effect of Pitch Rate}} \qquad (4.84)$$

$$\left.\frac{dC_m}{dC_L}\right|_{\text{Stick-Free Aircraft Pitching}} = \left.\frac{dC_m}{dC_L}\right|_{\text{Stick-Free Aircraft}} + \Delta \left.\frac{dC_m}{dC_L}\right|_{\text{Effect of Pitch Rate}} \qquad (4.85)$$

4.6 STICK-FREE MANEUVERING

The first analysis of stick-free maneuvering requires a review of longitudinal stability. It was determined in chapter 3 that the effect of freeing the elevator was to multiply the tail term by the free elevator factor F which equaled $(1 - \tau\, C_{h_\alpha}/C_{h_\delta})$. Consequently, to free the elevator in the maneuvering case and find the stick-free maneuver point, the tail effect of stick-fixed maneuvering must be multiplied by this free elevator factor. Recalling equation 4.47 from the stick-fixed maneuvering discussion,

$$h_m = \frac{X_{ac}}{c} - \frac{dC_m}{dC_{L_{Fus}}} + \frac{a_t}{a_w} V_H \eta_T \cdot \left(1 - \frac{d\epsilon}{d\alpha}\right) - \rho \frac{Sc}{4m} C_{m_q} \qquad (4.86)$$

Multiplying the tail terms by F,

$$h_m' = \frac{X_{ac}}{c} - \frac{dC_m}{dC_{L_{Fus}}} + \frac{a_t}{a_w} V_H \eta_T \left(1 - \frac{d\epsilon}{d\alpha}\right)F - \rho \frac{Sc}{4m} C_{m_q} F \qquad (4.87)$$

The first three terms on the right are the expression for stick-free neutral point, h'_n.

$$h'_m = h'_n - \rho\frac{Sc}{4m} C_{m_q} F \qquad (4.88)$$

This is the stick free maneuver point in terms of the stick-free neutral point for the pullup case. It may be extended to the turn case by using the term for the pitch rate of the tail in a turn.

$$h'_m = h'_n - \rho\frac{Sc}{4m} C_{m_q} F \left(1 + \frac{1}{n^2}\right) \qquad (4.89)$$

These equations do not give any flight test relationship and so it is necessary to derive this from stick forces, as was done in longitudinal static stability. The method used will be to relate the stick force-per-g to the stick-free maneuver point since stick forces can be related to the freeing of the elevator. Starting with the relationship of stick force, gearing, and hinge moments that was derived in chapter 3,

$$F_s = -GH_e \qquad (4.90)$$

$$H_e = q\, S_e\, c_e\, C_h \qquad (4.91)$$

$$F_s = -Gq\, S_e c_e\, C_h \qquad (4.92)$$

The change in stick force for a change in load factor becomes,

$$\frac{\Delta F_s}{\Delta n} = -Gq\, S_e\, c_e\, \frac{\Delta C_h}{\Delta n} \qquad (4.93)$$

where

$$\Delta C_h = C_{h_{\alpha_T}} \Delta \alpha_T + C_{h_{\delta_e}} \Delta \delta_e \qquad (4.94)$$

Stick-Free Pullup Maneuver:

ΔC_h must be written in terms of load factor and substituted back into equation 4.93. This will require defining $\Delta\alpha_T$ and $\Delta\delta_e$ in terms of load factor. The change in angle of attack of the tail comes partly from the change in angle of attack of the wing due to downwash and partly from the pitch rate.

$$\Delta \alpha_T = \Delta \alpha_W \left(1 - \frac{d\epsilon}{d\alpha}\right) + \Delta q \frac{\ell_T}{V} \qquad (4.95)$$

where $\Delta\alpha_W$ + Δq in the above equation are

$$\Delta \alpha_W = \frac{1}{a}\left[C_L (n - n_o) - C_{L_{\delta_e}} \Delta \delta_e\right] \qquad (4.96)$$

$$\Delta q = \frac{g}{V}(n - n_o) \qquad (4.97)$$

$$\Delta \delta_e = \frac{aC_L}{C_{m_\alpha} C_{L_{\delta_e}} - C_{m_{\delta_e}} a} (h - h_m)(n - n_o) \qquad (4.98)$$

If the equations above are substituted into 4.94, the results would be cumbersome at best. To simplify things $C_{L_{\delta_e}}$ will be assumed small enough to ignore. (Reasonable assumption since total change in lift of the aircraft when the elevator is deflected is small.) The above equations simplify to:

$$\Delta \alpha_T = \frac{C_L}{a}\left(1 - \frac{d\epsilon}{d\alpha}\right) \cdot (n - n_o) + \frac{g}{V^2} \ell_T (n - n_o) \qquad (4.99)$$

$$\Delta \delta_e = -\frac{C_L}{C_{m_{\delta_e}}} (h - h_m)(n - n_o) \qquad (4.100)$$

Substituting equations 4.99 and 4.100 into 4.94,

$$\frac{\Delta C_h}{n - n_o} = C_{h_\alpha} \frac{C_L}{a} \left(1 - \frac{d\epsilon}{d\alpha}\right) + C_{h_\alpha} g \frac{\ell_T}{V^2}$$

$$- C_{h_\delta} \frac{C_L}{C_{m_{\delta_e}}} (h - h_m) \qquad (4.101)$$

Substituting $V^2 = 2W/\rho S C_L$ and $C_{m_{\delta_e}} = -a_T V_H \tau$ and isolating the maneuver margin $(h - h_m)$ by factoring out $(- C_{h_\delta} C_L / C_{m_{\delta_e}})$, the result is:

$$\frac{\Delta C_h}{n - n_o} = - \frac{C_{h_\delta} C_L}{C_{m_{\delta_e}}} \left[\frac{C_{h_\alpha}}{C_{h_\delta} a_W} \left(1 - \frac{d\epsilon}{d\alpha}\right) a_T \tau V_H + \frac{C_{h_\alpha}}{C_{h_\delta}} \rho \frac{\ell_T}{2m} S a_T \tau V_H + h - h_m \right] \qquad (4.102)$$

From longitudinal stability,

$$h_n - h_n' = \frac{C_{h_\alpha}}{C_{h_\delta}} \frac{a_T \tau}{a_W} V_H \left(1 - \frac{d\epsilon}{d\alpha}\right) \qquad (4.103)$$

and if the second term in the parenthesis is multiplied by

$$\frac{-2ca_T}{-2ca_T} \text{ and knowing that}$$

$$C_{m_q} = -2a_T \frac{V_H}{c} \ell_T \qquad (4.104)$$

$$F = 1 - \tau \frac{C_{h_\alpha}}{C_{h_\delta}} \qquad (4.106)$$

The second term becomes:

$$(F-1) \rho \frac{Sc}{4m} C_{m_q} \qquad (4.107)$$

Rewriting equation 4.102,

$$\frac{\Delta C_h}{n - n_o} = + \frac{C_{h_\delta}}{C_{m_{\delta_e}}} C_L \cdot \left[h_n' - h_n + (1 - F) \rho \frac{Sc}{4m} C_{m_q} - h + h_m \right] \qquad (4.108)$$

but

$$h_m = h_n - \rho \frac{Sc}{4m} C_{m_q} \qquad (4.109)$$

Therefore:

$$\frac{\Delta C_h}{n - n_o} = - \frac{C_{h_\delta}}{C_{m_{\delta_e}}} C_L \cdot \left[h - h_n' + \rho \frac{Sc}{4m} C_{m_q} F \right] \qquad (4.110)$$

Substituting equation 4.110 back into 4.93 and taking the limit

$$\frac{dF_s}{dn} = G \, 1/2 \, \rho V^2 S_e c_e \frac{C_{h_\delta}}{C_{m_{\delta_e}}} C_L \left[h - h_n' + \rho \frac{Sc}{4m} C_{m_q} F \right] \qquad (4.111)$$

Defining the stick-free maneuver point as the cg position where dF_s/dn is equal to zero,

$$h_m' = h_n' - \rho \frac{Sc}{4m} C_{m_q} F \qquad (4.112)$$

which is the same equation as 4.88 previously derived. Equation 4.111 may be written,

$$\frac{dF_s}{dn} = G \, 1/2 \, \rho V^2 S_e c_e \frac{C_{h_\delta}}{C_{m_{\delta_e}}} C_L \left[h - h_m' \right] \qquad (4.113)$$

Equation 4.113 may be rearranged if the following substitution is made.

$$C_L = \frac{2W}{\rho V^2 S} \qquad (4.115)$$

The stick-force-per-g equation becomes:

$$\frac{dF_s}{dn} = G(S_e\, c_e\, C_{h_\delta})(\frac{W}{S})\;(\frac{1}{C_{m_{\delta_e}}})\;(h - h'_m)$$

(4.116)

Stick-Free Turn Maneuver:

The procedure used for determining the dF_s/dn equation and an expression for the stick-free maneuver point for the turning maneuver is practically identical to the pullup case. For the turn condition Δq is now,

$$\Delta q = \frac{g}{V}(n - n_o)\;(1 + \frac{1}{nn_o}) \qquad (4.117)$$

The change in angle of attack of the tail, $\Delta\alpha_T$ and $\Delta\delta_e$ become

$$\Delta\alpha_T = \frac{C_L}{a}(n - n_o)(1 - \frac{d\epsilon}{d\alpha}) +$$

$$\frac{g\,\ell_T}{V^2}(n - n_o)(1 + \frac{1}{nn_o}) \qquad (4.118)$$

$$\Delta\delta_e = \frac{-C_L}{C_{m_{\delta_e}}}\Big[(h - h_n)(n - n_o) +$$

$$\rho\frac{Sc}{4m}C_{m_q}(n-n_o)(1 + \frac{1}{nn_o})\Big] \qquad (4.119)$$

Substituting equations 4.118 and 4.119 into equation 4.94 and performing the same factoring and substitutions as in the pullup case;

then,

$$\frac{\Delta C_h}{n-n_o} = -\frac{C_{h_\delta}C_L}{C_{m_{\delta_e}}}\cdot$$

$$\left[h - h'_n + \rho\frac{Sc}{4m}C_{m_q}\;F\;(1 + 1/n^2)\right]$$

(4.120)

Substituting 4.120 into 4.93

$$\frac{dF_s}{dn} = G\;1/2\,\rho V^2\,S_e\,c_e\,\frac{C_{h_\delta}C_L}{C_{m_{\delta_e}}}\cdot$$

$$\left[h - h'_n + \rho\frac{Sc}{4m}C_{m_q}\;F\;(1 + \frac{1}{n^2})\right]$$

(4.121)

And solving for the stick-free maneuver point,

$$h'_m = h'_n - \rho\frac{Sc}{4m}C_{m_q}\;F\;(1 + 1/n^2) \qquad (4.122)$$

Further substitution puts equation 4.121 into the following form:

$$\frac{dF_s}{dn} = G(S_e\,c_e\,C_{h_\delta})(\frac{W}{S})\;(\frac{1}{C_{m_{\delta_e}}})(h - h'_m)$$

(4.123)

Again, the turning stick-force-per-g equation 4.123 appears identical to the stick-free pullup equation. However, the expression for the maneuver point h'_m is different.

The term in the first parenthesis represents the hinge moment of the elevator and the aircraft size. The second term in parenthesis is wing loading, and the third term in parenthesis is the reciprocal of elevator power. The last term is the negative value of the stick-free maneuver margin. The following conclusions are drawn from this equation.

1. The stick-force-per-g appears to vary directly with the wing loading. However, weight also appears inversely in h'_m. Therefore, the full effect of weight cannot be truly analyzed since one effect could cancel the other.

2. Since airspeed does not appear in the equation, the stick-force-per-g will be the same at all airspeeds for a fixed cg.

From equation 4.112, come the following conclusions.

1. The difference between the stick-fixed and stick-free maneuver point is a function of the free elevator factor, F.

2. The stick-free maneuver point, h'_m, varies directly with altitude, becoming closer to the stick-free neutral point, the higher the aircraft flies.

The location of the stick-free maneuver point occurs where $dF_s/dn = 0$. It is difficult to fly an aircraft with this type gradient. Consequently, military specifications limit the minimum value of dF_s/dn to three pounds per g.

The forward cg may be limited by stick force per g. The maximum value is limited by the type aircraft (bomber, fighter, or trainer); i.e., heavier gradients in bomber type and lighter ones in fighters.

4.7 EFFECT OF BOBWEIGHTS AND SPRINGS

The effect of bobweights and springs on the stick-free maneuver point and stick-force gradients is of interest. The result of adding a spring or a bobweight to the control system adds an incremental force to the system. The effect of the spring is different from the effect of the bobweight. The spring exerts a constant force on the stick no matter what load factor is applied. The bobweight exerts a force on the stick proportional to the load factor.

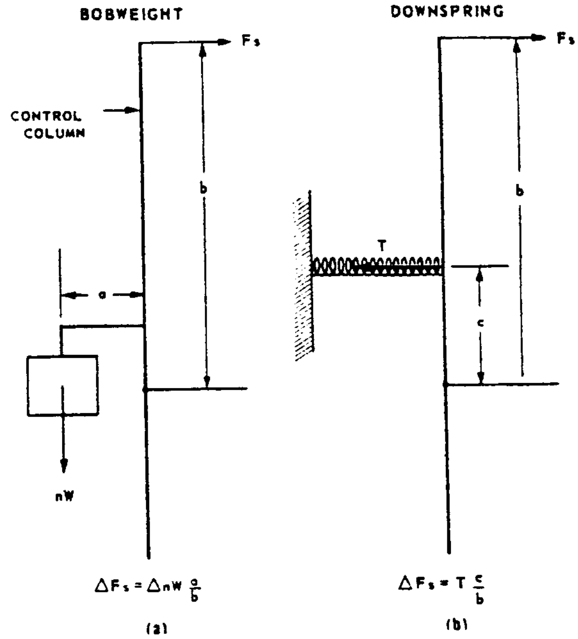


Figure 4.8 Bobweight and Downspring

The force increment for the downspring and bobweight are:

$$\Delta F_{s_{\text{Spring}}} = G(S_e c_e C_{h_\delta})\left(\frac{W}{S}\right)\left(\frac{1}{C_{m_{\delta_e}}}\right) \cdot (h - h'_m)(n - n_o) + T\frac{c}{b} \quad (4.124)$$

$$\Delta F_{s_{\text{Bobweight}}} = G(S_e c_e C_{h_\delta})(\frac{W}{S})(\frac{1}{C_{m_{\delta_e}}}) \cdot (h - h'_m)(n - n_o) + W \frac{a}{b} (n-n_o) \quad (4.125)$$

When the derivative is taken with respect to load factor, the effect on dF_s/dn of the spring is zero. The stick force gradient is not affected by the spring nor is the stick-free maneuver point changed.

$$\frac{dF_s}{dn}_{\text{Spring}} = G(S_e c_e C_{h_\delta})(\frac{W}{S})(\frac{1}{C_{m_{\delta_e}}}) (h - h'_m) \quad (4.126)$$

For the bobweight, the stick force gradient dF_s/dn becomes:

$$\frac{dF_s}{dn}_{\text{Bobweight}} = G(S_e c_e C_{h_\delta})(\frac{W}{S})(\frac{1}{C_{m_{\delta_e}}}) (h - h'_m) + W \frac{a}{b} \quad (4.127)$$

Consequently, the addition of the bobweight (positive) increases the stick force gradient, moves the stick-free maneuver point aft, and shifts the allowable cg spread aft (the minimum and maximum cg positions as specified by force gradients are moved aft). See figure 4.9.

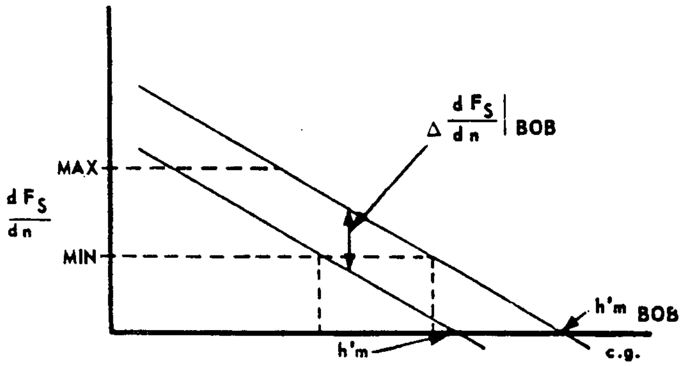


Figure 4.9 Effects of Adding a Bobweight

4.8 AERODYNAMIC BALANCING

Aerodynamic balancing is used to affect the stick force gradient and stick-free maneuver point. Aerodynamic balancing or varying values of C_{h_α} and C_{h_δ} affects the following stick-free equations.

$$\frac{dF_s}{dn} = G(S_e c_e C_{h_\delta})(\frac{W}{S})(\frac{1}{C_{m_{\delta_e}}}) (h - h'_m) \quad (4.128)$$

$$h'_m = h'_n - \rho \frac{Sc}{4m} C_{m_q} F \quad (4.129)$$

$$F = 1 - \tau \frac{C_{h_\alpha}}{C_{h_\delta}} \quad (4.130)$$

Decreasing C_{h_δ} and/or increasing C_{h_α} by using two such aerodynamic balanced devices as an overhang balance or a lagging balance tab, does the following:

1. The free elevator factor, F, decreases.
2. The stick-free maneuver point h'_m moves forward.
3. The maneuver margin term $(h - h'_m)$ decreases.
4. The stick force gradient decreases.
5. The forward and aft cg limits move forward.

Increasing C_{h_δ} and/or decreasing C_{h_α} by using a convex trailing edge or a leading balance tab does the following:

1. The free elevator factor, F, increases.
2. The stick-free maneuver point h'_m moves aft.

3. The maneuver margin term $(h - h'_m)$ increases.

4. The stick force gradient increases.

5. The forward and aft cg limits move aft.

4.9 cg RESTRICTIONS

The restrictions on the aircraft's center of gravity location may be examined by referring to the mean aerodynamic chord in figure 4.10.

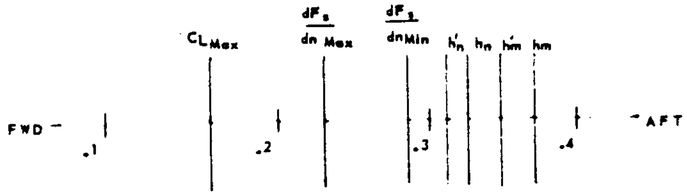


Figure 4.10 Restrictions to Center of Gravity Locations

The forward cg travel is normally limited by:

1. Maximum stick-force-per-g gradient - dF_s/dn.

or

2. Elevator required to land at $C_{L_{MAX}}$.

The aft cg travel is normally limited by:

1. Minimum stick-force-per-g - dF_s/dn.

or

2. Stick-free neutral point-power on - h'_n.

Additional considerations:

1. Freeing the elevator causes a destabilizing moment that locates the stick-free neutral and maneuver points ahead of their respective stick-fixed points.

2. The stick-free maneuver point, h'_m, can be moved aft with a bobweight but not a downspring.

3. The desired aft cg location may be unsatisfactory because it lies aft of the cg position giving minimum stick force gradient. The requirement for bobweight or a particular aerodynamic balancing would exist in order to shift the cg for minimum stick force gradient aft of the desired aft cg position.

The equations which pertain to maneuvering flight are repeated below:

<u>Pull Ups, Stick-Fixed</u>

$$h_m = h_n - \rho \frac{Sc}{4m} C_{m_q} \qquad (4.131)$$

$$\frac{d\delta_e}{dn} = \frac{a\, C_L}{C_{m_\alpha} C_{L_{\delta_e}} - C_{m_{\delta_e}} a} (h - h_m) \qquad (4.132)$$

<u>Pull Ups, Stick-Free</u>

$$h'_m = h'_n - \rho \frac{Sc}{4m} C_{m_q} F \qquad (4.133)$$

$$\frac{dF_s}{dn} = G(S_e\, c_e C_{h_\delta}) \left(\frac{W}{S}\right) \left(\frac{1}{C_{m_{\delta_e}}}\right) (h - h'_m) \qquad (4.134)$$

$$\frac{dF_s}{dn} = G(S_e\ c_e\ C_{h_\delta})(\frac{W}{S})(\frac{1}{C_{m_{\delta_e}}})(h - h'_m) + W\frac{a}{b}$$

Bobweight (4.135)

Turns, Stick-Fixed

$$h_m = h_n - \rho\frac{Sc}{4m} C_{m_q} (1 + \frac{1}{n^2}) \qquad (4.136)$$

$$\frac{d\delta_e}{dn} = \frac{a\ C_L}{C_{m_\alpha} C_{L_{\delta_e}} - C_{m_{\delta_e}} a} (h - h_m) \qquad (4.137)$$

Turns, Stick-Free

$$h'_m = h'_n - \rho\frac{Sc}{4m} C_{m_q} (1 + 1/n^2)\ F \qquad (4.138)$$

$$\frac{dF_s}{dn} = G(S_e\ c_e\ C_{h_\delta})(\frac{W}{S})(\frac{1}{C_{m_{\delta_e}}})(h - h'_m)$$

(4.139)

CHAPTER 5 LATERAL-DIRECTIONAL STATIC STABILITY

REVISED FEBRUARY 1977

5.1 INTRODUCTION

An analysis of the equations of aircraft motion leads to the following mathematical description of aircraft lateral-directional motion:

$$F_y = m\dot{v} + mru - pwm \quad (5.1)$$

$$G_x = \dot{p}I_x + qr(I_z - I_y) - (\dot{r} + pq)I_{xz} \quad (5.2)$$

$$G_z = \dot{r}I_z + pq(I_y - I_x) + (qr - \dot{p})I_{xz} \quad (5.3)$$

The right side of the equation represents the response of an aircraft to applied forces and moments. The forces and moments are expressed on the left side of the equation in terms of stability derivatives and small perturbations. As in "Long-Stat", an analysis of the lateral-directional static stability need only concern itself with the values of these derivatives. Further analysis of the aircraft equations of motion reveals the left side of the foregoing equations to be composed primarily of contributions from aerodynamic forces and moments, direct thrust, gravity, and gyroscopic moments. Of these, only the aerodynamic forces and moments ($Y, \mathcal{L}, \mathcal{N}$) will be analyzed because the other sources are usually eliminated through proper design.

It has been shown in Chapter 1 that when operating under a small disturbance assumption, aircraft lateral-directional motion can be considered independent of longitudinal motion and that it can be considered as a function of the following variables:

$$(Y, \mathcal{L}, \mathcal{N}) = f(\beta, \dot{\beta}, p, r, \delta_a, \delta_r) \quad (5.4)$$

The ensuing analysis is concerned with the question of lateral-directional static stability or the tendency of an airplane to return to stabilized flight after being perturbed in yaw or roll. This will be determined by the values of the yawing and rolling moments ($\mathcal{N}$ & $\mathcal{L}$). Since the side force equation governs only the aircraft translatory response and has no effect on the angular motion, the side force equation will not be considered.

The two remaining aerodynamic functions can be expressed in terms of non-dimensional stability derivatives, angular rates and angular displacements:

$$C_n = C_{n_\beta}\beta + C_{n_{\dot{\beta}}}\hat{\dot{\beta}} + C_{n_p}\hat{p} + C_{n_r}\hat{r} + C_{n_{\delta_a}}\delta_a + C_{n_{\delta_r}}\delta_r \quad (5.5)$$

$$C_\ell = C_{\ell_\beta}\beta + C_{\ell_{\dot{\beta}}}\hat{\dot{\beta}} + C_{\ell_p}\hat{p} + C_{\ell_r}\hat{r} + C_{\ell_{\delta_a}}\delta_a + C_{\ell_{\delta_r}}\delta_r \quad (5.6)$$

The analysis of aircraft lateral-directional motion is based on these two equations. A cursory examination of these equations reveals the presence of "cross-coupling" terms, e.g., $C_{n_p}p$ and $C_{n_{\delta_a}}\delta_a$ in the yawing moment equation (5.5). It is for this reason that aircraft lateral motions and directional motions must be considered together - each one influences the other.

Static directional stability will be considered first. Each stability derivative in equation (5.5) will be discussed and its contribution to aircraft stability will be analyzed. A summary of these stability derivatives is shown in figure 5.1.

DERIVATIVE	NAME	SIGN FOR A STABLE AIRCRAFT	CONTRIBUTING PARTS OF AIRCRAFT
C_{n_β}	Static Directional Stability or Weathercock Stability	(+)	Tail, Fuselage, Wing
$C_{n_{\dot\beta}}$	Lag Effects	(-)	Tail
C_{n_p}	Cross-Coupling	(+)	Wing, Tail
C_{n_r}	Yaw Damping	(-)	Tail, Wing, Fuselage
$C_{n_{\delta_a}}$	Adverse or Complimentary Yaw	"0" or slightly (+)	Lateral Control
$C_{n_{\delta_r}}$	Rudder Power	(+)	Rudder Control

Figure 5.1

5.2 C_{n_β} – STATIC DIRECTIONAL STABILITY OR WEATHERCOCK STABILITY

Static directional stability is defined as the initial tendency of an aircraft to return to or depart from its equilibrium angle of sideslip when disturbed. Although the static directional stability of an aircraft is determined through consideration of all the terms in equation 5.5, C_{n_β} is often referred to as "static directional stability" because it is the predominant term.

When an aircraft is placed in a sideslip, aerodynamic forces develop which create moments about all three axis. The moments created about the Z axis tend to turn the nose of the aircraft into or away from the relative wind. The aircraft is statically directionally stable if the moments created by a sideslip angle tend to align the nose of the aircraft with the relative wind. By convention, sideslip angle is defined as positive if the relative wind is displaced to the right of the fuselage reference line.

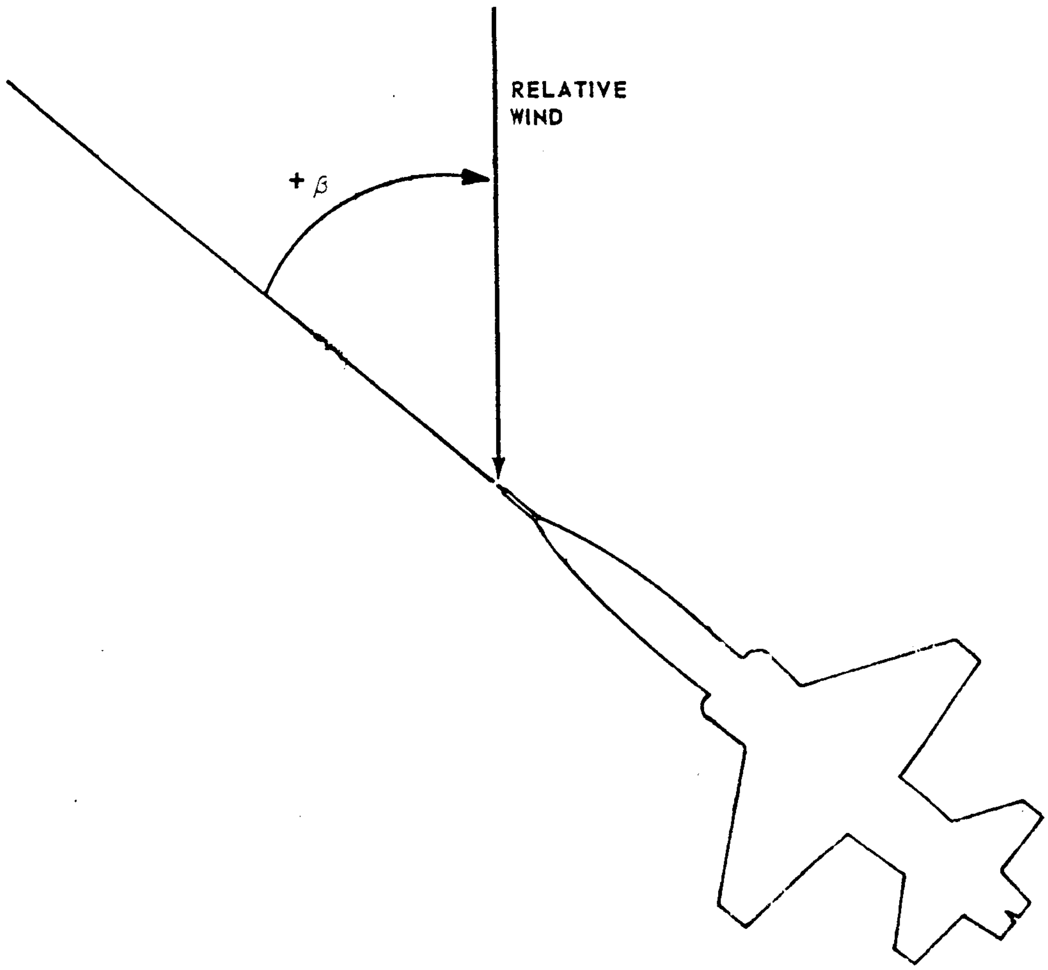


Figure 5.2

In figure 5.2 the aircraft is in a right sideslip. It is statically stable if it develops yawing moments that tend to align it with the relative wind, or, in this case, right (positive) yawing moments. Therefore, an aircraft is statically directionally stable if it develops positive yawing moments with a positive increase in sideslip. Thus, the slope of a plot of yawing moment coefficient, C_n, versus sideslip, β, is a quantitative measure of the static directional stability that an aircraft possesses. This plot would normally be determined from wind tunnel results.

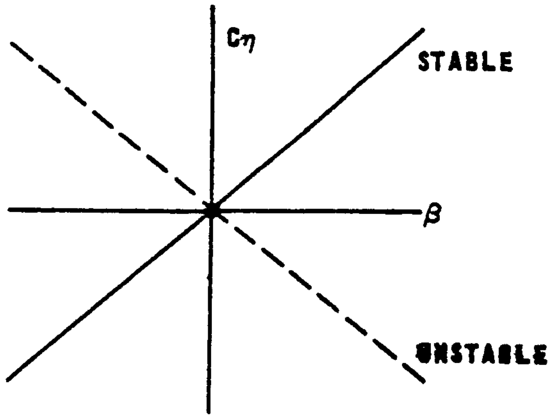


Figure 5.3
WIND TUNNEL RESULTS OF YAWING MOMENT COEFFICIENT vs SIDESLIP

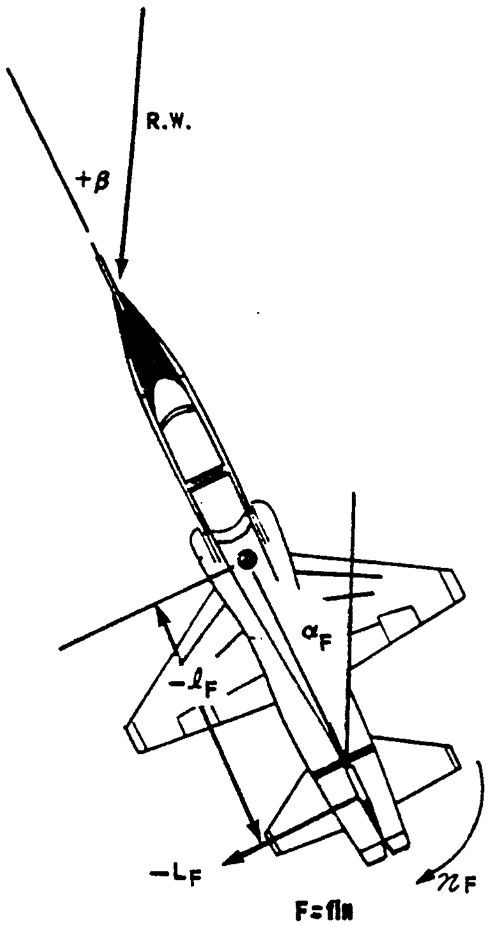


Figure 5.4

The total value of the directional stability derivative, C_{n_β}, at any sideslip angle, is determined by contributions from the vertical tail, the fuselage, and the wing. These contributions will be discussed separately.

●5.2.1 VERTICAL TAIL CONTRIBUTION TO C_{n_β}:

The vertical tail is the primary source of directional stability for virtually all aircraft. When the aircraft is yawed, the angle of attack of the vertical tail is changed. This change in angle of attack produces a change in lift on the vertical tail, and thus a yawing moment about the Z-axis.

Referring to figure 5.4, the yawing moment produced by the tail is:

$$n_F = (-\ell_F)\ (-L_F) = \ell_F L_F \tag{5.7}$$

The minus signs in this equation arise from the use of the sign convention adopted in the study of aircraft equations of motion. Forces to the left and distances behind the aircraft cg are negative.

As in other aerodynamic considerations, it is convenient to consider yawing moments in coefficient form so that static directional stability can be evaluated independent of weight, altitude and speed. Putting equation 5.7 in coefficient form:

$$C_{n_F} = \frac{\ell_F L_F}{q_w S_w b_w} = \frac{\ell_F C_{L_F}\ q_F\ S_F}{q_w S_w b_w} \tag{5.8}$$

Vertical tail volume ratio, V_v, is defined as:

$$V_v = \frac{S_F \ell_F}{S_w b_w} \tag{5.9}$$

The sign of V_v may be either positive or negative. Making this substitution in equation 5.8:

$$C_{n_F} = \frac{C_{L_F}\ q_F\ V_v}{q_w} \tag{5.10}$$

For a propeller-driven aircraft, q_w is greater than q_F. However, for a jet aircraft, these two quantities are equal. Thus, for a jet aircraft, equation 5.10 becomes:

$$C_{n_F} = C_{L_F}\ V_v \tag{5.11}$$

The lift curve for a vertical tail is presented in figure 5.5.

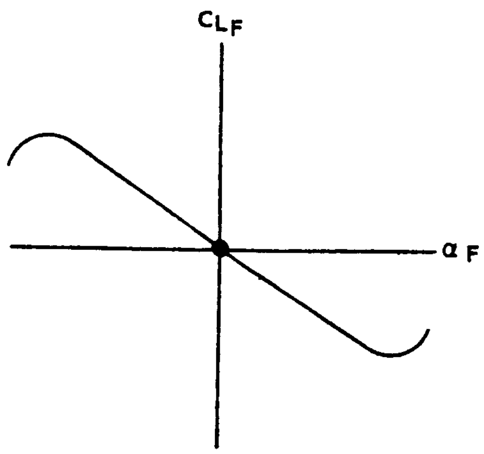


Figure 5.5 LIFT CURVE FOR A VERTICAL TAIL

The negative slope of this curve is a result of the sign convention used. Reference figure 5.4. When the relative wind is displaced to the right of the fuselage reference line, the vertical tail is placed at a positive angle of attack. However, this results in a lift force to the left, or a negative lift. Thus, the sign of the lift curve slope of a vertical tail, a_F, will always be negative below the stall.

$$C_{L_F} = a_F \alpha_F \tag{5.12}$$

Making this substitution in equation 5.11:

$$C_{n_F} = a_F \alpha_F V_v \tag{5.13}$$

The angle of attack of the vertical tail, α_F, is not merely β. If the vertical tail were placed alone in an airstream, the α_F would be equal to β. However, when the tail is installed on an aircraft, changes in both magnitude and direction of the local flow at the tail take place.

These changes may be caused by a propeller slipstream, or by the wing and the fuselage when the airplane is yawed. The angular deflection is allowed for by introducing the sidewash angle, σ, analogous to the downwash angle, ε. The value of σ is very difficult to predict, therefore suitable wind tunnel tests are required. The sign of σ is defined as positive if it causes α_F to be less than β. Thus,

$$\sigma = \beta - \alpha_F \tag{5.14}$$

Substituting in equation 5.13:

$$C_{n_F} = a_F V_v (\beta - \sigma) \tag{5.15}$$

The contribution of the vertical tail to weathercock stability is found by examining the change in C_{n_F} with a change in sideslip angle, β.

$$\frac{\partial C_{n_F}}{\partial \beta} = \left[C_{n_{\beta(\text{Tail})}} \right]_{\text{Fixed}} = V_v \, a_F \left(1 - \frac{\partial \sigma}{\partial \beta}\right) \tag{5.16}$$

The subscript "fixed" is added to emphasize that, thus far, the vertical tail has been considered as a surface with no movable parts, i.e., the rudder is "fixed."

Equation 5.16 reveals that the vertical tail contribution to directional stability can only be changed by varying the vertical tail volume ratio, V_v, or the vertical tail lift curve slope, a_F. The vertical tail volume ratio can be changed by varying the size of the vertical tail, or its distance from the aircraft cg. The vertical tail lift curve slope can be changed by altering the basic airfoil section of the vertical tail, or by end plating the vertical fin. An end plate on the top of the vertical tail is a relatively minor modification and yet it increases the directional stability of the aircraft significantly. This fact has been utilized in the case of the T-38 (figure 5.6). As can be seen in figure 5.7, the entire stabilator on the F-104 acts as an end plate and, therefore, adds greatly to the directional stability of the aircraft.

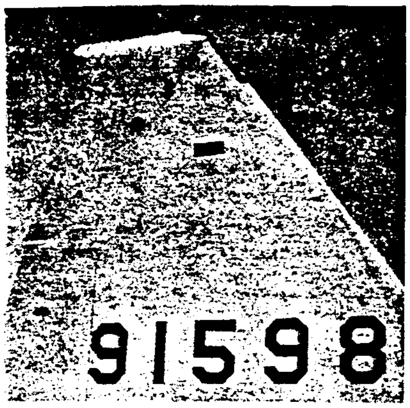


Figure 5.6

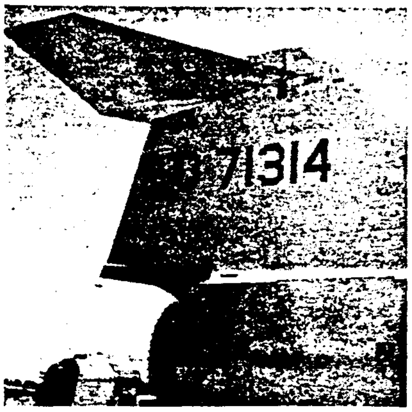
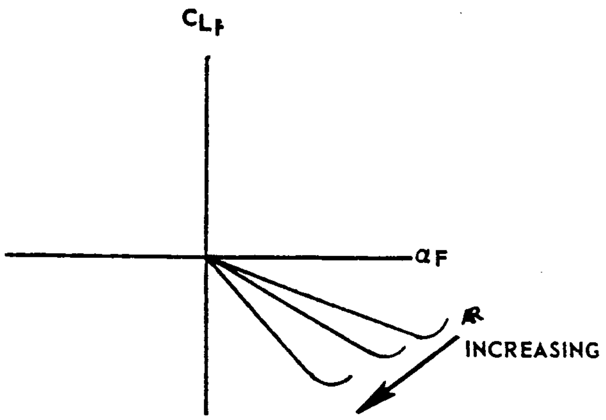


Figure 5.7

The effect of an end plate on the vertical stabilizer is to increase the effective aspect ratio of the vertical tail. As with any airfoil, this change in aspect ratio produces a change in the lift curve slope of the airfoil.



As the aspect ratio is increased, the α_F for stall is decreased. Thus, if the aspect ratio of the vertical tail is too high, the vertical tail will stall at low sideslip angles and a large decrease in directional stability will occur.

5.2.2 FUSELAGE CONTRIBUTION TO C_{n_β}:

The subsonic center of pressure of a typical fuselage occurs about one-fourth of the distance back from the nose. Since the aircraft center of gravity usually lies behind this point, the fuselage is generally destabilizing.

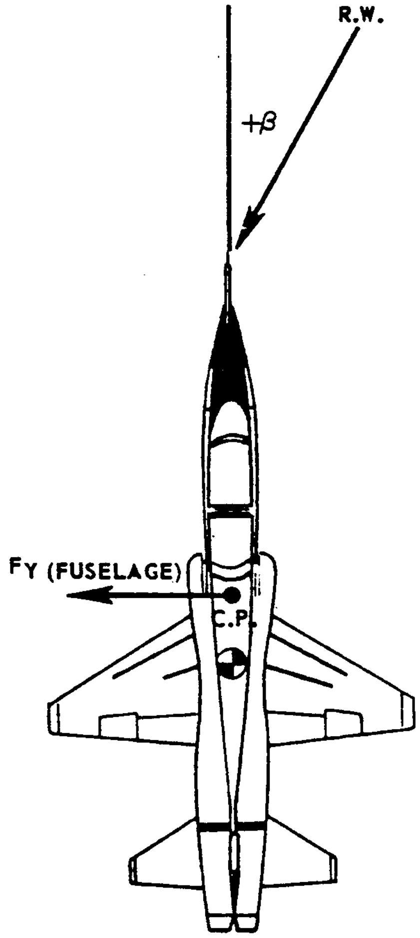


Figure 5.8

As can be seen from figure 5.8, a positive sideslip angle will produce a negative yawing moment about the cg, thus, $C_{n_{\beta(\text{fuselage})}}$ is negative or destabilizing. The destabilizing influence of the fuselage diminishes at large sideslip angles due to a decrease in lift as the fuselage stall angle of attack is exceeded, and also due to an increase in parasite drag acting at the center of equivalent parasite area which is located aft of the cg.

If the overall directional stability of an aircraft becomes too low, the fuselage-tail combination can be made more stabilizing by adding a dorsal fin or a ventral fin. A dorsal fin was added to the C-123 and a ventral fin was added to the F-104 to improve static directional stability.

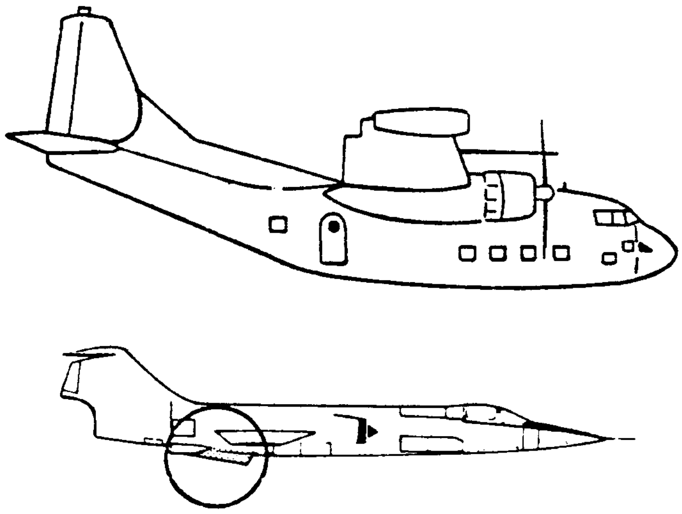


Figure 5.9

Since the addition of a dorsal fin decreases the effective aspect ratio of the tail, a higher sideslip angle can be attained before the vertical fin will stall. However, the major effect of the dorsal fin at large sideslip angles is to move the center of equivalent parasite area further aft of the cg, therefore producing a greater stabilizing moment at any given sideslip angle. Thus, a dorsal fin greatly increases directional stability at large sideslip angles. Figure 5.10 shows the effect on directional stability of adding a dorsal fin.

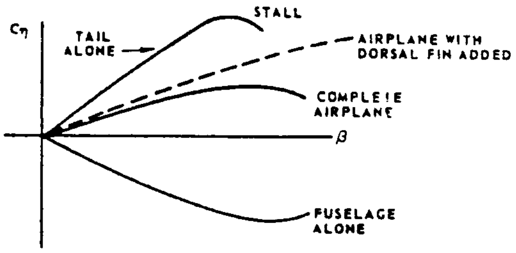


Figure 5.10 EFFECT OF ADDING A DORSAL FIN

C_{n_β} (fuselage) is difficult to estimate, and although some empirical formulas exist, it is usually measured directly by wind tunnel tests using a model without a tail.

5.2.3 WING CONTRIBUTION TO C_{n_β}:

The wing contribution to static directional stability is usually small. Straight wings make a slight positive contribution to static directional stability due to fuselage blanking in a sideslip. Effectively, the relative wind "sees" less of the downwind wing due to fuselage blanking. This reduces the lift of the downwind wing, and thus reduces the induced drag on the downwind wing. The difference in induced drag on the two wings tends to yaw the aircraft into the relative wind.

Swept back wings produce a greater positive contribution to static directional stability than do straight wings.

Reference figure 5.11. The wing sweep angle, Λ is defined as the angle between a perpendicular to the fuselage reference line and the quarter chord line of the wing.

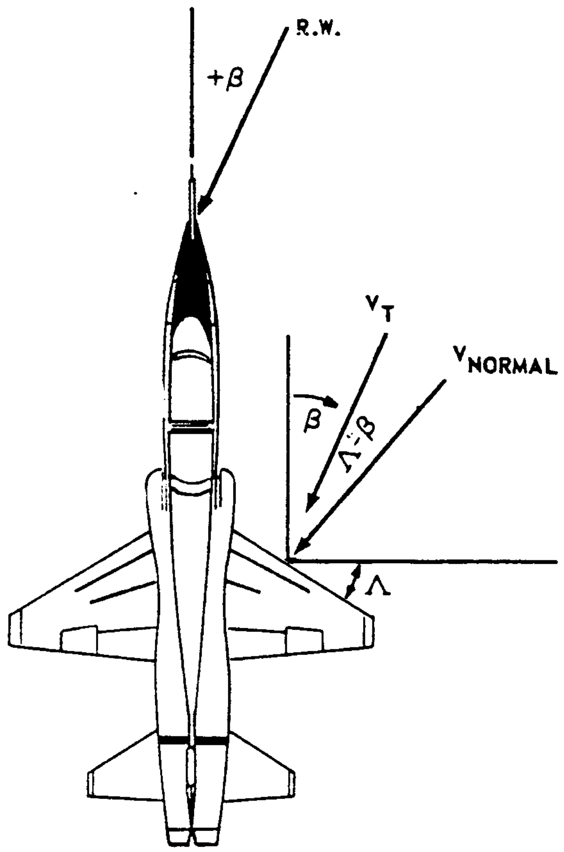


Figure 5.11

It can be seen that the component of free stream velocity normal to the wing is greater for swept back wings than for straight wings, and that is also greater on the upwind wing.

$$V_{N(\text{Upwind})} = V_T \cos (\Lambda - \beta) \tag{5.17}$$

$$V_{N(\text{Downwind})} = V_T \cos (\Lambda + \beta) \tag{5.18}$$

This difference in normal components creates a dissimilance of lift and therefore a disparity in induced drag on the two wings. Thus a stabilizing yawing moment is created. Similarly, forward swept wings would create an unstable contribution to static directional stability.

5.2.4 MISCELLANEOUS EFFECTS ON C_{n_β}:

A propeller can have large effects on an aircraft's static directional stability. The propeller contribution to directional stability arises from the side force component at the propeller disc created as a result of yaw (figure 5.12)

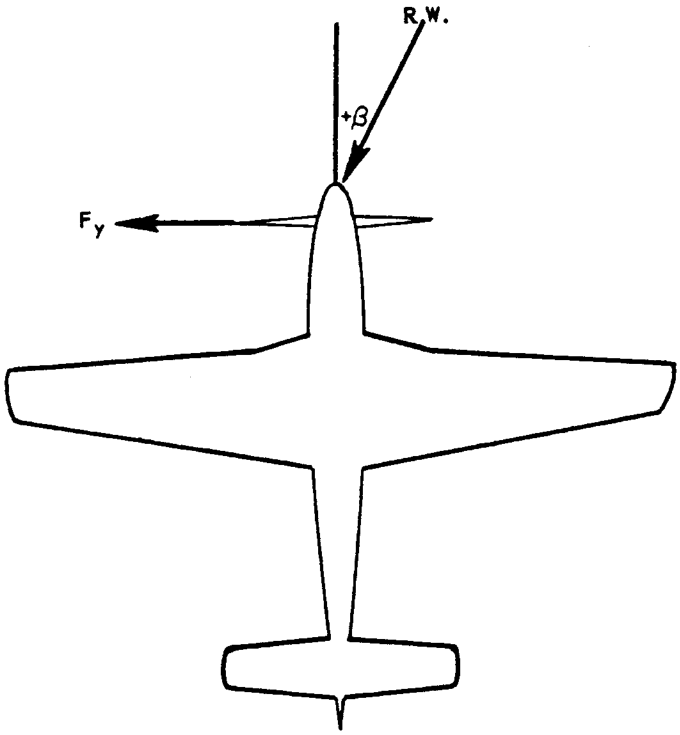


Figure 5.12

The propeller is destabilizing if a tractor and stabilizing if a pusher. Similarly, engine intakes have the same effects if they are located fore or aft of the aircraft cg.

Engine nacelles act like a small fuselage and can be stabilizing or destabilizing depending on whether their cp is located ahead or behind the cg.

Aircraft cg movement is restricted by longitudinal static stability considerations. However, within the relatively narrow limits established by longitudinal considerations, cg movements have no significant effects on static directional stability.

5.3 $C_{n_{\delta_r}}$ – RUDDER POWER

In most flight conditions, it is desired to maintain the sideslip angle equal to zero. If the aircraft has positive directional stability and is symmetrical, then it will tend to fly in this condition. However, yawing moments may act on the aircraft as a result of asymmetric thrust (one engine inoperative), slip stream rotation, or the unsymmetric flow field associated with turning flight. Under these conditions, sideslip angle can be kept to zero only by the application of a control moment. The control that provides this moment is the rudder.

Recall that,

$$C_{n_F} = a_F \, \alpha_F \, V_v \tag{5.13}$$

$$\frac{\partial C_{n_F}}{\partial \delta_r} = \frac{\partial C_n}{\partial \delta_r} = a_F \, V_v \, \frac{\partial \alpha_F}{\partial \delta_r} \tag{5.19}$$

Defining rudder effectiveness, T, as:

$$T = \frac{\partial \alpha_F}{\partial \delta_r} \tag{5.20}$$

$$\frac{\partial C_n}{\partial \delta_r} = C_{n_{\delta_r}} = a_F \, V_v \, T \tag{5.21}$$

The derivative, $C_{n_{\delta_r}}$, is called "rudder power" and by definition, its algebraic sign is always positive. This is because a positive rudder deflection, $+\delta_r$ is defined as one that produces a positive moment about the cg, $+C_n$. The magnitude of the rudder power can be altered by varying the size of the vertical tail and its distance from the aircraft cg, or by using different airfoils for the tail and/or rudder, or by varying the size of the rudder.

5.4 RUDDER FIXED STATIC DIRECTIONAL STABILITY

Having some knowledge of both C_{n_β} and $C_{n_{\delta_r}}$, it is now possible to work toward some relationship that can be used in flight to measure the static directional stability of the aircraft. In flight, the maneuver that will be used to determine the static directional stability of the aircraft is the "steady straight sideslip." In a steady straight sideslip, equation 5.5 reduces to,

$$C_{n_\beta}\beta + C_{n_{\delta_a}}\delta_a + C_{n_{\delta_r}}\delta_r = 0 \tag{5.22}$$

Thus,

$$\delta_r = -\frac{C_{n_\beta}}{C_{n_{\delta_r}}}\beta - \frac{C_{n_{\delta_a}}}{C_{n_{\delta_r}}}\delta_a \tag{5.23}$$

$$\frac{\partial\delta_r}{\partial\beta} = -\frac{C_{n_\beta (\text{Fixed})}}{C_{n_{\delta_r}}} \tag{5.24}$$

Again, the subscript "fixed" is added as a reminder that equation 5.24 is an expression for the static directional stability of an aircraft if the rudder is not free to float. Looking at equation 5.24, $C_{n_{\delta_r}}$ is a known quantity once an aircraft is built, therefore, $\partial\delta_r/\partial\beta$ can be taken as a direct indication of the rudder fixed static directional stability of an aircraft. The relationship, $\partial\delta_r/\partial\beta$, can easily be measured in flight. Since C_{n_β} has to be positive in order to have positive directional stability, and $C_{n_{\delta_r}}$ is positive by definition, $\partial\delta_r/\partial\beta$ must be negative to obtain positive directional stability.

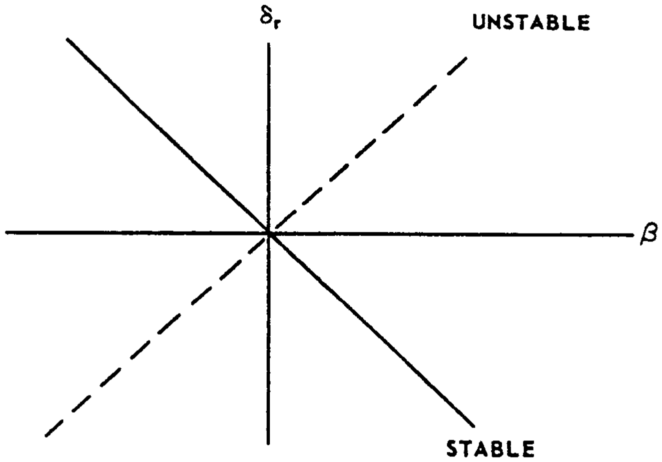


Figure 5.13 RUDDER DEFLECTION vs SLIDESLIP

5.5 RUDDER FREE DIRECTIONAL STABILITY

On aircraft with reversible control systems, the rudder is free to float in response to its hinge moments, and this floating can have large effects on the directional stability of the airplane. In fact, a plot of $\partial\delta_r/\partial\beta$ may be stable while an examination of the rudder free static directional stability reveals the aircraft to be unstable. Thus, if the rudder is free to float, there will be a change in the tail contribution to static directional stability. To analyze the nature of this change, recall that hinge moments are produced by the pressure distribution caused by angle of attack and control surface deflection. In the case of the rudder,

$$H_m = \cancel{H_{m_o}}^{\,o} + \frac{\partial H_m}{\partial \alpha_F}\alpha_F + \frac{\partial H_m}{\partial \delta_r}\delta_r \qquad (5.25)$$

In coefficient form

$$C_h = C_{h_{\alpha_F}} \cdot \alpha_F + C_{h_{\delta_r}} \cdot \delta_r \qquad (5.26)$$

It can be seen that when the vertical tail is placed at some angle of attack, α_F, the rudder will start to "float." However, as soon as it deflects, restoring moments are set up, and an equilibrium floating angle will be reached where the floating tendency is just balanced by the restoring tendency and $C_h = 0$. At this point,

$$C_{h_{\alpha_F}} \cdot \alpha_F = - C_{h_{\delta_r}} \cdot \delta_{r(Float)} \qquad (5.27)$$

Thus,

$$\delta_{r(Float)} = - \frac{C_{h_{\alpha_F}}}{C_{h_{\delta_r}}}\alpha_F \qquad (5.28)$$

With this background, it is now possible to develop a relationship that expresses the static directional stability of an aircraft with the rudder free to float.

Recall that,

$$C_{n_F} = V_v\, a_F\, \alpha_F \qquad (5.13)$$

$$\alpha_F = \beta - \sigma + \frac{\partial \alpha_F}{\partial \delta_r}\delta_{r(Float)} \qquad (5.29)$$

Therefore,

$$C_{n_F} = V_v\, a_F \left(\beta - \sigma + \frac{\partial \alpha_F}{\partial \delta_r}\delta_{r(Float)}\right) \qquad (5.30)$$

$$C_{n_{\beta}\,(Free)} = \frac{\partial C_{n_F}}{\partial \beta} = V_v \; a_F \left(1 - \frac{\partial \sigma}{\partial \beta} + T \; \frac{\partial \delta_{r\,(Float)}}{\partial \beta}\right) \tag{5.31}$$

$$C_{n_{\beta}\,(Free)} = V_v \; a_F \left(1 - \frac{\partial \sigma}{\partial \beta}\right) \cdot \left(1 + T \; \frac{\partial \delta_{r\,(Float)}}{\partial \beta} \cdot \frac{1}{1 - \frac{\partial \sigma}{\partial \beta}}\right) \tag{5.32}$$

From equation 5.14,

$$\frac{\partial \alpha_F}{\partial \beta} = 1 - \frac{\partial \sigma}{\partial \beta} \tag{5.33}$$

$$C_{n_{\beta}\,(Free)} = V_v \; a_F \left(1 - \frac{\partial \sigma}{\partial \beta}\right) \left(1 + T \; \frac{\partial \delta_{r\,(Float)}}{\partial \beta} \cdot \frac{\partial \beta}{\partial \alpha_F}\right) \tag{5.34}$$

$$C_{n_{\beta}\,(Free)} = V_v \; a_F \left(1 - \frac{\partial \sigma}{\partial \beta}\right) \cdot \left(1 + T \; \frac{\partial \delta_{r\,(Float)}}{\partial \alpha_F}\right) \tag{5.35}$$

Recall that,

$$\delta_{r\,(Float)} = - \frac{Ch_{\alpha_F}}{Ch_{\delta_r}} \; \alpha_F \tag{5.28}$$

Therefore,

$$\frac{\partial \delta_{r\,(Float)}}{\partial \alpha_F} = - \frac{Ch_{\alpha_F}}{Ch_{\delta_r}} \tag{5.36}$$

Thus,

$$C_{n_{\beta}\,(Free)} = V_v \; a_F \left(1 - \frac{\partial \sigma}{\partial \beta}\right) \cdot \left(1 - T \; \frac{Ch_{\alpha_F}}{Ch_{\delta_r}}\right) \tag{5.37}$$

It can be seen that this expression differs from equation 5.16, the expression for rudder fixed static directional stability by the term $(1 - T \; Ch_{\alpha_F}/Ch_{\delta_r})$. Since this term will always result in a quantity less than one, it can be stated that the effect of rudder float is to reduce the slope of the static directional stability curve.

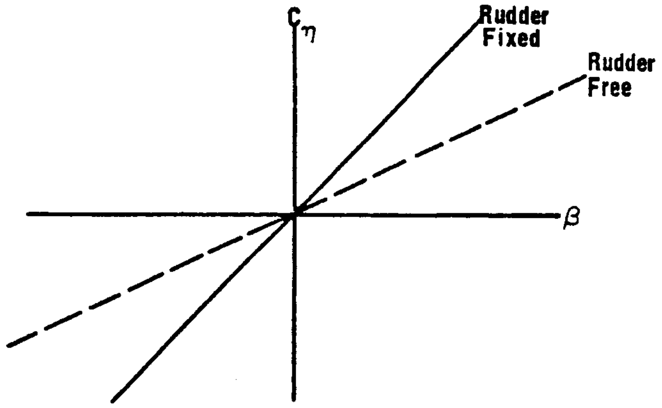


Figure 5.14

Equation 5.37 does not contain parameters that are easily measured in flight, therefore it is necessary to develop an expression that will be useful in flight test work.

Assuming a steady straight sideslip, figure 5.15 schematically represents the forces and moments at work.

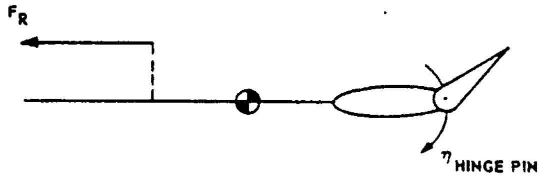


Figure 5.15

In a steady straight sideslip, $\Sigma\eta = 0$. Therefore, it follows that $\Sigma\eta_{Hinge\ Pin} = 0$. Now if moments are summed about the rudder hinge pin, the rudder force exerted by the pilot, F_R, acts through a moment arm and gearing mechanism, both accounted for by some constant, K, and must balance the other aerodynamic yawing moments so that $\Sigma\eta_{Hinge\ Pin} = 0$. The pilot is hindered in his task by the fact that the rudder floats. Thus, in steady straight flight,

$$\Sigma\eta_{Hinge\ Pin} = 0 = F_r \cdot K + H_m \quad (5.38)$$

$$F_r = - G \cdot H_M \quad (5.39)$$

Where G is merely 1/K.

Knowing,

$$H_m = C_h \, q_r \, S_r \, c_r \tag{5.40}$$

From equation 5.26,

$$H_m = q_r \, S_r \, c_r \, (C_{h_{\alpha_F}} \cdot \alpha_F + C_{h_{\delta_r}} \cdot \delta_r) \tag{5.41}$$

Thus, equation 5.39 becomes,

$$F_r = - G q_r \, S_r \, c_r \, (C_{h_{\alpha_F}} \cdot \alpha_F + C_{h_{\delta_r}} \cdot \delta_r) \tag{5.42}$$

Applying equation 5.27,

$$F_r = - G q_r \, S_r \, c_r \, (-C_{h_{\delta_r}} \cdot \delta_{r(Float)} + C_{h_{\delta_r}} \cdot \delta_r) \tag{5.43}$$

$$F_r = - G q_r \, S_r \, c_r \, C_{h_{\delta_r}} \, (\delta_r - \delta_{r(Float)}) \tag{5.44}$$

The difference between where the pilot pushes the rudder, δ_r, and the amount it floats, $\delta_{r(Float)}$, is the free position of the rudder, $\delta_{r(Free)}$

Therefore,

$$F_r = - G q_r \, S_r \, c_r \, C_{h_{\delta_r}} \, \delta_{r(Free)} \tag{5.45}$$

$$\frac{\partial F_r}{\partial \beta} = - G q_r \, S_r \, c_r \, C_{h_{\delta_r}} \cdot \frac{\partial \delta_{r(Free)}}{\partial \beta} \tag{5.46}$$

From equation 5.24, it can be shown that,

$$\frac{\partial \delta_{r(Free)}}{\partial \beta} = - \frac{C_{n_{\beta(Free)}}}{C_{n_{\delta_r}}} \tag{5.47}$$

Thus,

$$\frac{\partial F_r}{\partial \beta} = G q_r \, S_r \, c_r \, \frac{C_{h_{\delta_r}}}{C_{n_{\delta_r}}} C_{n_{\beta(Free)}} \tag{5.48}$$

This equation shows that the parameter, $\partial F_r / \partial \beta$, can be taken as an indication of the rudder free static directional stability of an aircraft. This parameter can be readily measured in flight.

An analysis of the components of equation 5.48 reveals that for static directional stability, the sign of $\partial F_r / \partial \beta$ should be negative.

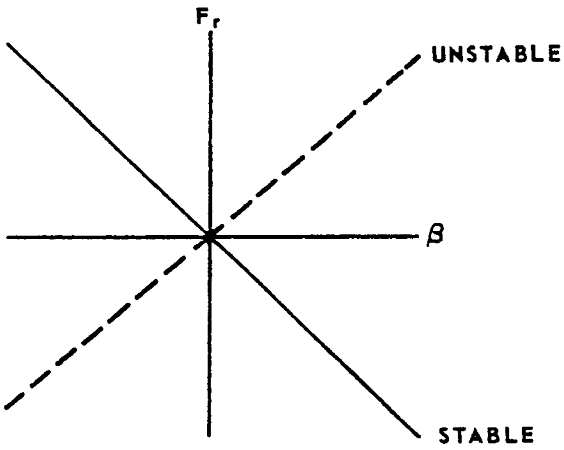


Figure 5.16

5.6 $C_{n_{\delta_a}}$ – YAWING MOMENT DUE TO LATERAL CONTROL DEFLECTION

The remaining derivatives in equation 5.5 that have not been studied thus far are called "cross derivatives." It is the existence of these cross derivatives that causes the rolling and yawing motions to be so closely coupled.

The first of these cross derivatives to be covered will be $C_{n_{\delta_a}}$, and is the yawing moment due to lateral control deflection. In order for a lateral control to produce a rolling moment, it must create an unbalanced lift condition on the wings. The wing with the most lift will also produce the most induced drag according to the equation $C_{D_i} = C_L^2/\pi e\, AR$. Also, any change in the profile of the wing due to a lateral control deflection will cause a change in profile drag. Thus, any lateral control deflection will produce a change in both induced and profile drag. The predominate effect will be dependent on the particular aircraft configuration and the flight condition. If induced drag predominates, the aircraft will tend to yaw away from the direction of roll. This phenomenon is known as "adverse yaw." The sign of $C_{n_{\delta_a}}$ for adverse yaw is negative. If profile drag predominates, the aircraft will tend to yaw into the direction of roll. This is known as "complimentary" or "proverse" yaw. The sign of $C_{n_{\delta_a}}$ for complimentary yaw is positive. Both ailerons and spoilers are capable of producing either adverse or complimentary yaw. To determine which condition will prevail, the particular aircraft configuration and flight condition must be analyzed. If design permits, it is desirable to have $C_{n_{\delta_a}} = 0$ or be slightly positive. A slight positive value will ease the pilot's turn coordination task.

■ 5.7 C_{n_p} – YAWING MOMENT DUE TO ROLL RATE

The derivative C_{n_p} is called yawing moment due to roll rate. Both the wing and the tail contribute to C_{n_p}. The wing contribution arises from two sources. The first comes from the change in profile drag associated with the change in wing angle of attack due to rolling. As an aircraft is rolled, the angle of attack on the downgoing wing is increased. Refer to figure 5.17. Conversely, the angle of attack on the upgoing wing is decreased.

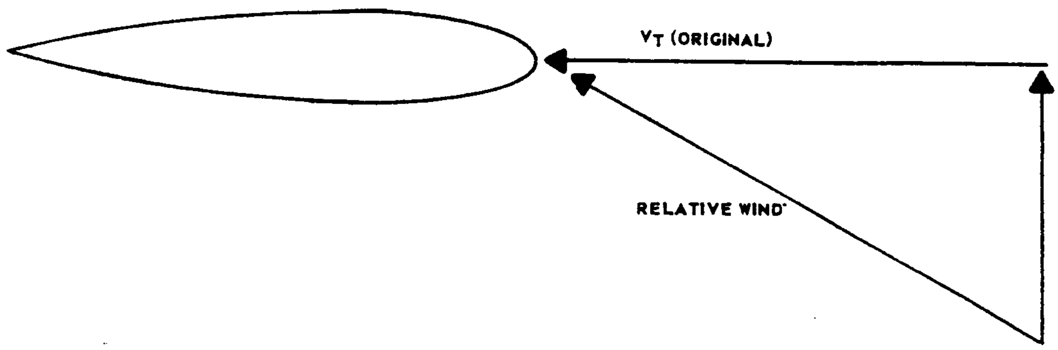


Figure 5.17

This increase in angle of attack on the donwgoing wing means that the relative wind "sees" more of the downgoing wing and that therefore the profile drag will be greater on this wing than on the upgoing wing. For the right roll depicted in figure 5.17, the increased profile drag would cause a yaw to the right. Thus, the sign of C_{n_p} due to this effect only is positive. However, the second wing effect is predominant and the foregoing effect exerts only a mitigating influence.

The local lift vector is always perpendicular to the local relative wind. As already discussed, the inclination of the relative wind is difference on the wings during a roll. Thus, there will be a difference in the inclination of the two wing lift vectors. The lift vector on the downgoing wing will be tilted forward, and the lift vector on the upgoing wing will be tilted aft. Refer to figure 5.18.

Since each lift vector has a component in the X-direction, a yawing moment will result. In the case depicted, for a right roll the yaw will be to the left. Thus, the sign of C_{n_p} due to this effect will be negative. As previously mentioned, this is the predominate wing effect and thus, overall, the sign of the wing contribution to C_{n_p} is negative.

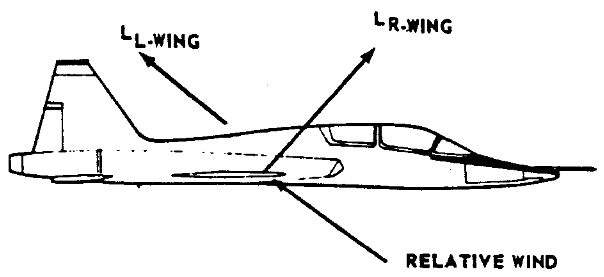


Figure 5.18 INCLINATION OF WING LIFT VECTORS DURING A RIGHT ROLL

The vertical tail makes a larger contribution to C_{n_p} than does either wing effect. Rolling changes the angle of attack on the vertical tail. Refer to figure 5.19.

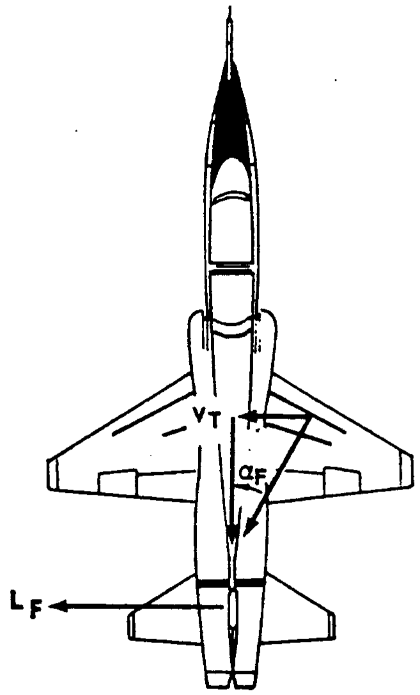


Figure 5.19 CHANGE IN ANGLE OF ATTACK OF THE VERTICAL TAIL DUE TO A RIGHT ROLL RATE

This change in angle of attack on the vertical tail will generate a lift force. In the situation depicted in figure 5.19, the change in angle of attack will generate a lift force, L_F, to the left. This will create a positive yawing moment. Thus, C_{n_p} for the vertical tail is positive.

Considering both wing and tail, a slight positive value of C_{n_p} is desired to aid in Dutch roll damping.

5.8 C_{n_r} YAW DAMPING

The derivative C_{n_r}, is called yaw damping and, by definition, its sign is always negative. The aircraft fuselage adds a negligible amount to C_{n_r} except when it is very large. The important contributions are those of the wing and tail.

The tail contribution to C_{n_r} arises from the fact that there is a change in angle of attack on the vertical tail whenever the aircraft is yawed. This change in α_F produces a lift force, L_F, that in turn produces a yawing moment that opposes the original yawing moment. Refer to figure 5.20. The tail contribution to C_{n_r} accounts for 80-90% of the total "yaw damping" on most aircraft.

The wing contribution to C_{n_r} arises from the fact that in a yaw, the outside wing experiences an increase in both induced drag and profile drag due to the increased dynamic pressure on the wing. An increase in drag on the outside wing creates a yawing moment that opposes the original direction of yaw.

5.9 $C_{n_{\dot\beta}}$ – YAW DAMPING DUE TO LAG EFFECTS IN SIDEWASH

The derivative $C_{n_{\dot\beta}}$ is yaw damping due to lag effects in sidewash, σ. Very little can be authoritatively stated about the magnitude or algebraic sign of $C_{n_{\dot\beta}}$ due to the wide variations of opinion in interpreting the experimental data concerning it.

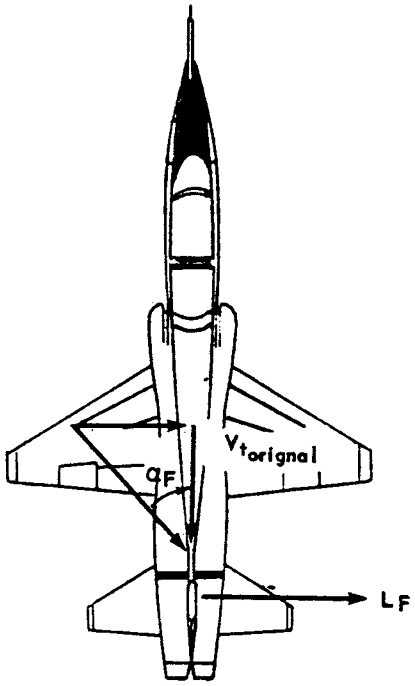


Figure 5.20
CHANGE IN ANGLE OF ATTACK OF VERTICAL FIN DUE TO YAWING RATE

During any change in β, the angle of attack of the vertical fin will always be less than it will be at steady state. This is due to lag effects in sidewash. Since this phenomenon reduces the angle of attack of the vertical tail, it also reduces the yawing moment created by the vertical tail. This reduction in yawing moment is, effectively, a contribution to yaw damping. Thus the description, "yaw damping due to lag effects in sidewash."

5.10 HIGH SPEED ASPECTS OF STATIC DIRECTIONAL STABILITY

C_{n_β} - The effectiveness of an airfoil decreases as the velocity increases supersonically. Thus, for a given β, as Mach increases, the restoring moment generated by the tail diminishes. The wing-fuselage combination continues to be destabilizing throughout the flight envelope. Thus, the overall C_{n_β} of the aircraft will decrease with increasing Mach.

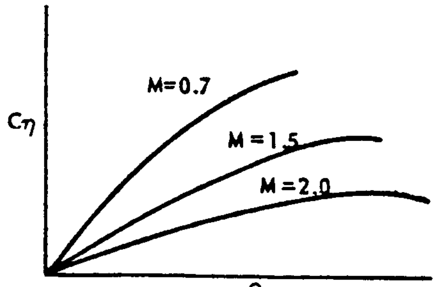


Figure 5.21 CHANGE IN C_{n_β} WITH MACH NUMBER

The requirement for large values of C_{n_β} is compounded by the tendency of high speed aerodynamic designs toward divergencies in yaw due to roll coupling effects. This problem can be combated by designing an extremely large tail (F-104, F-111, T-38), by endplating the tail (F-104, T-38), by using ventral fins (F-104), or by using fore body strakes.

The F-104 employs a ventral fin in addition to a sizeable vertical stabilizer to increase supersonic directional stability. The efficiency of underbody surfaces is not affected by wing wake at high angles of attack, and supersonically, they are located in a high energy compression pattern.

Fore body strakes located radially along the horizontal center line in the x-y plane of the aircraft have also been employed effectively to increase directional stability at supersonic speeds. This increase in C_{n_β} by the employment of strakes is a result of a more favorable pressure distribution over the fore body surface, and in addition, the creation of improved flow effects at the vertical tail location by virtue of diminished flow circulation. In addition, even small sideslip angles will produce fuselage blanking of the downwind strake and create a dissimilance of induced drag, and thus a stable contribution to C_{n_β}.

$C_{n_{\delta_r}}$ - In the transonic region, flow separation will decrease the effectiveness of any trailing edge control surface. On most aircraft however, this is offset by an increase in the C_{L_α} curve in the transonic region. As a result, flight controls are usually the most effective in this region. However, as Mach number continues to increase, the C_{L_α} curve will decrease, and thus, control surface effectiveness will continue to decrease. In addition, once the flow over the surface is supersonic, a trailing edge control cannot influence the pressure distribution on the surface itself, due to the fact that pressure disturbances cannot be transmitted forward in a supersonic environment. Thus, the rudder power will decrease as Mach increases above the transonic region.

$C_{n_{\delta_a}}$ - For the same reasons discussed under rudder power, a given aileron deflection will not produce as much lift at high Mach number as it did transonically. Therefore, induced drag will be less. In addition, the profile drag, for a given aileron deflection, increases with Mach number. Thus, the tendency toward complimentary yaw increases with Mach.

C_{n_r} - The development of yaw damping depends on the ability of the wing and tail to develop lift. Thus, as Mach number increases and the ability of all surfaces to develop lift decreases, yaw damping will also decrease.

C_{n_p} - The slope of a curve of C_{n_p} normally doesn't change with Mach number. However, the magnitude of attainable roll rate will decrease with decreasing aileron effectiveness. Therefore, the magnitude of C_{n_p} encountered at higher Mach numbers will normally be less.

$C_{n_{\dot\beta}}$ - This derivative normally will not change with Mach number.

5.11 STATIC LATERAL STABILITY

The analysis of aircraft lateral static stability is based on equation 5.6, which is repeated here for reference.

$$C_\ell = C_{\ell_\beta}\beta + C_{\ell_{\dot\beta}}\hat{\dot\beta} + C_{\ell_p}\hat{p} + C_{\ell_r}\hat{r} + C_{\ell_{\delta_a}}\delta_a + C_{\ell_{\delta_r}}\delta_r \qquad (5.6)$$

It can be seen that the rolling moment, C_ℓ, is not a function of bank angle, ϕ. In other words, a change in bank angle will produce no change in rolling moment. In fact, ϕ produces no moment at all. Thus, $C_{\ell_\phi} = 0$, and although it is analogous to C_{m_α} and C_{n_β}, it contributes nothing to the lateral static stability analysis.

Bank angle, ϕ, does have an indirect effect on rolling moment. As the aircraft is rolled into a bank angle, a component of aircraft weight will act along the Y-axis, and will thus produce an unbalanced force. Refer to figure 5.22. This unbalanced force in the Y direction, F_Y, will produce a sideslip, β, and as seen from equation 5.6, this will influence the rolling moment produced.

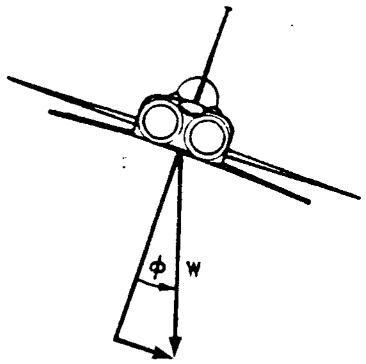


Figure 5.22 SIDE FORCE PRODUCED BY BANK ANGLE

Each stability derivative in equation 5.6 will be discussed and its contribution to aircraft stability will be analyzed. A summary of these stability derivatives follows:

DERIVATIVE	NAME	SIGN FOR A STABLE AIRCRAFT	CONTRIBUTING PARTS OF AIRCRAFT
C_{ℓ_β}	Dihedral Effect	(-)	Wing, Tail
$C_{\ell_{\dot\beta}}$	C_ℓ due to $\dot\beta$	(+)	Wing, Tail
C_{ℓ_p}	Roll Damping	(-)	Wing, Tail
C_{ℓ_r}	C_ℓ due to Yaw Rate	(+)	Wing, Tail
$C_{\ell_{\delta_a}}$	Lateral Control Power	(+)	Lateral Control
$C_{\ell_{\delta_r}}$	C_ℓ due to Rudder Deflection	(-)	Rudder

Figure 5.23

5.12 C_{ℓ_β} – DIHEDRAL EFFECT

The tendency of an aircraft to fly wings level is related to the derivative C_{ℓ_β}, which is known as "Dihedral Effect." Although the static lateral stability of an aircraft is a function of all the derivatives in equation 5.6, C_{ℓ_β} is the predominant term. Therefore, static lateral stability is often referred to as "Stable Dihedral Effect."

An aircraft has stable dihedral effect if a positive sideslip produces a negative rolling moment. Thus, the algebraic sign of C_{ℓ_β} must be negative for stable dihedral effect.

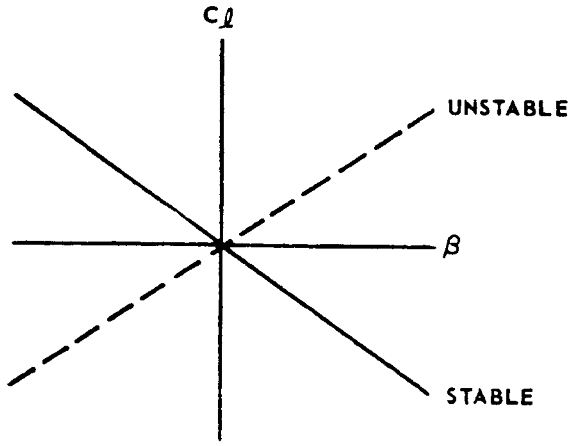


Figure 5.24 WIND TUNNEL RESULTS OF ROLLING MOMENT COEFFICIENT vs SIDESLIP

It is possible to have too much or too little dihedral effect. High values of dihedral effect give good spiral stability. If an aircraft has a large amount of positive dihedral effect, the pilot is able to pick up a wing with top rudder. This also means that in level flight a small amount of sideslip will cause the aircraft to roll and this can be annoying to the pilot. This is known as a high ϕ/β ratio. In multi-engine aircraft, an engine failure will normally produce a large sideslip angle. If the aircraft has a great deal of dihedral effect, the pilot must supply an excessive amount of aileron force and deflection to overcome the rolling moment due to sideslip. Still another detrimental effect of too much dihedral effect may be encountered when the pilot rolls an aircraft. If an aircraft in rolling to the right tends to yaw to the left, the resulting right sideslip, together with stable dihedral effect, creates a rolling moment to the left. This effect could materially reduce the maximum roll rate available. The pilot, then wants a certain amount of dihedral effect, but not too much. The end result is usually a design compromise.

Both the wing and the tail exert an influence on C_{ℓ_β}. The various effects on C_{ℓ_β} can be classified as "direct" or "indirect." A direct effect actually produces some increment of C_{ℓ_β} while an indirect effect merely alters the value of the existing C_{ℓ_β}.

The discrete wing and tail effects that will be considered are classified as follows:

Effects on C_{ℓ_β}

DIRECT	INDIRECT
Geometric Dihedral	Aspect Ratio
Wing Sweep	Taper Ratio
Wing-Fuselage Interference	Tip Tanks
Vertical Tail	Wing Flaps

Figure 5.25

Geometric dihedral, γ, is defined as positive when the chord lines of the wing tip are above those at the wing root. To understand the effect of geometric dihedral on static lateral stability, consider figure 5.26.

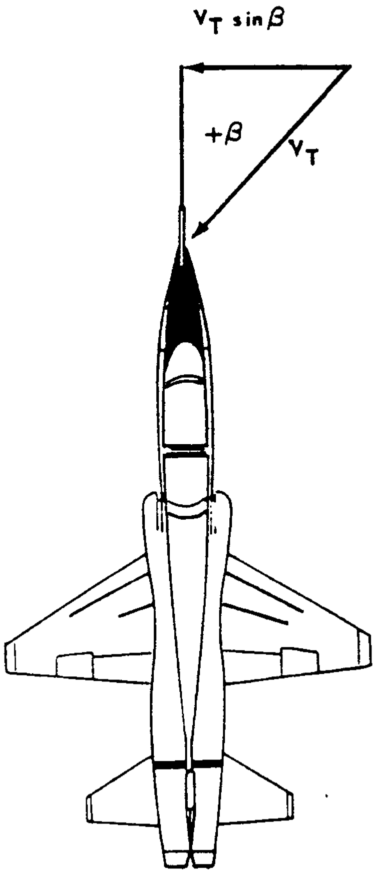


Figure 5.26a

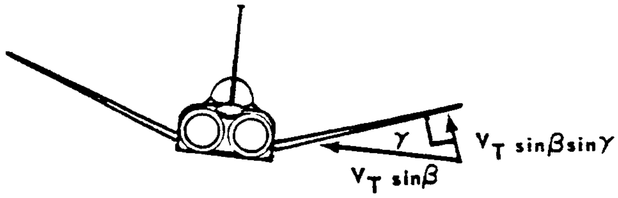


Figure 5.26b

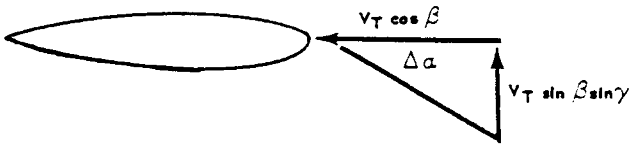


Figure 5.26c

It can be seen that when an aircraft is placed in a sideslip, positive geometric dihedral causes the component, $V_T \sin \beta \sin \gamma$ to be added to the lift producing component of the relative wind, $V_T \cos \beta$. Thus, geometric dihedral causes the angle of attack on the upwind wing to be increased by $\Delta\alpha$.

$$\tan \Delta\alpha = \frac{V_T \sin \beta \sin \gamma}{V_T \cos \beta} = \tan \beta \sin \gamma \qquad (5.49)$$

Making the small angle assumption,

$$\Delta\alpha = \tan \beta \sin \gamma \qquad (5.50)$$

Conversely, the angle of attack on the downwind wing will be reduced. These changes in angle of attack tend to increase the lift on the upwind wing and decrease the lift on the downwind wing, thus producing a roll away from the sideslip. In figure 5.26, for example, a positive sideslip, $+\beta$, will increase the angle of attack on the upwind, or right, wing, thus producing a roll to the left. Therefore, it can be seen that this effect produces a stable, or negative, contribution to C_{ℓ_β}.

5.12.1 WING SWEEP:

The wing sweep angle, Λ, is measured from a perpendicular to the aircraft x-axis at the forward wing root, to a line connecting the quarter cord points of the wing. Wing sweep back is defined as positive.

Aerodynamic theory shows that the lift of a yawed wing is determined by the component of the free stream velocity normal to wing. That is, $L = C_L\ 1/2\ \rho V_N^2 S$ where, V_N is the normal velocity.

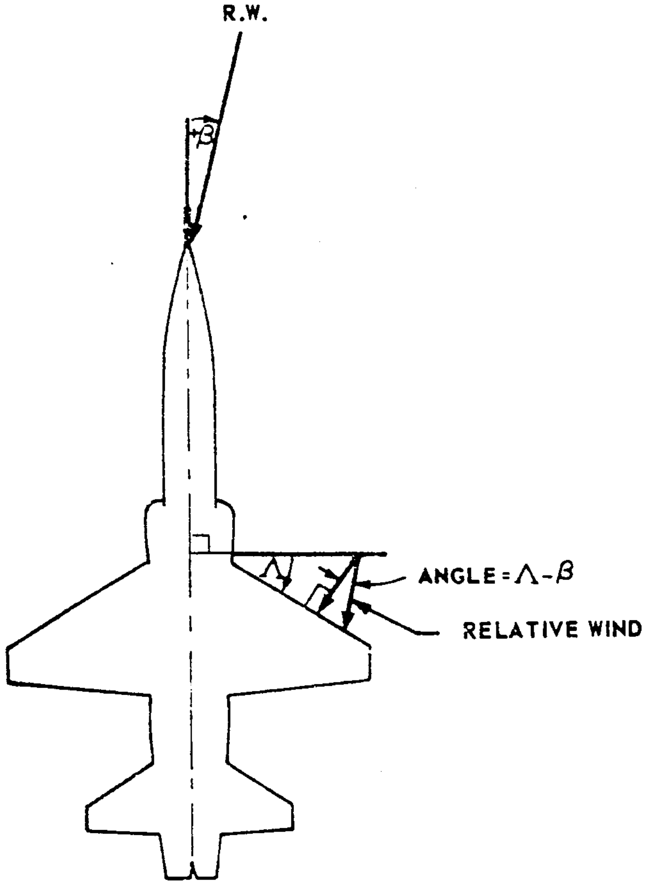


Figure 5.27 EFFECT OF WING SWEEP ON C_{ℓ_β}

It can be seen from figure 5,27 that on a swept wing aircraft, the normal component of free stream velocity on the upwind wing is,

$$V_N = V_T \cos (\Lambda - \beta) \tag{5.51}$$

Conversely, on the downwind wing,

$$V_N = V_T \cos (\Lambda + \beta) \tag{5.52}$$

Therefore, V_N will be greater on the upwind wing. This will cause the upwind wing to produce more lift and will thus create a roll away from the direction of the sideslip. In other words, a right sideslip will produce a roll to the left. Thus, wing sweep makes a stable contribution to C_{ℓ_β} and produces the same effect as geometric dihedral.

To fully appreciate the effect of wing sweep on static lateral stability, it will be necessary to develop an equation relating the two.

$$L_{(\text{Upwind Wing})} = C_L \frac{S}{2} 1/2 \, \rho V_N^{\,2} \tag{5.53}$$

$$L_{(\text{Upwind Wing})} = C_L \frac{S}{2} 1/2 \, \rho \left[V_T \cos (\Lambda - \beta)\right]^2 \tag{5.54}$$

$$\Delta L = C_L \frac{S}{2} 1/2 \, \rho \left[V_T \cos (\Lambda - \beta)\right]^2 - C_L \frac{S}{2} 1/2 \, \rho \left[V_T \cos (\Lambda + \beta)\right]^2 \tag{5.55}$$

$$\Delta L = C_L \frac{S}{2} 1/2 \, \rho \, V_T^{\,2} \left[\cos^2 (\Lambda - \beta) - \cos^2 (\Lambda + \beta)\right] \tag{5.56}$$

Applying a trigonometric identity,

$$\left[\cos^2 (\Lambda - \beta) - \cos^2 (\Lambda + \beta)\right] = \sin 2 \Lambda \sin 2 \beta \tag{5.57}$$

Making the assumption of a small sideslip angle,

$$\left[\cos^2 (\Lambda - \beta) - \cos^2 (\Lambda + \beta)\right] = 2 \beta \sin 2 \Lambda \tag{5.58}$$

Therefore, equation 5.56 becomes,

$$\Delta L = C_L \frac{S}{2} 1/2 \, \rho \, V_T^{\,2} \, 2 \beta \sin 2 \Lambda = C_L S \, 1/2 \, \rho \, V_T^{\,2} \beta \sin 2 \Lambda \tag{5.59}$$

The rolling moment produced by this change in lift is,

$$\mathcal{L} = - \Delta L \cdot \bar{Y} \tag{5.60}$$

Where $\bar{Y}$ is the distance from the wing cp to the aircraft cg. The minus sign arises from the fact that equation 5.59 assumes a positive sideslip, $+\beta$, and for an aircraft with stable dihedral effect, this will produce a negative rolling moment.

$$C_{\ell} = \frac{\mathcal{L}}{q_w S_w b_w} \tag{5.61}$$

$$C_{\ell} = \frac{\bar{Y}\, C_L\, S\, \rho\, V_T^2\, \beta\, \sin 2\Lambda}{\rho\, V_T^2\, Sb} = -\frac{C_L\, \bar{Y}\, \beta}{b} \sin 2\Lambda \tag{5.62}$$

$$\frac{\partial C_{\ell}}{\partial \beta} = C_{\ell_\beta} = -\frac{\bar{Y}}{b} C_L \sin 2\Lambda = -\text{CONST}\,(C_L \sin 2\Lambda) \tag{5.63}$$

Where the constant will be on the order of 0.2. Equation 5.63 should not be used above $\Lambda = 45°$ because highly swept wings are subject to leading edge separation at high angles of attack, and this can result in reversal of the dihedral effect. Therefore, it's best to use empirical results above $\Lambda = 45°$.

From equation 5.63, it can be seen that at low speeds, high C_L, sweepback makes a large contribution to stable dihedral effect. However, at high speeds, low C_L, sweepback makes a relatively small contribution to stable dihedral effect.

For angles of sweep on the order of 45°, the wing sweep contribution to C_{ℓ_β} may be on the order of $-1/5\ C_L$. For large values of C_L, this is a very large contribution, equivalent to nearly ten degrees of geometric dihedral. At very high angles of attack, such as during landing and takeoff, this effect can be very helpful to a swept wing fighter encountering downwash.

Since the effect of sweepback varies with C_L, becoming extremely small at high speeds, it can help keep the proper ratio of C_{ℓ_β} to C_{n_β} at high speeds and reduce poor Dutch roll characteristics at these speeds.

5.12.2 WING ASPECT RATIO:

The wing aspect ratio exerts an indirect effect on dihedral effect. On a high aspect ratio wing, the center of pressure is further from the cg than on a low aspect ratio wing. This results in high aspect ratio planforms having a longer moment arm and thus, greater rolling moments for a given asymmetric lift distribution. Refer to figure 5.28. It should be noted that aspect ratio, in itself, does not create dihedral effect, but that it merely alters the magnitude of the existing dihedral effect.

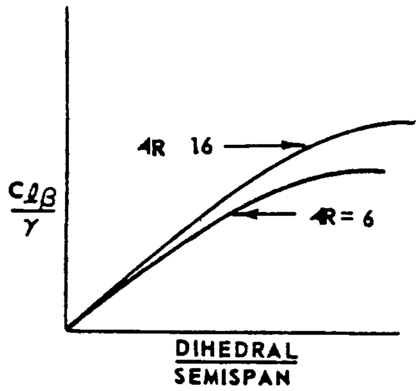


Figure 5.28

CONTRIBUTION OF ASPECT RATIO TO DIHEDRAL EFFECT.

5.12.3 WING TAPER RATIO:

Taper ratio, λ, is a measure of how fast the wing chord shortens. Taper ratio is the ratio of the tip chord to the root chord. Therefore, the lower the taper ratio, the faster the chord shortens. On highly tapered wings, the center of pressure is closer to the aircraft cg than on untapered wings. This results in a shorter moment arm and thus, less rolling moment for a given asymmetric lift distribution. Refer to figure 5.29. Taper ratio does not create dihedral effect, but merely alters the magnitude of the existing dihedral effect. Thus it has an "indirect" effect on dihedral effect.

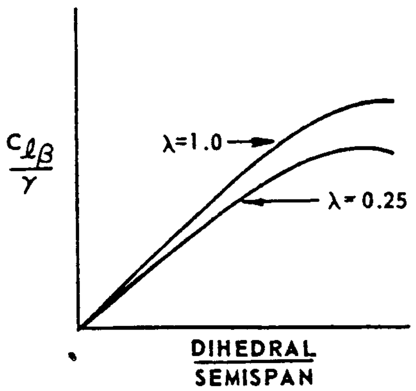


Figure 5.29 CONTRIBUTION OF TAPER RATIO TO DIHEDRAL EFFECT

5.12.4 TIP TANKS

Tip tanks, pylon tanks and other external stores will generally **exert** an indirect influence on C_{ℓ_β}. Adding external stores creates an end-plating effect on the wing, and this, in turn, alters the effective aspect ratio of the wing. The effect of a given external store configuration is hard to predict analytically, and it is usually necessary to rely on empirical results. To illustrate the effect of various external store configurations, data for the F-80 is presented in figure 5.30. The data is for a clean F-80 230 gallon centerline tip tanks, and 165 gallon underslung tanks. This data shows that the centerline tanks increase dihedral effect while the underslung tanks reduce stable dihedral effect considerably.

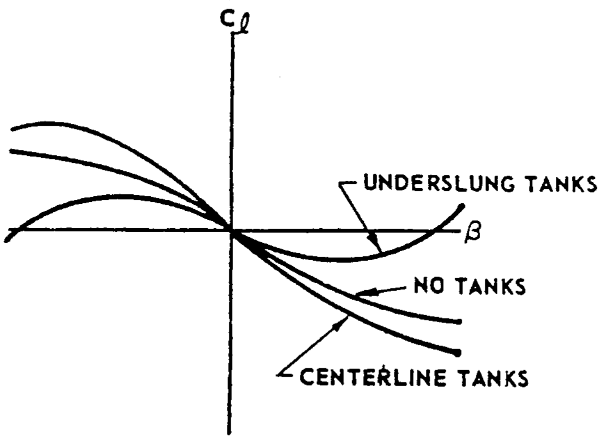


Figure 5.30 EFFECT OF TIP TANKS ON C_{ℓ_β} OF F-80

5.12.5 PARTIAL SPAN FLAPS:

Partial-span flaps indirectly exert a detrimental effect on static lateral stability. Refer to figure 5.31.

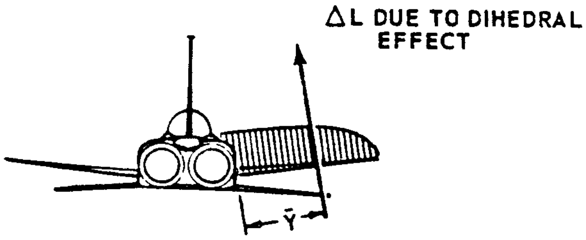


Figure 5.31a WING LIFT DISTRIBUTION, NO FLAPS

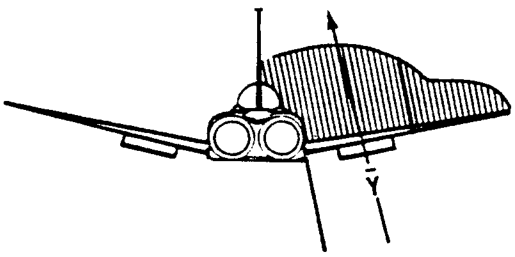


Figure 5.31b WING LIFT DISTRIBUTION, FLAPS EXTENDED

Partial-span flaps shift the center of lift of the wing inboard, reducing the effective moment arm $\bar{Y}$. Therefore, although the values of ΔL remain the same, the rolling moment will decrease. The higher the effectiveness of the flaps in increasing the lift coefficient, the greater will be the change in span lift distribution and the more detrimental will be the effect of the flaps. Therefore, the decrease in lateral stability due to flap deflection may be large.

Deflected flaps cause a secondary variation in the effective dihedral that depends on the planform of the flaps themselves. If the shape of the wing gives a sweepback to the leading edge of the flaps, a slight positive dihedral effect results when the flaps are deflected. If the leading edge of the flaps are swept forward, flap deflection causes a slight negative dihedral effect. These effects are produced by the same phenomenon that produced a change in C_{ℓ_β} with wing sweep. The effect of flap platform on C_{ℓ_β} is generally small.

● 5.12.6 WING - FUSELAGE INTERFERENCE :

Of the various interference effects between parts of the aircraft, probably the most important is the change in angle of attack of the wing near the root due to the flow pattern about the fuselage in a sideslip. To visualize the change in angle of attack, refer to figure 5.32.

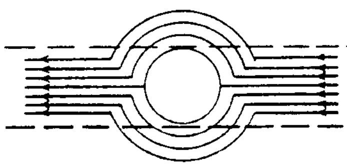


Figure 5.32 INFLUENCE OF WING – FUSELAGE INTERFERENCE ON C_{ℓ_β}

The fuselage induces vertical velocities in a sideslip which, when combined with the mainstream velocity, alter the local angle of attack of the wing. When the wing is located at the top of the fuselage (high-wing), then the angle of attack will be increased at the wing root, and a positive sideslip will produce a negative rolling moment: i.e., the dihedral effect will be enhanced. Conversely, when the aircraft has a low wing, the dihedral effect will be diminished by the fuselage interference. Generally, this explains why high-wing airplanes often have little or no geometric dihedral, whereas low-wing aircraft may have a great deal of geometric dihedral.

● 5.12.7 VERTICAL TAIL:

When an aircraft sideslips, the angle of attack of the vertical tail is changed. This change in angle of attack produces a lift force on the vertical tail. If the center of pressure of the vertical tail is above the aircraft cg, this lift force will produce a rolling moment. Refer to figure 5.33.

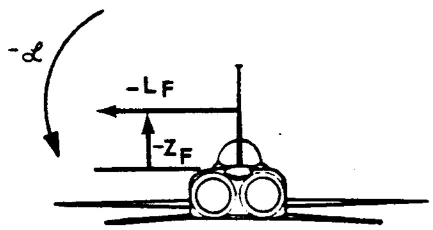


Figure 5.33 ROLLING MOMENT CREATED BY VERTICAL TAIL AT A POSITIVE ANGLE OF SIDESLIPE

In the situation depicted in figure 5.33, the negative rolling moment was created by a positive sideslip angle, thus, the vertical tail contributes a stable increment to dihedral effect. This contribution can be quite large. In fact, it can be the major contribution to C_{ℓ_β} on aircraft with large vertical tails such as the F-104 and the T-38. This effect can be calculated in the same manner yawing moments were calculated in the directional case.

Assuming a positive sideslip angle,

$$-\mathcal{L}_F = (-L_F)\ (-Z_F) \tag{5.64}$$

$$C_{\ell_F} = \frac{-Z_F L_F}{q_w S_w b_w} \tag{5.65}$$

$$C_{\ell_F} = \frac{-Z_F C_{L_F} q_F S_F}{q_w S_w b_w} \tag{5.66}$$

Define V_F as,

$$V_F = \frac{S_F Z_F}{S_w b_w} \tag{5.67}$$

Assume that for a jet aircraft,

$$q_F = q_w \tag{5.68}$$

And equation 5.66 becomes,

$$C_{\ell_F} = - C_{L_F} V_F = - a_F \alpha_F V_F \tag{5.69}$$

Knowing

$$\alpha_F = (\beta - \sigma) \tag{5.14}$$

$$C_{\ell_F} = - a_F V_F \; (\beta - \sigma) \tag{5.70}$$

$$C_{\ell_{\beta_{\text{Vertical tail}}}} = \frac{\partial C_{\ell_F}}{\partial \beta} = - a_F V_F \left(1 - \frac{\partial \sigma}{\partial \beta}\right) \tag{5.71}$$

Equation 5.71 reveals that a vertical tail contributes a stable increment to C_{ℓ_β}, whereas a ventral fin [V_F = (+)] would contribute an unstable increment to C_{ℓ_β}. Also, if the lift curve slope of the vertical tail is increased, by end plating for example, the stable dihedral effect would be greatly increased. For example, the F-104 has a high horizontal stabilizer that acts as an end plate on the vertical tail and this increases the stable dihedral effect. In fact, the increase is so large that it is necessary to add negative geometric dihedral to the wings and a ventral fin to maintain a reasonable value of stable dihedral effect.

5.13 $C_{\ell_{\delta_a}}$ - LATERAL CONTROL POWER

Lateral control is achieved by altering the lift distribution so that the total lift on the two wings differ, thereby creating a rolling moment. This may be done simply by destroying a certain amount of lift on one wing by means of a spoiler, or by altering the lift on both wings by means of ailerons. This discussion will be limited to the use of ailerons as the means of lateral control.

Since the purpose of the ailerons is to create a rolling moment, a logical measure of aileron power would be the rolling moment created by a given aileron deflection. Before progressing to the actual development of this relationship, it is necessary to make several definitions. A positive deflection of either aileron, $+\delta_a$, is defined as one which produces a positive rolling moment, (right wing down). Thus, by definition, $C_{\ell_{\delta_a}}$ is positive. Also, in this discussion, total aileron deflection is defined as the sum of the two individual aileron deflections. Thus,

$$\delta_{a_{Total}} = \delta_{a_{Left}} + \delta_{a_{Right}} \tag{5.72}$$

The assumption will be made that the wing cp shift due to aileron deflection will not alter the value of C_{ℓ_β}. The distance from the x-axis to the cp of the wing will be labeled $\overline{Y}$. When the ailerons are deflected, they produce a change in lift on both wings. This total change in lift, ΔL, produces a rolling moment, $\mathcal{L}$.

$$\mathcal{L} = \Delta L \cdot \overline{Y} \tag{5.73}$$

$$\mathcal{L} = \frac{\partial C_{L_a}}{\partial \alpha_a} \cdot \Delta \alpha_a \cdot q_a \cdot S_a \cdot \overline{Y} \tag{5.74}$$

Where the "a" subscripts refer to "aileron" values.

$$\mathcal{L} = a_a \Delta \alpha_a q_a S_a \overline{Y} \tag{5.75}$$

$$C_\ell = \frac{a_a \Delta \alpha_a S_a \overline{Y}}{S_w b_w} \tag{5.76}$$

Where $\Delta\alpha_a = \delta_{a_{Total}}$

$$C_\ell = \frac{a_a \delta_{a_{Total}} S_a \overline{Y}}{S_w b_w} \tag{5.77}$$

$$\frac{\partial C_\ell}{\partial \delta_a} = C_{\ell_{\delta_a}} = \frac{a_a S_a \overline{Y}}{b_w S_w} \tag{5.78}$$

Thus, from equation 5.78, it can be seen that lateral control power is a function of the aileron airfoil section, the area of the aileron in relation to the area of the wing, and the location of the wing cp.

5.14 IRREVERSIBLE CONTROL SYSTEMS

Now that both C_{ℓ_β} and $C_{\ell_{\delta_a}}$ have been discussed, it is possible to develop a parameter which can be measured in flight to determine the static lateral stability of an aircraft. As in the directional case, the maneuver that will be flown will be a steady straight sideslip. Considering this maneuver, equation 5.6 reduces to,

$$C_\ell = C_{\ell_\beta}\beta + C_{\ell_{\delta_a}}\delta_a + C_{\ell_{\delta_r}}\delta_r = 0 \tag{5.79}$$

$$\delta_a = -\frac{C_{\ell_\beta}\beta}{C_{\ell_{\delta_a}}} - \frac{C_{\ell_{\delta_r}}}{C_{\ell_{\delta_a}}}\delta_r \tag{5.80}$$

$$\frac{\partial\delta_{a\,(\text{Fixed})}}{\partial\beta} = -\frac{C_{\ell_\beta\,(\text{Fixed})}}{C_{\ell_{\delta_a}}} \tag{5.81}$$

Thus, since $C_{\ell_{\delta_a}}$ is known once the aircraft is built, $\partial\delta_a/\partial\beta$, can be taken as a direct measure of the static lateral stability of an aircraft. Again, the subscript "Fixed" has been added as a reminder that in this development the aileron has not been free to "float."

Equation 5.81 reveals that for static lateral stability, a plot of $\partial\delta_a/\partial\beta$ should have a positive slope. Refer to figure 5.34.

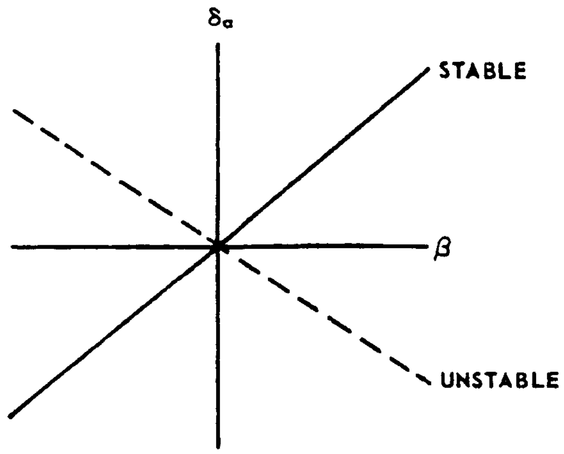


Figure 5.34 AILERON DEFLECTION VERSUS SIDESLIP ANGLE

5.15 REVERSIBLE CONTROL SYSTEMS

It is now necessary to consider an aircraft with a reversible control system. On this type aircraft, the ailerons are free to float in response to their hinge moments. Using the same approach as in the directional case, it is possible to derive an expression that will relate the "Aileron Free" static lateral stability to parameters that can be easily measured in flight.

In a steady straight sideslip, $\Sigma\mathcal{L} = 0$. Therefore, it follows that $\Sigma\mathcal{L}_{\substack{\text{Aileron}\\\text{Hinge Pin}}} = 0$. Now if moments are summed about the aileron hinge pin, the aileron force exerted by the pilot, F_a, acts through a moment arm and gearing mechanism, both accounted for by some constant, K, and must balance the other aerodynamic rolling moments so that $\Sigma\mathcal{L}_{\substack{\text{Aileron}\\\text{Hinge Pin}}} = 0$. Thus, in steady straight flight,

$$\Sigma\mathcal{L}_{\substack{\text{Aileron}\\\text{Hinge Pin}}} = 0 = F_a \cdot K + H_a \tag{5.82}$$

$$F_a = - G \cdot H_a \tag{5.83}$$

Where G is merely 1/K.

Knowing.

$$H_a = C_h \, q_a \, S_a \, c_a \tag{5.84}$$

From equation 5.26

$$H_a = q_a \, S_a \, c_a \, (C_{h_{\alpha_a}} \cdot \alpha_a + C_{h_{\delta_a}} \cdot \delta_a) \tag{5.85}$$

Thus, equation 5.83 becomes,

$$F_a = - G \, q_a \, S_a \, c_a \, (C_{h_{\alpha_a}} \cdot \alpha_a + C_{h_{\delta_a}} \cdot \delta_a) \tag{5.86}$$

From equation 5.27,

$$C_{h_{\alpha_a}} \cdot \alpha_a = - C_{h_{\delta_a}} \cdot \delta_{a(\text{Float})} \tag{5.87}$$

Equation 5.86 becomes,

$$F_a = - G \, q_a \, S_a \, c_a \, (-C_{h_{\delta_a}} \cdot \delta_{a(\text{Float})} + C_{h_{\delta_a}} \cdot \delta_a) \tag{5.88}$$

$$F_a = - G \, q_a \, S_a \, c_a \, C_{h_{\delta_a}} \, (\delta_a - \delta_{a(\text{Float})}) \tag{5.89}$$

The difference between where the pilot pushes the aileron, δ_a, and the amount it floats, $\delta_{a(Float)}$, is the free position of the aileron, $\delta_{a(Free)}$.

Therefore,

$$F_a = - G\, q_a\, S_a\, c_a\, C_{h_{\delta_a}}\, \delta_{a(Free)} \tag{5.90}$$

$$\frac{\partial F_a}{\partial \beta} = - G\, q_a\, S_a\, c_a\, C_{h_{\delta_a}} \frac{\partial \delta_{a(Free)}}{\partial \beta} \tag{5.91}$$

From equation 5.81, it can be shown that,

$$\frac{\partial \delta_{a(Free)}}{\partial \beta} = - \frac{C_{\ell_{\beta(Free)}}}{C_{\ell_{\delta_a}}} \tag{5.92}$$

Thus,

$$\frac{\partial F_a}{\partial \beta} = G\, q_a\, S_a\, c_a \frac{C_{h_{\delta_a}}}{C_{\ell_{\delta_a}}} C_{\ell_{\beta}(Free)} \tag{5.93}$$

This equation shows that the parameter $\partial F_a/\partial \beta$, can be taken as an indication of the aileron free static lateral stability of an aircraft. This parameter can be readily measured in flight.

An analysis of equation 5.93 reveals that for stable dihedral effect, a plot of $\partial F_a/\partial \beta$ would have a positive slope. Refer to figure 5.35.

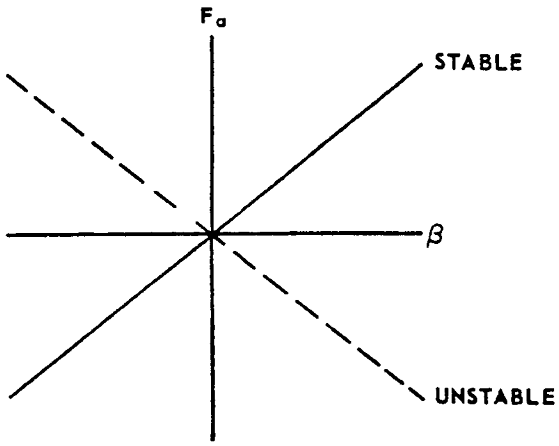


Figure 5.35 AILERON FORCE VERSUS SIDESLIP ANGLE

5.16 ROLLING PERFORMANCE

It has been shown how aileron force and aileron deflection can be used as a measure of the stable dihedral effect of an aircraft. However, it is now necessary to consider how aileron force and aileron deflection affect the rolling capability of the aircraft. For example, full aileron deflection may produce excellent rolling characteristics on certain aircraft, however, because of the large aileron forces required, the pilot may not be able to fully deflect the ailerons, thus making the overall rolling performance unsatisfactory. Thus, it is necessary to evaluate the rolling performance of the aircraft.

The rolling qualities of an aircraft can be evaluated by examining the parameters F_a, δ_a, p and (pb/2V). Although the importance of the first three parameters is readily apparent, the parameter (pb/2V) needs some additional explanation. Physically, (pb/2V) may be described as the helix angle that the wing tip of a rolling aircraft describes. Refer to figure 5.36.

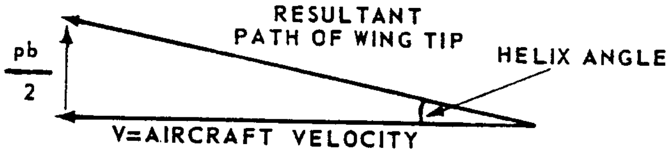


Figure 5.36
WING TIP HELIX ANGLE

It can be seen that,

$$\tan\text{(Helix Angle)} = \frac{pb}{2V} \tag{5.94}$$

Assuming a small angle,

$$\text{Helix Angle} = \frac{pb}{2V} \tag{5.95}$$

Figure 5.36 is a vectorial presentation of the wind forces acting on the downgoing wing during a roll. It shows that the angle of attack of the downgoing wing is increased due to roll rate. Thus (pb/2V) represents a damping term.

With the foregoing background, it is possible to discuss the effect of the parameters, F_a, δ_a, p, (pb/2V) throughout the flight envelope of the aircraft.

From equation 5.90, it can be seen that

$$F_a = (f)\ V^2\ \delta_{a(Free)} \qquad (5.96)$$

$$\delta_{a(Free)} = (f)\ F_a \frac{1}{V^2} \qquad (5.97)$$

To derive a functional relationship for (pb/2V), it is necessary to start with,

$$C_\ell = C_{\ell_\beta}\beta + C_{\ell_{\dot\beta}}\hat{\dot\beta} + C_{\ell_p}\hat{p} + C_{\ell_r}\hat{r} + C_{\ell_{\delta_a}}\delta_a + C_{\ell_{\delta_r}}\delta_r \qquad (5.6)$$

and examine the effects of roll terms only, i.e., assume that the roll moment developed is due to the interaction of moments due to δ_a and roll damping only. Therefore, equation 5.6 becomes,

$$C_\ell = C_{\ell_p}\hat{p} + C_{\ell_{\delta_a}}\delta_a = C_{\ell_p}\left(\frac{pb}{2V}\right) + C_{\ell_{\delta_a}}\delta_a \qquad (5.98)$$

Below Mach or aerolastic effects, $C_{\ell_{Max}}$ = constant, so if it is desired to evaluate an aircraft's maximum rolling performance, equation 5.98 becomes,

$$C_{\ell_p}\left(\frac{pb}{2V}\right) + C_{\ell_{\delta_a}}\delta_a = \text{constant} \qquad (5.99)$$

$$\left(\frac{pb}{2V}\right) = \frac{\text{Constant} - C_{\ell_{\delta_a}}\delta_a}{C_{\ell_p}} \qquad (5.100)$$

$$\left(\frac{pb}{2V}\right) = (f)\ \delta_a \qquad (5.101)$$

From equation 5.97,

$$\left(\frac{pb}{2V}\right) = (f)\ \delta_a = (f)\ F_a \frac{1}{V^2} \qquad (5.102)$$

A function relationship for roll rate, p, can be derived from equation 5.100,

$$p = \frac{\text{Constant} - C_{\ell_{\delta_a}}\delta_a}{C_{\ell_p}} \cdot \frac{2}{b} \cdot V \qquad (5.103)$$

$$p = (f)\ V\ \delta_a \qquad (5.104)$$

From equation 5.97,

$$p = (f)\ V\ \delta_a = (f)\ F_a\ \frac{1}{V} \qquad (5.105)$$

To summarize, the rolling performance of an aircraft can be evaluated by examining the parameters, F_a, δ_a, p, and (pb/2V). Functional relationships have been developed in order to look at the variance of these parameters below Mach or aeroelastic effects. These functional relationships are:

$$F_a = (f)\ V^2\ \delta_a \qquad (5.96)$$

$$\delta_a = (f)\ F_a\ \frac{1}{V^2} \qquad (5.97)$$

$$(\frac{pb}{2V}) = (f)\ \delta_a = (f)\ F_a\ \frac{1}{V^2} \qquad (5.102)$$

$$p = (f)\ V\ \delta_a = (f)\ F_a\ \frac{1}{V} \qquad (5.105)$$

These relationships are expressed graphically in figure 5.37 for a case in which the pilot desires the maximum roll rate at all airspeeds.

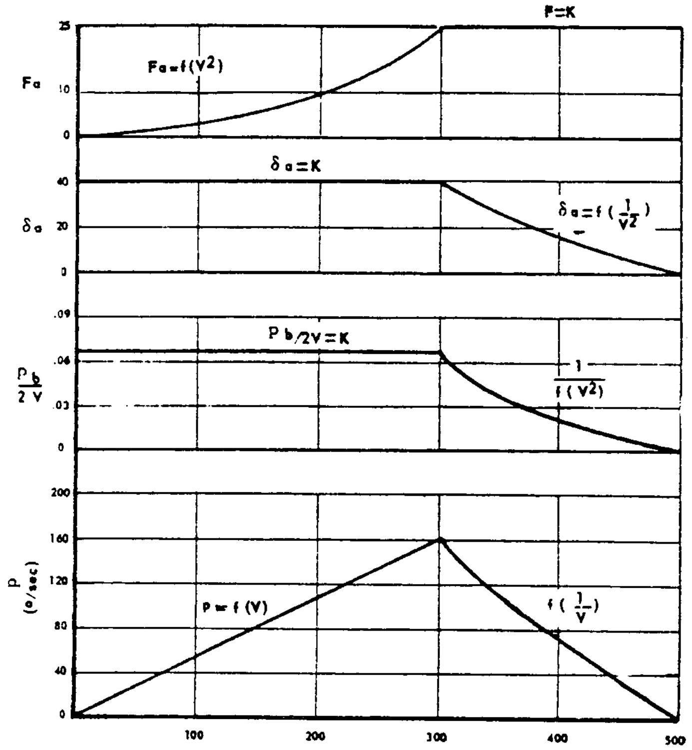


Figure 5.37 ROLLING PERFORMANCE

As indicated in equation 5.96, the force required to hold a constant aileron deflection will vary as the square of the airspeed. The force required by the pilot to hold full aileron deflection will increase in this manner until the aircraft reaches V_{Max} or until the pilot is unable to apply any more force. In figure 5.37, it is assumed that the pilot can supply a maximum of 25 pounds force and that this force is reached at 300 knots. If the speed is increased further, the aileron force will remain at this 25 pound maximum value. The curve of aileron deflection versus airspeed shows that the pilot is able to maintain full aileron deflection out to 300 knots. Inspection of equation 5.97 shows that if aileron force is constant beyond 300 knots, then aileron deflection will be proportional to $(1/V^2)$. Equation 5.102 shows that (pb/2V) will vary in the same manner as aileron deflection. Inspection of equation 5.105 shows that the maximum roll rate available will increase linearly as long as the pilot can maintain maximum aileron deflection; up to 300 knots in this case. Beyond this point, the maximum roll rate will fall off hyperbolically. That is, above 300 knots, p is proportional to 1/V. It follows, then, that at high speeds the maximum roll rate may become unacceptably low. One method of combating this problem is to increase the pilot's mechanical advantage by adding boosted or fully powered ailerons.

5.17 ROLL DAMPING C_{ℓ_p}

Aircraft roll damping comes from the wing and the vertical tail. The algebraic sign of C_{ℓ_p} is negative as long as the local angle of attack remains below the local stall angle of attack.

The wing contribution to C_{ℓ_p} arises from the change in wing angle of attack that results from the rolling velocity. It has already been shown that the downgoing wing in a rolling maneuver experiences an increase in angle of attack and that this increased α tends to develop a rolling moment that opposes the original rolling moment. However, when the wing is near the aerodynamic stall, a rolling motion may cause the downgoing wing to exceed the stall angle of attack. In this case, the local lift curve slope may fall to zero or even reverse sign. The algebraic sign of the wing contribution to C_{ℓ_p} may then become positive. This is what occurs when a wing "autorotates," as in spinning.

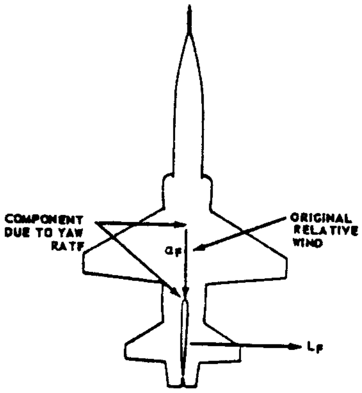
The vertical tail contribution to C_{ℓ_p} arises from the fact that when the aircraft is rolled, the angle of attack on the vertical tail is changed. This change in angle of attack develops a lift force. If the vertical tail cg is above or below the aircraft cg, the rolling moment developed will oppose the original rolling moment and C_{ℓ_p} due to a conventional vertical tail or a ventral fin will be negative.

5.18 ROLLING MOMENT DUE TO YAW RATE – C_{ℓ_r}

The contributions to this derivative come from two sources, the wings and the vertical tail.

As the aircraft yaws, the velocity of the relative wind is increased on the outboard wing and decreased on the inboard wing. This causes the outboard wing to produce more lift and thus produces a rolling moment. A right yaw would produce more lift on the left wing and thus a rolling moment to the right. Thus, the algebraic sign of the wing contribution to C_{ℓ_r} is positive.

The tail contribution to C_{ℓ_r} arises from the fact that as the aircraft is yawed, the angle of attack on the vertical tail is changed. Refer to figure 5.38.



CHANGE IN α_F DUE TO YAW RATE

Figure 5.38

The lift force thus produced, L_F, will create a rolling moment if the vertical tail cg is above or below the cg. For a conventional vertical tail, the sign of C_{ℓ_r} will be positive while for a ventral fin the sign will be negative.

5.19 ROLLING MOMENT DUE TO RUDDER DEFLECTION – $C_{\ell_{\delta_r}}$

When the rudder is deflected, it creates a lift force on the vertical tail. If the cp of the vertical tail is above or below the aircraft cg a rolling moment will result. Refer to figure 5.39.

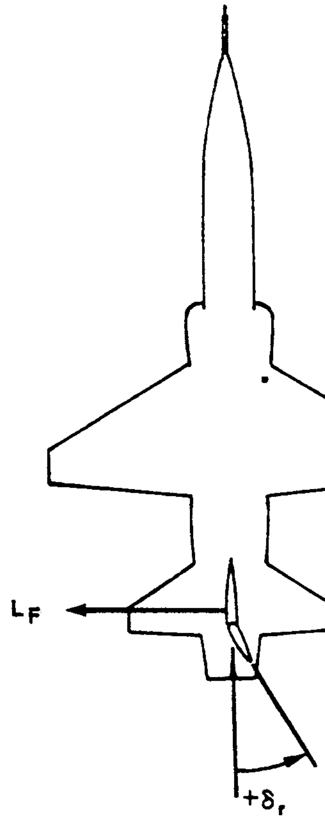


Figure 5.39 LIFT FORCE DEVELOPED AS A RESULT OF δ_r

It can be seen that if the cp of the vertical tail is above the cg, as with a conventional vertical tail, the sign of $C_{\ell_{\delta_r}}$ will be negative. However, with a ventral fin, the sign would be positive.

It is interesting to note that the effects of $C_{\ell_{\delta_r}}$ and C_{ℓ_β} are opposite in nature. When the rudder is deflected to the right, initially, a rolling moment to the left is created due to $C_{\ell_{\delta_r}}$. However, as sideslip develops due to the rudder deflection, dihedral effect, C_{ℓ_β}, comes into play and causes a resulting rolling moment to the right. Therefore, when a pilot applies right rudder to pick up a left wing, he initially creates a rolling moment to the left and finally, to the right.

5.20 ROLLING MOMENT DUE TO LAG EFFECTS IN SIDEWASH $-C_{\ell_{\dot{\beta}}}$

In the discussion of $C_{n_{\dot{\beta}}}$, it was pointed out that during an increase in β, the angle of attack of the vertical tail will be less than it will finally be in steady state conditions. If the cp of the vertical tail is displaced from the aircraft cg, this change in α_F due to lag effects will alter the rolling moment created during the β build up period. Because of lag effects, C_{ℓ_F} will be less during the β build up period than at steady state. Thus, for a conventional vertical tail, the algebraic sign of $C_{\ell_{\dot{\beta}}}$ is positive.

Again, it should be pointed out that there is widespread disagreement over the interpretation of data concerning lag effects in sidewash and that the foregoing is only one basic approach to a many faceted and complex problem.

5.21 HIGH SPEED CONSIDERATIONS OF STATIC LATERAL STABILITY

C_{ℓ_β} - Generally, C_{ℓ_β} is not greatly affected by Mach number. However, in the transonic region the increase in the lift curve slope of the vertical tail increases this contribution to C_{ℓ_β} and usually results in an overall increase in C_{ℓ_β} in the transonic region.

$C_{\ell_{\delta_a}}$ - Because of the decrease in the lift curve slope of all aerodynamic surfaces in supersonic flight, lateral control power decreases as Mach number increases supersonically.

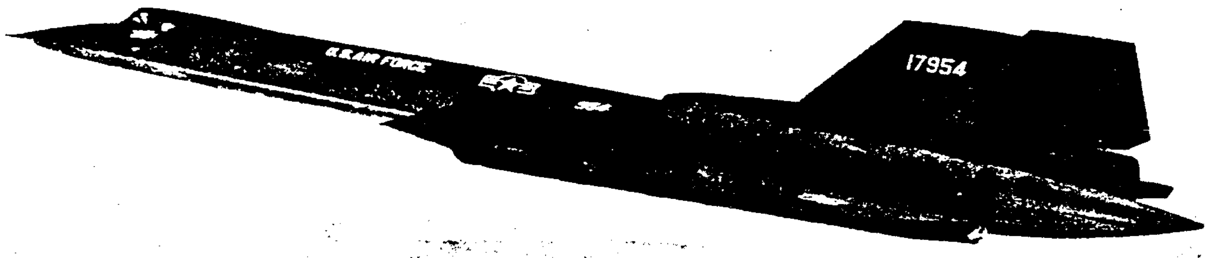
Aeroelasticity problems have been quite predominant in the lateral control system, since in flight at very high dynamic pressures the wing torsional deflections which occur with aileron usage are considerable and cause noticeable changes in aileron effectiveness. At some high dynamic pressures, dependent upon the given wing structural integrity, the twisting deformation might be great enough to nullify the effect of aileron deflection and the aileron effectiveness will be reduced to zero. Since at speeds above the point where this phenomenon occurs, rolling moments are created which are opposite in direction to the control deflection, this speed is termed "aileron reversal speed." In order to alleviate this characteristic the wing must have a high torsional stiffness which presents a significant design problem in sweptwing aircraft. For an aircraft design of the B-47 type, it is easy to visualize how aeroelastic distortion might result in a considerable reduction in lateral control capability at high speeds. In addition, lateral control effectiveness at transonic Mach numbers may be reduced seriously by flow separation effects as a result of shock formation. However, modern high speed fighter designs have been so successful in introducing sufficient rigidity into wing structures and employing such design modifications as split ailerons, inboard ailerons, spoiler systems, etc., that the resulting high control power, coupled with the low C_{ℓ_p} of low aspect ratio planforms, has resulted in the lateral control becoming an accelerating device rather than a rate control. That is to say, a steady state rolling velocity is normally not reached prior to attaining the desired bank angle. Consequently, many high speed aircraft have a type of differential aileron system to provide the pilot with much more control surface during approach and landings and to restrict his degree of control in other areas of flight.

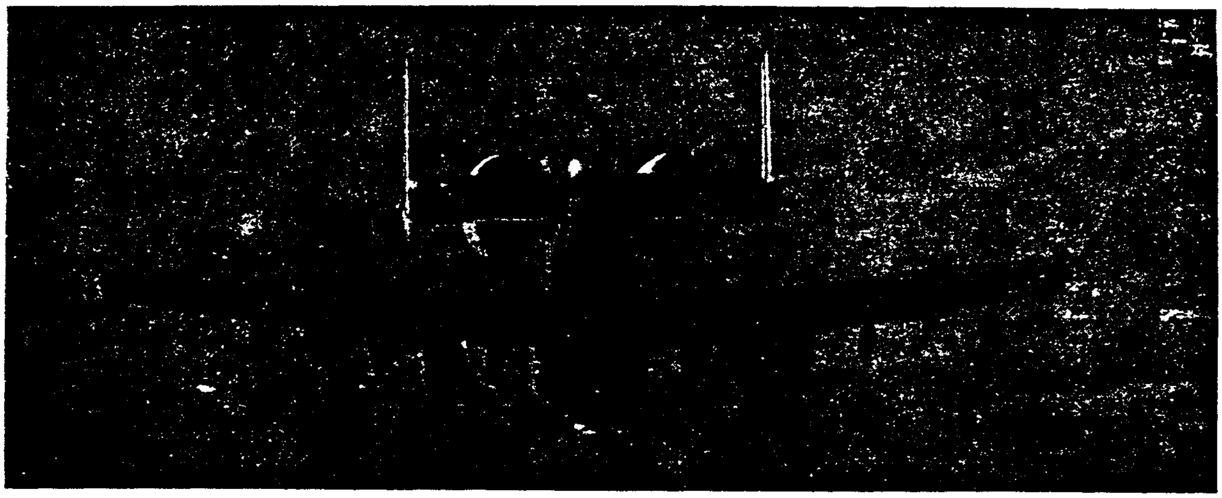
Spoiler controls are quite effective in reducing aeroelastic distortions since the pitching moment changes due to spoilers are generally smaller than those for a flap type control surface. However, a problem associated with spoilers is their tendency to reverse the roll direction for small stick inputs during transonic flight. This occurs as a result of re-energizing the boundary layer by a vortex generator effect for very small deflections of the spoiler, which can reduce the magnitude of the shock induced separation and actually increase the lift on the wing. This difficulty can be eliminated by proper design techniques.

C_{ℓ_p} - Since the development of "damping" requires the development of lift on either the wing or the tail, it is dependent on the value of the lift curve slope. Thus, as the lift curve slope of both the wing and tail decrease supersonically, C_{ℓ_p} will decrease. Also, since most supersonic designs make use of low aspect ratio surfaces, C_{ℓ_p} will tend to be less for these designs.

C_{ℓ_r} and $C_{\ell_{\delta_r}}$ - Both of these derivatives depend on the development of lift and will decrease as the lift curve slope decreases supersonically.

$C_{\ell_{\dot{\beta}}}$ - Data on the supersonic variation of this derivative is sketchy, but it probably will not change significantly with Mach number.





VOLUME I

CHAPTER 6

DYNAMICS

REVISED FEBRUARY 1977

6.1 INTRODUCTION

The study of dynamics is concerned with the time history of the motion of some physical system. An aircraft is such a system, and its equations of motion can be derived from theory. In their basic form these equations comprise a set of six simultaneous, nonlinear differential equations with ill-defined forcing functions such as $F_X = f$ (Aero, Gravity, Thrust). Recall from Chapter 1 that two methods were used to get these equations into a set of workable simultaneous linear differential equations:

1. Small perturbations were assumed such that products of perturbations were negligible.

2. Forcing functions were approximated by the linear part of a Taylor Series expansion for the forcing function.

This set of linear differential equations can then be operated on by Laplace transforms so that simple algebraic solutions followed by inverse transformations back to the time domain result in equations which describe the aircraft's motion as a function of time.

In the good old days when aircraft were simple, all aircraft exhibited the five characteristic dynamic modes of motion, two longitudinal and three lateral-directional modes. The two longitudinal modes are the short period and the phugoid; the three lateral-directional modes are the Dutch roll, the spiral, and the roll mode. Theoretical solutions for these modes of motion can be obtained by the methods listed above.

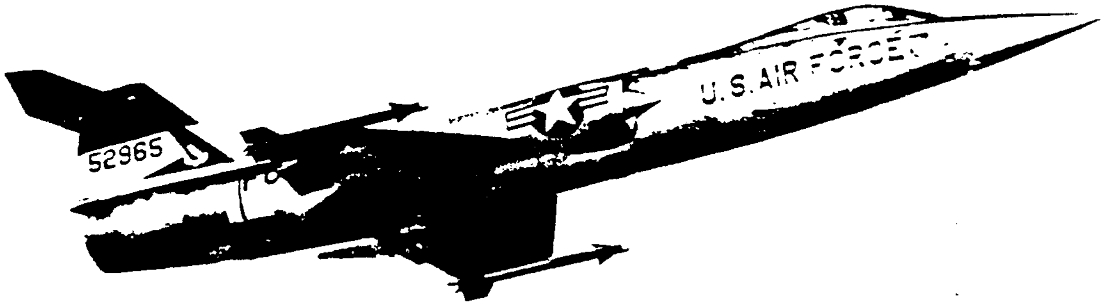
As aircraft control systems have increased in complexity, it is conceivable that one or more of these modes may not exist as a dominant longitudinal or lateral-directional mode. It can be expected, however, that frequently the higher order effects of complex control systems will be quick to die out and will leave the basic five dynamic modes of motion. When this is not the case, the development of special procedures may be required to meaningfully describe an aircraft's dynamic motion. For the purposes of this chapter, aircraft will be assumed to possess these five basic modes of motion.

During this study of aircraft dynamics, the solutions to both first order and second order systems will be of interest, and several important descriptive parameters will be used to define either a first order system or a second order system.

The quantification of handling qualities, that is, specify how the magnitude of some of these descriptive parameters can be used to indicate how well an aircraft can be flown, has been an extensive investigation which is by no means complete. Flight test, simulators, variable stability aircraft, engineering know-how, and pilot opinion surveys have all played major roles in this investigation. The military specification on aircraft handling qualities, MIL-F-8785B, is the culmination of this effort and has the intent of insuring that an aircraft will handle well if compliance has been achieved. This chapter will not attempt to evaluate how satisfactory MIL-F-8785B is for this purpose, but suffice it to say that even this comprehensive document has some room for improvement.

6.2 DYNAMIC STABILITY

When it is necessary to investigate the dynamic stability characteristics of a physical system, the time history of its motion must be known.



As indicated earlier, this time history can be obtained theoretically and with good accuracy in many cases, depending on the depth of the theoretical analysis.

A particular mode of an aircraft's motion is defined to be "dynamically stable" if the parameters of interest tend toward finite values as time increases without limit. Some examples of dynamically stable time histories and some terms used to describe them are shown in figures 6.1 and 6.2

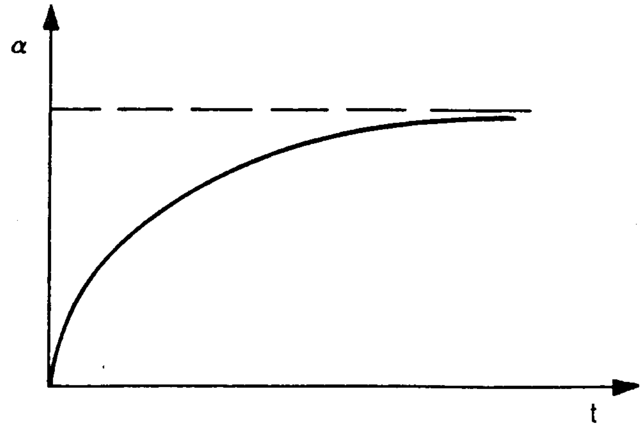


Figure 6.1 Exponentially Decreasing

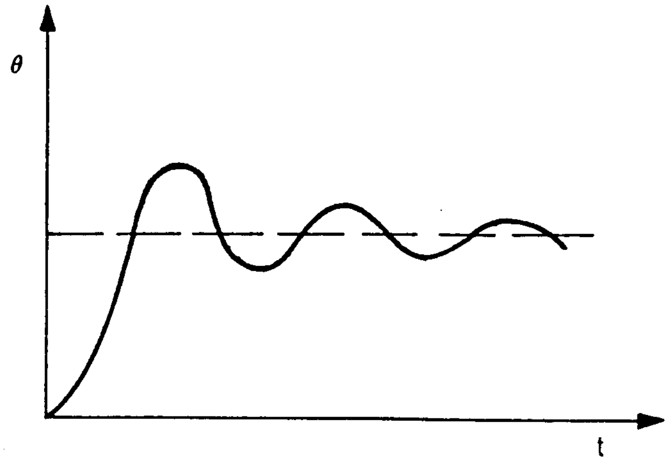


Figure 6.2 Damped Sinusoidal Oscillation

A mode of motion is defined to be "dynamically unstable" if the parameters of interest increase without limit as time increases without limit. Some examples of dynamic instability are shown in figures 6.3 and 6.4.

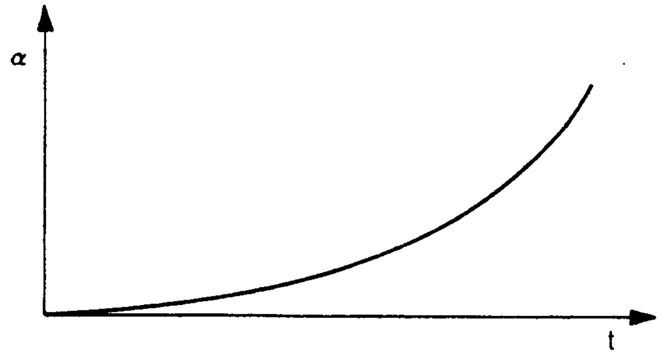


Figure 6.3 Exponentially Increasing

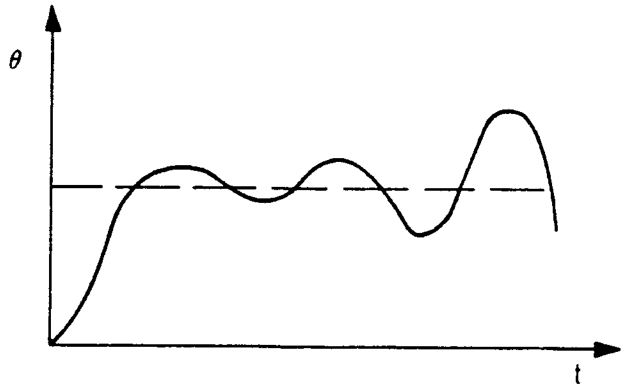


Figure 6.4 Divergent Sinusoidal Oscillation

A mode of motion is said to have "neutral dynamic stability" if the parameters of interest exhibit an undamped sinusoidal oscillation as time increases without limit. A sketch of such motion is shown in figure 6.5.

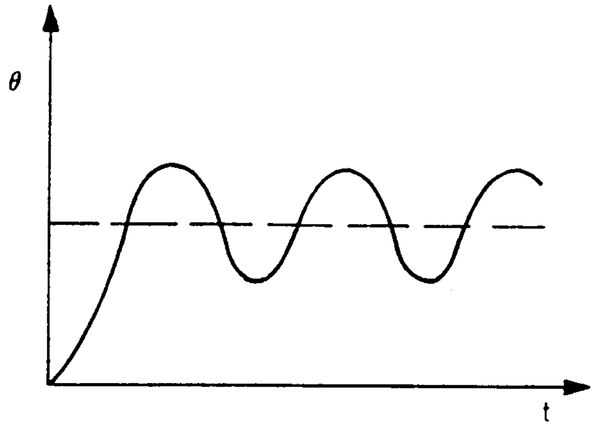


Figure 6.5 Undamped Oscillation

● 6.2.1 EXAMPLE PROBLEM

To emphasize the difference between static stability and dynamic stability the simple physical system shown in figure 6.6 consisting of a mass and a spring will be examined for both static stability and dynamic stability.

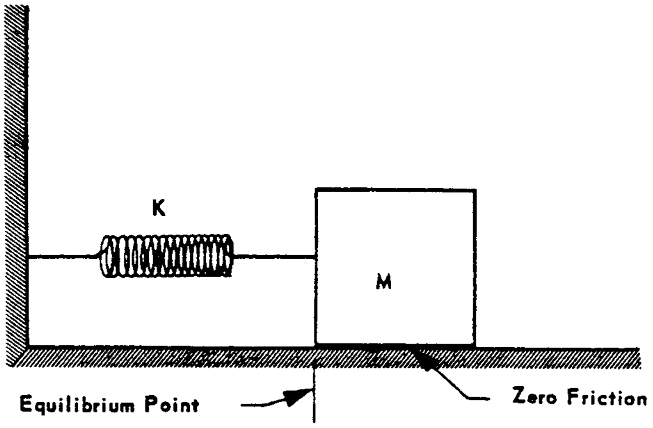


Figure 6.6

● 6.2.1.1 STATIC STABILITY ANALYSIS

If the mass were displaced from its equilibrium position, then a spring force would exist to return M toward its initial position. Thus, this physical system has positive static stability.

● 6.2.1.2 DYNAMIC STABILITY ANALYSIS

The motion of the system as a function of time must be known to describe its dynamic stability. Two methods could be used to find the time history of the motion of the block:

1. A test could be devised to perturb the block from its equilibrium position and the resulting motion would be observed for analysis.

2. If a good enough mathematical model of the system could be obtained, the equation of motion could be analyzed to describe its dynamic stability.

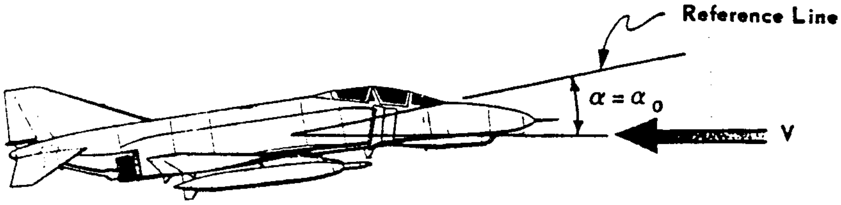
Using the theoretical approach, the equation of motion for this physical system is

$$x(t) = C_1 \cos \left(\sqrt{\frac{K}{M}}\, t + \phi\right)$$

where C_1 and ϕ are constants dependent on the initial velocity and displacement of the mass from its equilibrium condition. Examination of its equation of motion shows that this system has "neutral dynamic stability."

6.2.2 EXAMPLE PROBLEM

A similar analysis must be accomplished to analyze the aircraft shown in figure 6.7 for longitudinal static stability and dynamic stability. This aircraft is operating at a constant α_o in 1-g flight.



(Specified That $C_{M_\alpha} < 0$)

Figure 6.7

6.2.2.1 STATIC STABILITY ANALYSIS

If the aircraft were displaced from its equilibrium flight conditions by increasing the angle of attack to $\alpha = \alpha_o + \Delta\alpha$, then the change in pitching moment due to the increase in angle of attack would be nose down because $C_{M_\alpha} < 0$. Thus the aircraft has positive static longitudinal stability.

6.2.2.2 DYNAMIC STABILITY ANALYSIS

The motion of the aircraft as a function of time must be known to describe its dynamic stability. Two methods could be used to find the time history of the motion of the aircraft:

1. A flight test could be flown in which the aircraft is perturbed from its equilibrium condition and the resulting motion is recorded and observed.

2. Solutions to the aircraft equations of motion could be obtained and analyzed.

A sophisticated solution to the aircraft equations of motion with valid aerodynamic inputs can result in good theoretically obtained time histories. However, the fact remains that the <u>only</u> way to discover the aircraft's actual dynamic motion is to flight test and record its motion for analysis.

6.3 EXAMPLES OF FIRST AND SECOND ORDER DYNAMIC SYSTEMS

6.3.1 SECOND ORDER SYSTEM WITH POSITIVE DAMPING

The problem of finding the motion of the block shown in figure 6.8 encompasses many of the methods and ideas that will be used in finding the time history of an aircraft's motion from its equations of motion.

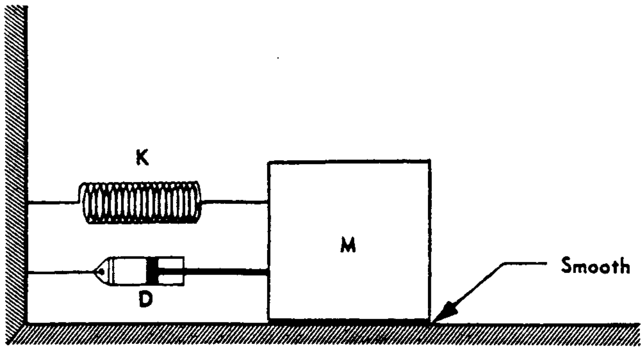


Figure 6.8 Second Order System

The differential equation of motion for this physical system is

$$0 = M\ddot{x} + D\dot{x} + Kx \tag{6.1}$$

After Laplace transforming equation 6.1 and solving for X(S), the displacement of the block in the Laplace domain, the result is

$$X(S) = \frac{Sx_o + \dot{x}_o + \frac{D}{M}x_o}{S^2 + \frac{D}{M}S + \frac{K}{M}} \tag{6.2}$$

The denominator of the S domain equation which gives the response of a system will be referred to as its "characteristic equation," and the symbol $\Delta(S)$ will be used to indicate the characteristic equation.

The $\Delta(S)$ of a second order system will frequently be written in a standard notation

$$0 = S^2 + 2\zeta\omega_n S + \omega_n^2 \tag{6.3}$$

where

ω_n = natural frequency

ζ = damping ratio

The two terms natural frequency and damping ratio are frequently used to characterize the motion of second order systems.

Also, knowing the location of the roots of $\Delta(S)$ on the complex plane makes it possible to immediately specify and sketch the dynamic motion associated with a system. Continuing to discuss the problem shown in figure 6.8 and making an identity between equations 6.2 and 6.3 results in

$$\omega_n = \sqrt{\frac{K}{M}} \tag{6.4}$$

$$\zeta = \frac{D}{2M\sqrt{\frac{K}{M}}} \tag{6.5}$$

The roots of $\Delta(S)$ can be found from equation 6.3 to be

$$S_{1,2} = -\zeta\omega_n \pm i\omega_d \tag{6.6}$$

Where

$$\omega_d = \omega_n\sqrt{1 - \zeta^2}$$

Note that if $(-1 < \zeta < 1)$, then the roots of $\Delta(S)$ comprise a complex conjugate pair, and for positive ζ would result in root locations as shown in figure 6.9.

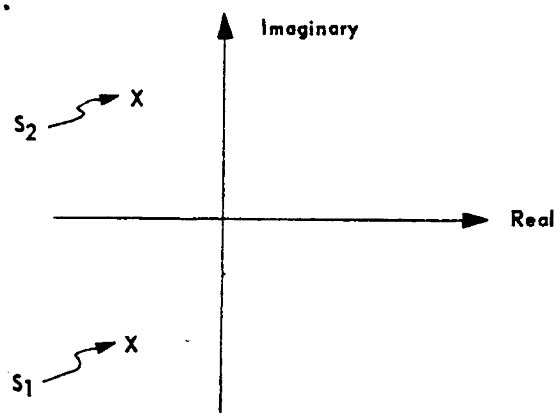


Figure 6.9 Complex Plane

The equation describing the time history of the block's motion can be written by knowing the roots of $\Delta(S)$ given in equation 6.6.

$$x(t) = C_1\, e^{-\zeta\omega_n t} \sin(\omega_d t + \phi) \tag{6.7}$$

Where

C_1 and ϕ are constants determined by boundary conditions.

Knowing either the $\Delta(S)$ root location shown in figure 6.9 or equation 6.7 makes it possible to sketch or describe the time history of the motion of the block. The motion of the block shown in figure 6.8 as a function of time is a sinusoidal oscillation within an exponentially decaying envelope and is dynamically stable.

6.3.2 SECOND ORDER SYSTEM WITH NEGATIVE DAMPING

A similar procedure to that used in Section 6.3.1 can be used to find the motion of the block shown in figure 6.10.

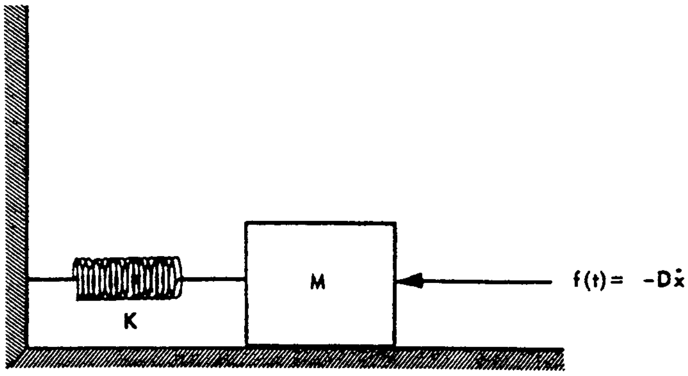


Figure 6.10

The differential equation of motion for this block is

$$0 = M\ddot{x} - D\dot{x} + Kx \tag{6.8}$$

and the equation for X(S) is

$$X(S) = \frac{Sx_o + \dot{x}_o - \frac{D}{M}x_o}{S^2 - \frac{D}{M}S + \frac{K}{M}} \tag{6.9}$$

By inspection, for this system

$$\omega_n = \sqrt{\frac{K}{M}} \tag{6.10}$$

$$\zeta = \frac{-D}{2M\sqrt{\frac{K}{M}}} \tag{6.11}$$

From equation 6.11 note that the damping ratio has a negative value. The equation giving the time response of this system is

$$x(t) = C_1 e^{(\text{pos. value})t} \sin(\omega_d t + \phi) \tag{6.12}$$

where

$$-\zeta\omega_n = \text{pos. value}$$

For the range $(-1 < \zeta < 0)$, the roots of $\Delta(S)$ for this system could again be plotted on the complex plane from equation 6.6 as shown in figure 6.11.

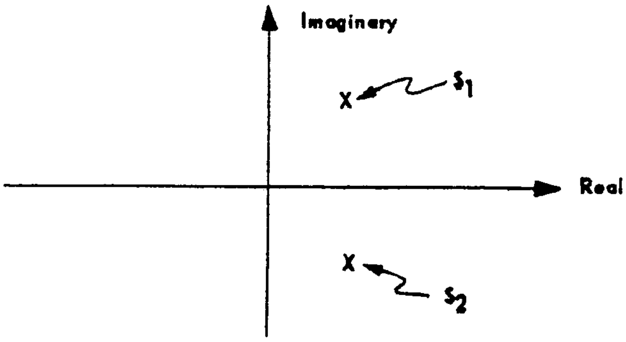


Figure 6.11 Complex Plane

The motion of this system can now be sketched or described. The motion of this system is a sinusoidal oscillation within an exponentially diverging envelope and is dynamically unstable.

6.3.3 UNSTABLE FIRST ORDER SYSTEM

Assume that some physical system has been mathematically modeled and its equation of motion in the S domain is

$$\phi(S) = \frac{0.5}{0.4S - 0.7} \qquad (6.13)$$

For this system the characteristic equation is

$$\Delta(S) = S - 1.75$$

And its root is shown plotted on the complex plane in figure 6.12.

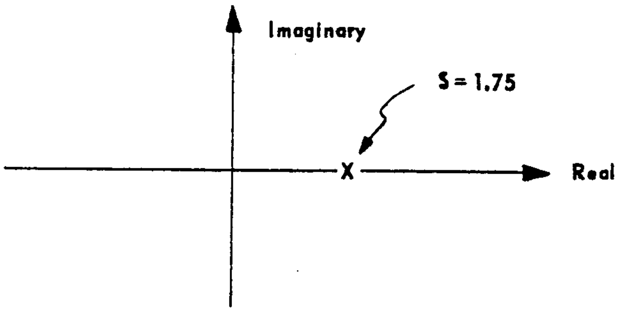


Figure 6.12 Complex Plane

The equation of motion in the time domain becomes

$$\phi(t) = 1.25\, e^{1.75\, t} \qquad (6.14)$$

Note that it is possible to sketch or describe the motion for this system by knowing the location of the root of $\Delta(S)$ or its equation of motion.

For an unstable first order system such as this, one parameter that can be used to characterize its motion is T_2, defined as the time to double amplitude. Without proof,

$$T_2 = \frac{.693}{a} \tag{6.15}$$

For a first order system described by

$$Z = C_1 \, e^{at} \tag{6.16}$$

Note that for a stable first order system a similar parameter has been defined: $T_{1/2}$ is the time to half amplitude.

$$T_{1/2} = \frac{-0.693}{a} \tag{6.17}$$

Where the term a must have a negative value for a stable system.

6.3.4 ADDITIONAL TERMS USED IN DYNAMICS

The time constant, τ, is defined for a stable first order system as the time when the exponent of e in the system equation is -1. From equation 6.16,

$$\tau = \frac{-1}{a} \tag{6.18}$$

The time constant can be thought of as the time required for the parameter of interest to accomplish $(1 - \frac{1}{e})$th of its total value change. Note that

$$\frac{1}{e} = \frac{1}{2.718} \approx \frac{1}{3}$$

so that

$$(1 - \frac{1}{e}) \approx \frac{2}{3}$$

With this in mind, it is easy to visualize an approximate value for a system time constant from a time history. Thus, the magnitude of the time constant gives a measure of how quickly the dynamic motion of a first order system occurs.

The following list contains some terms commonly used to describe second order systems based on damping ratio values:

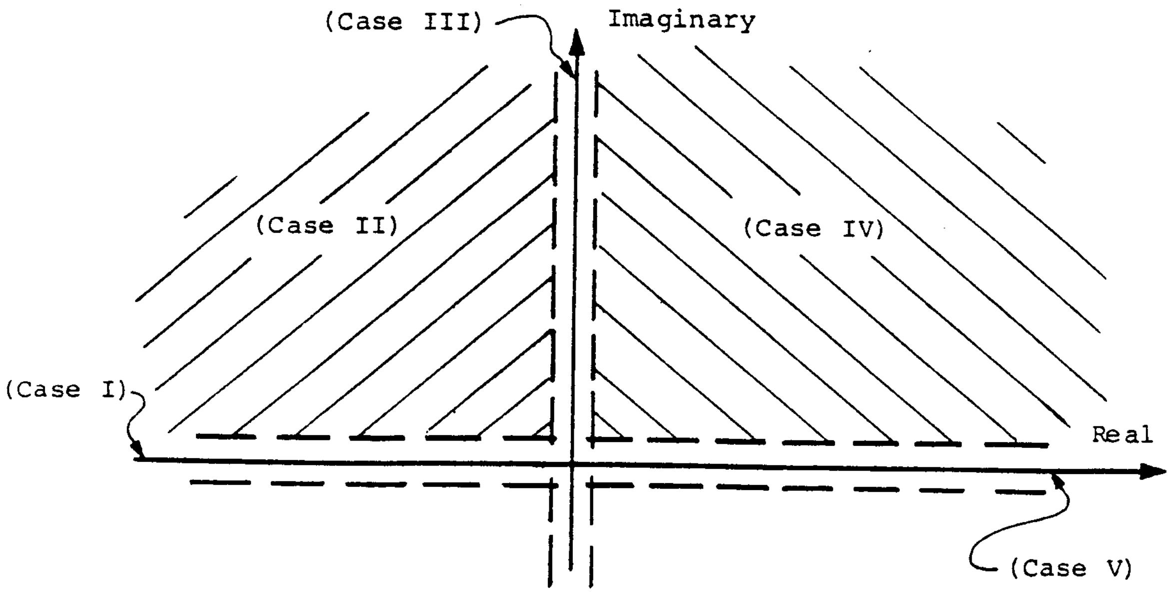
Term	Damping Ratio Value
Overdamped	$1 < \zeta$
Critically damped	$1 = \zeta$
Underdamped	$0 < \zeta < 1$
Undamped	$0 = \zeta$
Negatively damped	$\zeta < 0$

Typical responses can be visualized after the value of ζ has been established.

6.4 THE COMPLEX PLANE

It is possible to describe the type response a system will have by knowing the location of the roots of its characteristic equation on the complex plane. A first order response will be associated with each real root, and a complex conjugate pair will have a second order response that is either stable, neutrally stable, or unstable. A complicated system such as an aircraft might have a characteristic equation with several roots, and the total response of such a system will be the sum of the responses associated with each root. A summary of root location and associated response is presented in the following list and sketch below.

	Root Location	Associated Response
Case I	On the negative Real axis (1st Order Response)	Dynamically stable with exponential decay
Case II	In the left half plane off the negative Real axis (2nd Order Response)	Dynamically stable with sinusoidal oscillation in exponentially decaying envelope
Case III	On the Imaginary axis (2nd Order Response)	Neutral dynamic stability
Case IV	In the right half plane off the positive Real axis (2nd Order Response)	Dynamically unstable with sinusoidal oscillation in exponentially increasing envelope
Case V	On the positive Real axis (1st Order Response)	Dynamically unstable with exponential increase



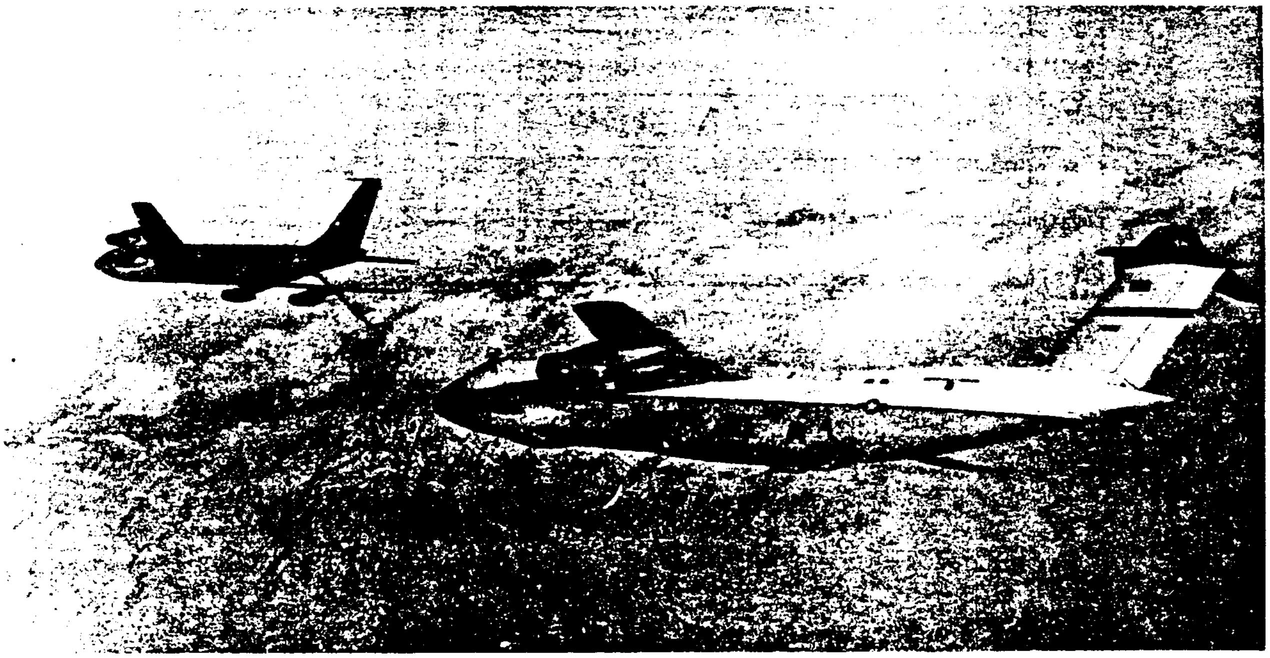
Note that any roots appearing in the upper half plane must have reflected roots in lower half plane.

6.5 HANDLING QUALITIES

Because the "goodness" with which an aircraft flies is often stated as a general appraisal . . . "My F-69 is the best damn fighter ever built, and it can outfly and outshoot any other airplane." "It flies good." "That was really hairy." . . . you probably can understand the difficulty of measuring how well an aircraft handles. The basic question of what parameters to measure and how those parameters relate to good handling qualities has been a difficult one, and the total answer is not yet available. The current best answers for military aircraft are found in MIL-F-8785B, the specification for the "Flying Qualities of Piloted Airplanes."

When an aircraft is designed for performance, the design team has definite goals to work toward . . . a particular takeoff distance, a minimum time to climb, or a specified combat radius. If an aircraft is also to be designed to handle well, it is necessary to have some definite handling quality goals to work toward. Success in attaining these goals can be measured by flight tests for handling qualities when some rather firm standards are available against which to measure and from which to recommend.

To make it possible to specify acceptable handling qualities it was necessary to evolve some flight test measurable parameters. Flight testing results in data which yield values for the various handling quality parameters, and the military specification gives a range of values that should insure good handling qualities. Because MIL-F-8785 is not the ultimate answer, the role of the test pilot in making accurate qualitative observations and reports in addition to generating the quantitative data is of great importance in handling qualities testing.



One method that has been extensively used in handling qualities quantification is the use of pilot opinion surveys and variable stability aircraft. For example, a best range of values for the short period damping ratio and natural frequency could be identified by flying a particular aircraft type to accomplish a specific task while allowing the ζ and ω_n to vary. From the opinions of a large number of pilots, a valid best range of values for ζ and ω_n could thus be obtained, as shown in figure 6.13.

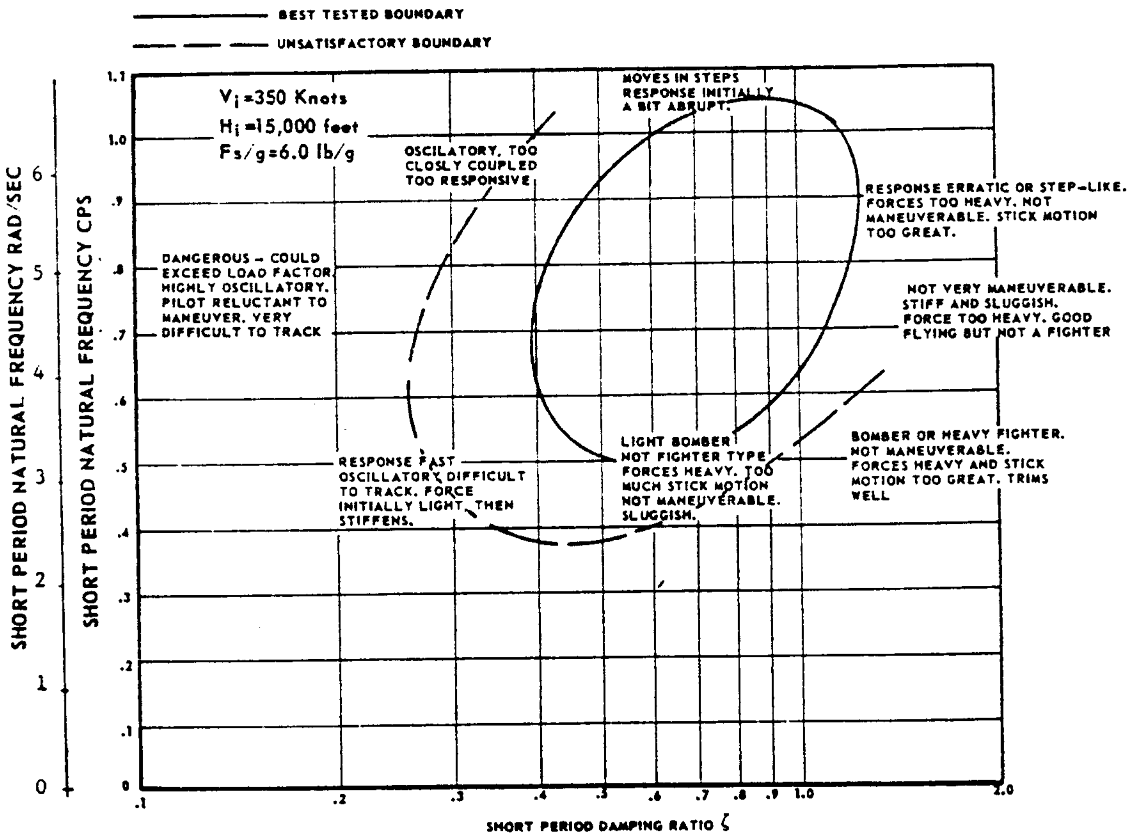


Figure 6.13

6.5.1 FREE RESPONSE

The ζ and ω_n being discussed here are the aircraft free response characteristics which describe aircraft motion without pilot inputs. With the pilot in the loop, the free response of the aircraft is hidden as pilot inputs are continually made. The closed loop block diagram shown in figure 6.14 should be used to understand aircraft closed loop and open loop response.

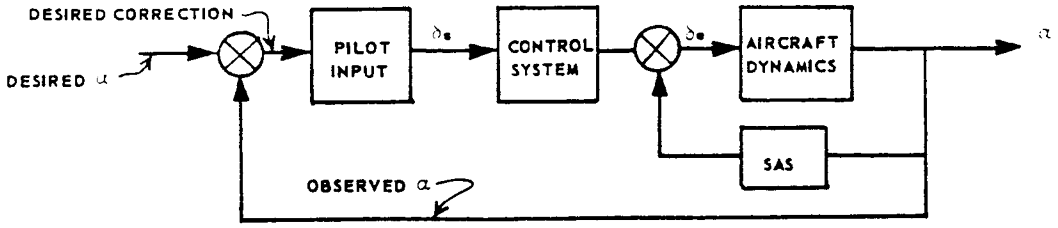


Figure 6.14

The free response of an aircraft does relate directly to how well the aircraft can be flown with a pilot in the loop, and many of the pertinent handling qualities parameters are for the open loop aircraft.

It must be kept in mind that the real test of an aircraft's handling qualities is how well it can be flown closed loop to accomplish a particular mission. Closed loop handling quality evaluations such as air-to-air tracking in a simulated air combat maneuvering mission play an important part of determining how well an aircraft handles.

6.5.2 PILOT RATING SCALES[1]

The Calspan Corporation (formerly Cornell Aeronautical Laboratory) has made notable contributions to the use and understanding of pilot rating scales and pilot opinion surveys. Except for minor variations between pilots, which sometimes prevent a sharp delineation between acceptable and unacceptable flight characteristics, there is very definite consistency and reliability in pilot opinion. In addition, the opinions of well qualified test pilots can be exploited because of their engineering knowledge and experience in many different aircraft types.

The stability and control characteristics of airplanes are generally established by wind tunnel measurement and by technical analysis as part of the airplane design process. The handling qualities of a particular airplane are related to the stability and control characteristics. The relationship is a complex one which involves the combination of the airplane and its human pilot in the accomplishment of the intended mission. It is important that the effects of specific stability and control characteristics be evaluated in terms of their ultimate effects on the suitability of the pilot-vehicle combination for the mission. On the basis of

[1] A Revised Pilot Rating Scale for the Evaluation of Handling Qualities, CAL Report No. 153, Robert P. Harper and George E. Cooper.

this information, intelligent decisions can be made during the airplane design phase which will lead to the desired handling qualities of the final product.

There are three general ways in which the relationship between stability and control parameters and the degree of suitability of the airplane for the mission may be examined:

1. Theoretical analysis

2. Experimental performance measurement

3. Pilot evaluation

Each of the three approaches has an important role in the complete evaluation. One might ask, however, why is the pilot assessment necessary? At present a mathematical representation of the human operator best lends itself to analysis of specific simple tasks. Since the intended use is made up of several tasks and several modes of pilot-vehicle behavior, difficulty is experienced first in accurately describing all modes analytically, and second in integrating the quality of the subordinate parts into a measure of overall quality for the intended use. In spite of these difficulties, theoretical analysis is fundamental to understanding pilot-vehicle difficulties, and pilot evaluation without it remains a purely experimental process.

The attainment of satisfactory performance in fulfillment of a designated mission is, of course, a fundamental reason for our concern with handling qualities. Why cannot the experimental measurement of performance replace pilot evaluation? Why not measure pilot-vehicle performance in the intended use - isn't good performance consonant with good quality? A significant difficulty arises here in that the performance measurement tasks may not demand of the pilot all that the real mission demands. The pilot is an adaptive controller whose goal (when so instructed) is to achieve good performance. In a specific task, he is capable of attaining essentially the same performance for a wide range of vehicle characteristics, at the expense of significant reductions in his capacity to assume other duties and planning operations. Significant differences in task performance may not be measured where very real differences in mission suitability do exist.

The questions which arise in using performance measurements may be summarized as follows: (1) For what maneuvers and tasks should measurements be made to define the mission suitability? (2) How do we integrate and weigh the performance in several tasks to give an overall measure of quality if measurable differences do exist? (3) Is it necessary to measure or evaluate pilot workload and attention factors for performance to be meaningful? If so, how are these factors weighed with those in (2)? (4) What disturbances and distractions are necessary to provide a realistic workload for the pilot during the measurement of his performance in the specified task?

Pilot evaluation still remains the only method of assessing the interactions between pilot performance and workload in determining suitability of the airplane for the mission. It is required in order to provide a basic measure of quality and to serve as a standard against which pilot-airplane system theory may be developed, against which performance measurements may be correlated and with which significant airplane design parameters may be determined and correlated.

The technical content of the pilot evaluation generally falls into two categories: one, the identification of characteristics which interfere with the intended use, and two, the determination of the extent to which these characteristics affect mission accomplishment. The latter judgment may be formalized as a pilot rating.

In 1956, the newly formed Society of Experimental Test Pilots accepted responsibility for one program session at the annual meeting of the Institute of Aeronautical Sciences. For this purpose, a paper, entitled "Understanding and Interpreting Pilot Opinion" was prepared, which represented an attempt to create better understanding and utilization of pilot opinion and evaluation in the field of aeronautical research and development. The widespread use of rating systems has indicated a general need for some uniform method of assessing aircraft handling qualities through pilot opinion.

Several rating scales were independently developed during the early use of variable stability aircraft. These vehicles, as well as the use of ground simulation, made possible systematic studies of aircraft handling qualities through pilot evaluation and rating of the effects of specific stability and control parameters.

Figure 6.15 shows the 10-point Harper-Cooper Rating Scale that is widely used today.

CONTROLLABLE CAPABLE OF BEING CONTROLLED OR MANAGED IN CONTEXT OF MISSION, WITH AVAILABLE PILOT ATTENTION	ACCEPTABLE MAY HAVE DEFICIENCIES WHICH WARRANT IMPROVEMENT, BUT ADEQUATE FOR MISSION. PILOT COMPENSATION, IF REQUIRED TO ACHIEVE ACCEPTABLE PERFORMANCE, IS FEASIBLE.	SATISFACTORY MEETS ALL REQUIREMENTS AND EXPECTATIONS, GOOD ENOUGH WITHOUT IMPROVEMENT CLEARLY ADEQUATE FOR MISSION.	EXCELLENT, HIGHLY DESIRABLE	A1
			GOOD, PLEASANT, WELL BEHAVED	A2
			FAIR. SOME MILDLY UNPLEASANT CHARACTERISTICS. GOOD ENOUGH FOR MISSION WITHOUT IMPROVEMENT.	A3
		UNSATISFACTORY RELUCTANTLY ACCEPTABLE. DEFICIENCIES WHICH WARRANT IMPROVEMENT. PERFORMANCE ADEQUATE FOR MISSION WITH FEASIBLE PILOT COMPENSATION.	SOME MINOR BUT ANNOYING DEFICIENCIES. IMPROVEMENT IS REQUESTED. EFFECT ON PERFORMANCE IS EASILY COMPENSATED FOR BY PILOT.	A4
			MODERATELY OBJECTIONABLE DEFICIENCIES. IMPROVEMENT IS NEEDED. REASONABLE PERFORMANCE REQUIRES CONSIDERABLE PILOT COMPENSATION.	A5
			VERY OBJECTIONABLE DEFICIENCIES. MAJOR IMPROVEMENTS ARE NEEDED. REQUIRES BEST AVAILABLE PILOT COMPENSATION TO ACHIEVE ACCEPTABLE PERFORMANCE.	A6
	UNACCEPTABLE DEFICIENCIES WHICH REQUIRE MANDATORY IMPROVEMENT. INADEQUATE PERFORMANCE FOR MISSION EVEN WITH MAXIMUM FEASIBLE PILOT COMPENSATION.		MAJOR DEFICIENCIES WHICH REQUIRE MANDATORY IMPROVEMENT FOR ACCEPTANCE. CONTROLLABLE. PERFORMANCE INADEQUATE FOR MISSION, OR PILOT COMPENSATION REQUIRED FOR MINIMUM ACCEPTABLE PERFORMANCE IN MISSION IS TOO HIGH.	U7
			CONTROLLABLE WITH DIFFICULTY. REQUIRES SUBSTANTIAL PILOT SKILL AND ATTENTION TO RETAIN CONTROL AND CONTINUE MISSION.	U8
			MARGINALLY CONTROLLABLE IN MISSION. REQUIRES MAXIMUM AVAILABLE PILOT SKILL AND ATTENTION TO RETAIN CONTROL.	U9
UNCONTROLLABLE CONTROL WILL BE LOST DURING SOME PORTION OF MISSION.			UNCONTROLLABLE IN MISSION.	10

Figure 6.15 Ten-Point Harper-Cooper Pilot Rating Scale

A flow chart is shown in figure 6.16 which traces the series of dichotomous decisions which the pilot makes in arriving at the final rating. As a rule, the first decision may be fairly obvious. Is the configuration controllable or uncontrollable? Subsequent decisions become less obvious as the final rating is approached.

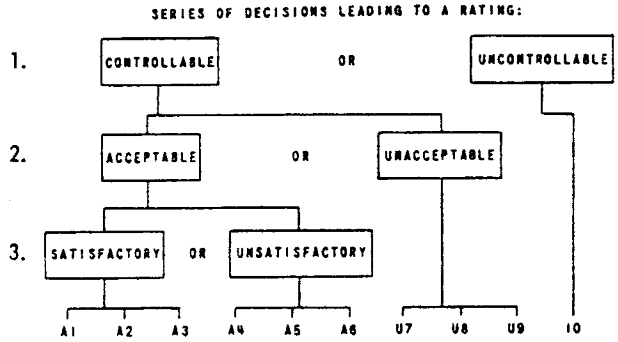


Figure 6.16 Sequential Pilot Rating Decisions

If the airplane is uncontrollable in the mission, it is rated 10. If it is controllable, the second decision examines whether it is acceptable or unacceptable. If unacceptable, the ratings U7, U8, and U9 are to be considered (rating 10 has been excluded by the "controllable" answer to the first decision). If it is acceptable, the third decision must examine whether it is satisfactory or unsatisfactory. If unsatisfactory, the ratings A4, A5 and A6 are to be considered; if satisfactory, the ratings A1, A2, and A3 are to be considered.

The basic categories must be described in carefully selected terms to clarify and standardize the boundaries desired. Following a careful review of dictionary definitions and consideration of the pilot's requirement for clear, concise descriptions, the category definitions shown in figure 6.17 were selected. When considered in conjunction with the structural outline presented in figure 6.16 a clearer picture of the series of decisions which the pilot must make is obtained.

CATEGORY	DEFINITION
CONTROLLABLE	CAPABLE OF BEING CONTROLLED OR MANAGED IN CONTEXT OF MISSION, WITH AVAILABLE PILOT ATTENTION.
UNCONTROLLABLE	CONTROL WILL BE LOST DURING SOME PORTION OF MISSION.
ACCEPTABLE	MAY HAVE DEFICIENCIES WHICH WARRANT IMPROVEMENT BUT ADEQUATE FOR MISSION. PILOT COMPENSATION, IF REQUIRED TO ACHIEVE ACCEPTABLE PERFORMANCE, IS FEASIBLE.
UNACCEPTABLE	DEFICIENCIES WHICH REQUIRE MANDATORY IMPROVEMENT. INADEQUATE PERFORMANCE FOR MISSION, EVEN WITH MAXIMUM FEASIBLE PILOT COMPENSATION.
SATISFACTORY	MEETS ALL REQUIREMENTS AND EXPECTATIONS; GOOD ENOUGH WITHOUT IMPROVEMENT. CLEARLY ADEQUATE FOR MISSION.
UNSATISFACTORY	RELUCTANTLY ACCEPTABLE. DEFICIENCIES WHICH WARRANT IMPROVEMENT. PERFORMANCE ADEQUATE FOR MISSION WITH FEASIBLE PILOT COMPENSATION.

Figure 6.17 Major Category Definitions

6.5.2.1 MAJOR CATEGORY DEFINITIONS

Let us examine what is meant by controllable. To control is to exercise direction of, or to command. Control also means to regulate. The determination as to whether the airplane is controllable or not must be made within the framework of the defined mission or intended use. An example of the considerations of this decision would be the evaluation of fighter handling qualities during which the evaluation pilot encounters a configuration over which he can maintain control only with his complete and undivided attention. The configuration is "controllable" in the sense that the pilot can maintain control by restricting the tasks and maneuvers which he is called upon to perform, and by giving the configuration his undivided attention. However, for him to answer "Yes, it is controllable in the mission," he must be able to retain control in the mission tasks with whatever effort and attention are available from the totality of his mission duties.

The dictionary shows acceptable to mean that a thing offered is received with a consenting mind; unacceptable means that it is refused or rejected. Acceptable means that the mission can be accomplished; it means that the evaluation pilot would agree to buy it for the mission: for him to fly, for his son to fly, or for either to ride in as a passenger. "Acceptable" in the rating scale doesn't say how good it is for the mission, but it does say it is good enough. With these characteristics, the mission can be accomplished. It may be accomplished with considerable expenditure of effort and concentration on the part of the pilot, but the levels of effort and concentration required in order to achieve this acceptable performance are feasible in the intended use. By the same token, unacceptable does not necessarily mean that the mission cannot be accomplished; it does mean that the effort, concentration, and workload necessary to accomplish the mission are of such a magnitude that the evaluation pilot rejects that airplane for the mission.

Consider now a definition of satisfactory. The dictionary defines this as adequate for the purpose. A pilot's definition of satisfactory might be that it isn't necessarily perfect or even good, but it is good enough that he wouldn't ask that it be fixed. It meets a standard, it has sufficient goodness; it can meet all requirements of a mission. Acceptable but unsatisfactory implies that it is reluctantly acceptable even though objectionable characteristics should be improved, that it is deficient in a limited sense, or that there is insufficient goodness. Thus, the quality is either:

a. Completely acceptable (satisfactory) and therefore of the best category, or

b. Reluctantly acceptable (unsatisfactory) and of the next best category, or

c. Unacceptable. Not suitable for the mission, but still controllable, or

d. Unacceptable for the mission and uncontrollable.

6.5.2.2 EXPERIMENTAL USE OF RATING OF HANDLING QUALITIES

The evaluation of handling qualities has a similarity to other scientific experiments in that the output data are only as good as the care taken in the design and execution of the experiment itself and in the analysis and reporting of the results. There are two basic categories of output data in a handling qualities evaluation: the pilot comment data and the pilot ratings. Both items are important output data. An experiment which ignores one of the two outputs is discarding a substantial part of the output information.

As one might expect, the output data which are most often neglected are the pilot comments, primarily because they are quite difficult to deal with due to their qualitative form and, perhaps, their bulk. Ratings, however, without the attendant pilot objections, are only part of the story. Only if the deficient areas can be identified, can one expect to devise improvements to eliminate or attenuate the shortcomings. The pilot comments are the means by which the identification can be made.

There are several factors which have a strong influence on the quality of pilot evaluation data and a brief discussion of them follows.

6.5.2.3 MISSION DEFINITION

Explicit definition of the mission is probably the most important contributor to the objectivity of the pilot evaluation data. The mission is defined here as a use to which the pilot-airplane combination is to be put. The mission must be very carefully examined, and a clear definition and understanding must be reached between the engineer and the evaluation pilot as to their interpretation of this mission. This definition must include:

a. What the pilot is required to accomplish with the airplane, and

b. the conditions or circumstances under which he must perform the mission.

For example, the conditions or circumstances might include instrument or visual flight or both, type of displays in the cockpit, input information to assist the pilot in the accomplishment of the mission, etc. The environment in which the mission is to be accomplished must also be defined and considered in the evaluation, and could include, for example, the presence or absence of turbulence, day versus night, the frequency with which the mission has to be repeated, the variability in the preparedness of the pilot for the mission, and his level of proficiency.

6.5.2.4 SIMULATION SITUATION

The pilot evaluation is seldom conducted under the circumstances of the real mission. The evaluation almost inherently involves simulation to some degree because of the absence of the real situation. As an example, the evaluation of a day fighter is seldom carried out under the circumstances of a combat mission in which the pilot is not only shooting at real targets, but also being shot back at by real guns. Therefore, after the mission has been defined, the relationship of the simulation situation to the real mission must be explicitly stated for both the engineer and the evaluation pilot so that each may clearly understand the limitations of the simulation situation.

The pilot and engineer must both know what is left out of the evaluation program, and also what is in that should not be in. The fact that the anxiety and tension of the real situation are missing, and that the airplane is flying in the clear blue of calm daylight air, instead of in the icing, cloudy, turbulent, dark situation of the real mission, will affect results. Regardless of the evaluation tasks selected, the pilot must use his knowledge and experience to provide a rating which includes all considerations which are pertinent to the mission, whether provided in the tasks or not.

● 6.5.2.5 PILOT COMMENT DATA

One of the fallacies resulting from the use of a rating scale which is considered for universal handling qualities application is the assumption that the numerical pilot rating can represent the entire qualitative assessment. Extreme care must be taken against this oversimplification because it does not constitute the full data gathering process.

The pilot objections to the handling qualities are important, particularly to the airplane designer who is responsible for the improvement of the handling qualities. But, even more important, the pilot comment data are essential to the engineer who is attempting to understand and use the pilot rating data. If ratings are the only output data, one has no real way of assessing whether the objectives of the experiment were actually realized. Pilot comments supply a means of assessing whether the pilot objections (which lead to his summary rating) were related to the mission or resulted from some extraneous uncontrolled factor in the execution of the experiment, or from individual pilots focusing on and weighing differently various aspects of the mission. In order that the pilot comments be most useful, several details are important.

The comments must be given by the pilot in the simplest language. Engineering terms are generally to be avoided, unless they are carefully defined. The pilot should report what he sees and feels, and describe his difficulties in carrying out that which he is attempting. It is then important for the pilot to relate the difficulties which he is having in executing specific tasks to their effect on the accomplishment of the mission.

The pilot should be required to make specific comments in evaluating each configuration. These comments generally are in response to questions which have been developed in the discussions of the mission and simulation situation. The pilot must also be free to make comments regarding his difficulties over and above the answers to the specific questions asked of him. In this regard, the test pilot should strive for a balance between a continuous running commentary and occasional comment in the form of an explicit adjective. The former often requires so much editing to find the substance that it is often ignored, while the latter may add nothing to the numerical rating itself.

The pilot comments must be taken during or immediately after each evaluation. For in-flight evaluations, this means that the comments should be recorded on a tape recorder. Experience has shown that the best free comments are often given during the evaluation. If the comments are left until the conclusion of the evaluation, they are often forgotten. A useful procedure is to permit free comment during the evaluation itself and to require answers to specific questions in the summary comments at the end of the evaluation.

Questionnaires and supplementary pilot comments are most necessary to ensure that: (a) all important or suspected aspects are considered and not overlooked, (b) information is provided relative to why a given rating has been given, (c) an understanding is provided of the tradeoffs with which pilots must continually contend, and (d) supplementary comment that might not be offered otherwise is stimulated. It is recommended that the pilots participate in the preparation of the questionnaires. The questionnaires should be modified if necessary as a result of the pilots' initial evaluations.

● 6.5.2.6 PILOT RATING DATA

The pilot rating is an overall summation of the net effect of all of the objections which the pilot has observed during the evaluation as they relate to the mission. It is emphasized that the basic question that is asked of the pilot conditions the answer that he provides. For this reason, it is most important to ensure that the objectives of the program are clearly stated and understood by all concerned, and that all criteria, whether established or assumed, be clearly defined. In other words, it is extremely important that the basis upon which the evaluation is established be firmly understood by pilots and engineers. Unless a common basis is used, one cannot hope to achieve comparable pilot ratings, and confusing disagreement will often result. Care must also be taken that criteria established at the beginning of the program carry through to the end. If the pilot finds it necessary to modify his tasks, technique or mission definition during the program, he must make it clear just when this change occurred.

A discussion of the specific use of a rating scale tends to indicate some disagreement among pilots as to how they actually arrive at a specific numerical rating. There is general agreement that the numerical rating is only a shorthand for the word definition. Some pilots, however, lean heavily on the specific adjective description and look for that description which best fits their overall assessment. Other pilots prefer to make the dichotomous decisions sequentially, thereby arriving at a choice between two or three ratings. The decision among the two or three ratings is then based upon the adjective description. In concept, the latter technique is much to be preferred since it emphasizes the relationship of all decisions to the mission.

It is suggested that the actual technique used is somewhere between the two techniques above and not so different among pilots. In the past, the pilot's choice has probably been strongly influenced by the relative usefulness of the descriptions provided for the categories on one hand, and the numerical ratings on the other. The evaluation pilot is more or less continuously considering the rating decision process during his evaluation. He proceeds through the dichotomous decisions to the adjective descriptors enough times that his final decision is a blend of both techniques. It is therefore obvious that descriptors should not be contradictory to the mission-oriented framework.

Half ratings are permitted (e.g., rating 4.5) and are generally used by the evaluation pilot to indicate a reluctance to assign either of the adjacent ratings to describe the configuration. Any finer breakdown than half ratings is prohibited since any number greater than or less than the half rating implies that it belongs in the adjacent group. Any distinction between configurations assigned the same rating must be made in the pilot comments. Use of the 3.5, 6.5, and 9.5 ratings is discouraged as they must be interpreted as evidence that the pilot is unable to make the fundamental decision with respect to category.

As noted previously, the pilot rating and comments must be given on the spot in order to be most meaningful. If the pilot should later want to change his rating, the engineer should record the reasons and the new rating for consideration in the analysis, and should attempt to repeat the configuration later in the evaluation program. If the configuration cannot be repeated, the larger weight (in most circumstances) should be given to the on-the-spot rating since it was given when all the characteristics were freshest in the pilot's mind.

● 6.5.2.7 EXECUTION OF HANDLING QUALITIES EXPERIMENTS

Probably the most important item is the admonition to execute the experiment as it was planned. This requires careful attention to the conduct of the experiment so that the plans are actually executed in the manner intended. It is valuable for the engineer to monitor the pilot comment data as the experiment is conducted in order that he becomes aware of evaluation difficulties as soon as they occur. These difficulties may take a variety of forms. The pilot may use words which the engineer needs to have defined. The pilot's word descriptions may not convey a clear, understandable picture of the piloting difficulties. Direct communication between pilot and engineer is most important in clarifying such uncertainties. In fact, communication is probably the most important single element in the evaluation of handling qualities. Pilot and engineer must endeavor to understand one another, and cooperate to achieve and retain this understanding. The very nature of the experiment itself makes this somewhat difficult. The engineer is usually not present during the evaluation and, hence, he has only the pilot's word description of any piloting difficulty. Often, these described difficulties are contrary to the intuitive judgments of the engineer based on the characteristics of the airplane by itself. Mutual confidence is required. The engineer should be confident that the pilot will give him accurate, meaningful data; the pilot should be confident that the engineer is vitally interested in what he has to say and trusts the accuracy of his comments.

It is important that the pilot have no foreknowledge of the specific characteristics of the configuration being investigated. This does not exclude information which can be provided to help shorten certain tests (e.g., the parameter variations are lateral-directional, only). But it does exclude foreknowledge of the specific parameters under evaluation. The pilot must be free to examine the configuration without prejudice, learn all he can about it from meeting it as an unknown for the first time, look clearly and accurately at his difficulties in performing the evaluation task, and freely associate these difficulties with their effects on the ultimate success of the mission. A considerable aid to the pilot in this assessment is to present the configuration in a random-appearing fashion.

The amount of time which the pilot should use for the evaluation is difficult to specify a priori. He is normally asked to examine each configuration for as long as is necessary to feel confident that he can give a reliable and repeatable assessment. Sometimes, however, it is necessary to limit the evaluation time to a specific period of time because of circumstances beyond the control of the researcher. If the evaluation time per pilot is limited, a larger sample of pilots or repeat evaluations will be required for similar accuracy, and the pilot comment data will be of poorer quality.

One final point is the state of mind of the evaluation pilot. He must be confident of the importance of the simulation program and join wholeheartedly into the production of data which will supply answers to the questions. Pilots as a group are strongly motivated toward the production of data to improve the handling qualities of the airplanes they fly. It isn't usually necessary to explicitly motivate the pilot, but it is very important to inspire in him confidence in the structure of the experiment and the usefulness of his rating and comment data. Pilot evaluations are probably one of the most difficult tasks that a pilot undertakes. To produce useful data involves a lot of hard work, tenacity, and careful thought. There is a strong tendency for the pilot to become discouraged in the course of his evaluations about their ultimate usefulness. He worries constantly about his assessments: their accuracy and repeatability. The pilot may feel that the engineer has the answers on a sheet of paper and he is merely testing the pilot as to his ability to search out the correct answers. Such feelings are added to by a lack of communication between the piloting and engineering organizations and are to be avoided. Probably the best approach is to explicitly state to the pilot that only he knows the answers to the questions which are being asked, and he can arrive at these correct answers by carrying out the evaluation program. He must be reassured in the course of the program that his assessments are good, so that he gains confidence in the manner in which he is carrying out the program.

6.6 CONTROL INPUTS

There are several different control inputs that could be used to excite the dynamic modes of motion of an aircraft. To accomplish the task of obtaining the free response of an aircraft, the pilot makes an appropriate control input, removes himself from the loop, and observes the resulting aircraft motion. Three inputs that are frequently used in stability and control investigations will be discussed in this section: the step input, the pulse, and the doublet.

6.6.1 STEP INPUT

When a step input is made, the applicable control is rapidly moved to a desired new position and steadily held there. The aircraft motion resulting from this suddenly applied new control position can then be recorded for analysis. A mathematical representation of a step input assumes the deflection occurs in zero time and is contrasted to a typical actual control position time history in figure 6.18. The "unit step" input is frequently used in theoretical analysis and has the magnitude of one radian, which is equivalent to 57.3 degrees. Specifying control inputs in dimensionless radians instead of degrees is convenient for use in the non-dimensional equations of motion.

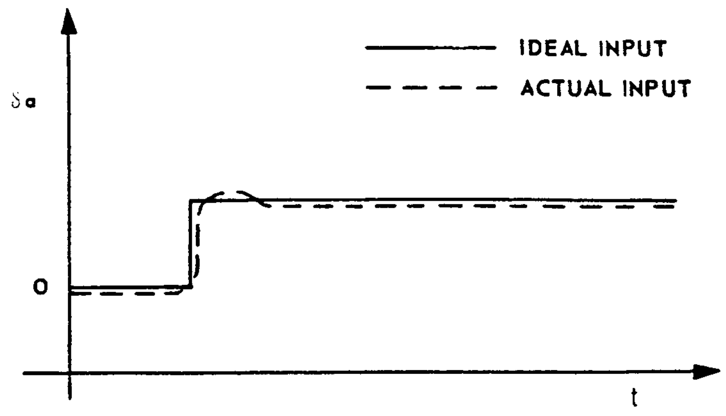


Figure 6.18 Step Input

● 6.6.2 PULSE

When a pulse, or singlet, input is applied, the control is moved rapidly to a desired position, held momentarily, and then rapidly returned to its original position. The pilot can then remove himself from the loop and observe the free aircraft response. Again, deflections are theoretically assumed to occur instantaneously, and an example of a pulse, or singlet is shown in figure 6.19.

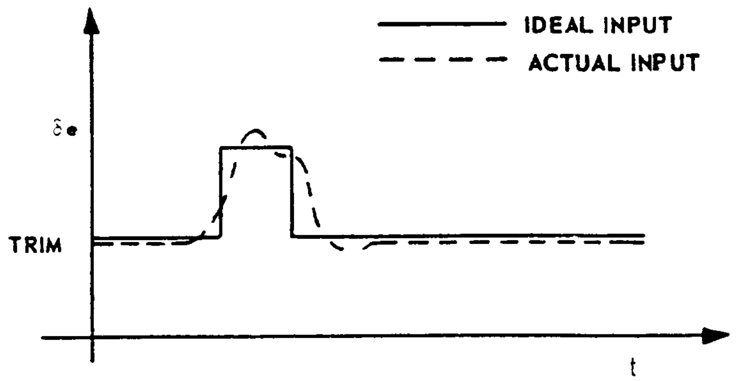


Figure 6.19 Pulse Input

The "unit impulse" input is frequently used in theoretical analysis and is related to the pulse input. The unit impulse is the mathematical result of a limiting process which begins with a pulse having an area of unity under the rectangle formed by the input and ends with an infinitely large magnitude input applied in zero time.

● 6.6.3 DOUBLET

A doublet input is a double pulse which is skew symmetric with time. After exciting a dynamic mode of motion with this input and removing himself from the control loop, the pilot can record the aircraft open loop motion. Figure 6.20 depicts a theoretical doublet input.

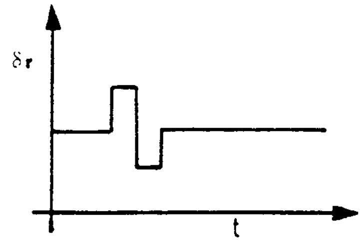


Figure 6.20 Doublet Input

■ 6.7 EQUATIONS OF MOTION

Six equations of motion (three translational and three rotational) for a rigid body flight vehicle are required to solve its motion problem. A rigid body aircraft and constant mass were assumed, and the equations of motion were derived and expressed in terms of a coordinate system fixed in the body. Solving for the motion of a rigid body in terms of a body fixed coordinate system is particularly convenient in the case of an aircraft when the applied forces are most easily specified in the body axis system.

"Stability axes" were used to specify the body fixed coordinate system. With the vehicle at reference flight conditions the x axis is aligned into the relative wind; the z axis is 90 degrees from the x axis in the aircraft plane of symmetry, with positive direction down relative to the vehicle; the y axis completes the orthogonal triad. This xyz coordinate system is then fixed in the vehicle and rotates with it when perturbed from the reference equilibrium conditions. The solid lines in figure 6.21 depict initial alignment of the stability axes, and the dashed lines show the perturbed coordinate system.

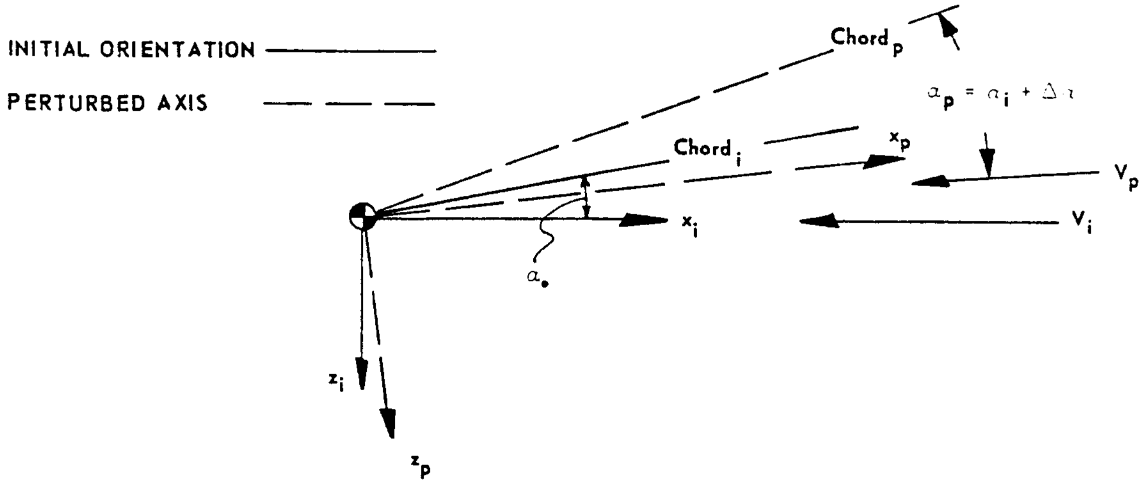


Figure 6.21 Stability Axis System

Chapter 1 contains the derivation of the complete equations of motion, and the results are listed here for your convenience.

$$\left.\begin{aligned} F_x &= m\,(\dot{u} + qw - rv) \\ F_y &= m\,(\dot{v} + ru - pw) \\ F_z &= m\,(\dot{w} + pv - qu) \\ L &= \dot{p}I_x + qr\,(I_z - I_y) - (\dot{r} + pq)\,I_{xz} \\ M &= \dot{q}I_y - pr\,(I_z - I_x) + (p^2 - r^2)\,I_{xz} \\ N &= \dot{r}I_z + pq\,(I_y - I_x) + (qr - \dot{p})\,I_{xz} \end{aligned}\right\} \quad (6.19)$$

where F_x, F_y, and F_z are forces in the x, y, and z direction, and L, M, and N are moments about the x, y, and z axes taken at the vehicle center of mass.

6.7.1 SEPARATION OF THE EQUATIONS OF MOTION

When all lateral-directional forces, moments, and accelerations are constrained to be zero, the equations which govern pure longitudinal motion result from the six general equations of motion. That is, substituting

$$\left.\begin{aligned} p &= 0 = r \\ \dot{p} &= 0 = \dot{r} \\ L &= 0 = N \\ F_y &= 0 \\ v &= 0 \\ \dot{v} &= 0 \end{aligned}\right\} \quad (6.20)$$

into the equations labeled 6.19 results in the longitudinal equations of motion:

$$\left.\begin{aligned} F_x &= m\,(\dot{u} + qw) \\ F_z &= m\,(\dot{w} - qu) \\ M &= \dot{q}\,I_y \end{aligned}\right\} \quad (6.21)$$

Linearization of the equations labeled 6.21 by Taylor series first order approximations of the forcing functions and small perturbation assumptions for the variables u, q, and w result in a set of workable equations for longitudinal motion. Note that the resulting equations are the longitudinal perturbation equations, and that the unknowns are the perturbed values of α, u, and θ from the equilibrium condition. These equations in coefficient form are[2]

[2] *Automatic Control of Aircraft and Missiles,* Blakelock, Wylie, 1965.

$$\left.\begin{aligned}
&\left(\frac{mU_o}{Sq}\dot{u} - C_{x_u} u\right) + \left(\frac{-c}{2U_o} C_{x_{\dot{\alpha}}}\dot{\alpha} - C_{x_\alpha}\alpha\right) + \left(\frac{-c}{2U_o} C_{x_q}\dot{\theta} + \frac{mg}{Sq}\theta\right) = C_{x_{\delta e}}\delta e \\
&\left(-C_{z_u} u\right) + \left[\left(\frac{mU_o}{Sq} - \frac{c}{2U_o} C_{z_{\dot{\alpha}}}\right)\dot{\alpha} - C_{z_\alpha}\alpha\right] + \left[\left(\frac{-mU_o}{Sq} - \frac{c}{2U_o} C_{z_q}\right)\dot{\theta}\right] = C_{z_{\delta e}}\delta e \\
&\left(-C_{M_u} u\right) + \left(\frac{-c}{2U_o} C_{M_{\dot{\alpha}}}\dot{\alpha} - C_{M_\alpha}\alpha\right) + \left[\frac{I_y}{Sqc}\ddot{\theta} - \frac{c}{2U_o} C_{M_q}\dot{\theta}\right] = C_{M_{\delta e}}\delta e,
\end{aligned}\right\} \quad (6.22)$$

In the equations labeled 6.22,

$q = \frac{1}{2}\rho U_o^2$ (except when q is a subscript denoting a partial derivative with respect to pitch rate.)

$u = \frac{\Delta U}{U_o}$ (A dimensionless velocity parameter has been defined for convenience.)

α, θ are perturbations about their equilibrium values.

C_{x_u}, C_{z_α}, etc., are partial derivatives evaluated at the reference conditions with respect to force coefficients. Derivations for these terms may be found in Blakelock.

Note that the equations labeled 6.22 are for pure longitudinal motion and that the unknowns are perturbation values about the reference conditions.

Laplace transforms can be used to facilitate solutions to the longitudinal perturbation equations. For example, taking Laplace transforms of the x force equation and stating that initial perturbation values are zero results in

$$\left[\frac{mU_o}{Sq} S - C_{x_u}\right]u(S) + \left[\frac{-c}{2U_o} C_{x_{\dot{\alpha}}} S - C_{x_\alpha}\right]\alpha(S) + \left[\frac{-c}{2U_o} C_{x_q} S + \frac{mg}{Sq}\right]\theta(S)$$

$$= C_{x_{\delta e}}\delta e(S) \qquad (6.23)$$

The other two equations could similarly be Laplace transformed to obtain a set of longitudinal perturbation equations in the S domain.

6.8 LONGITUDINAL MOTION

The equations labeled 6.22 describe the perturbed longitudinal motion of an aircraft about some equilibrium conditions. The theoretical solutions for aircraft motion can be quite good, depending on the accuracy of the various aerodynamic parameters. For example,

$$C_{x_\alpha} = C_L - C_{D_\alpha}$$

is one parameter appearing in the x force equation, and the goodness of the solution will certainly depend on how accurately the values of C_L and C_{D_α} are known. Before an aircraft flies, such values result from theoretical predictions and wind tunnel data. After appropriate flight tests have been flown, values for the various stability derivatives can be extracted from flight test data.

6.8.1 EXAMPLE PROBLEM

Blakelock presents an example problem for a four-engine jet transport using the longitudinal equations to solve for the perturbed aircraft motion. The reference flight conditions are straight and level at 40,000 feet with a velocity of 600 feet per second. Values for the various aerodynamic parameters are specified, and the set of longitudinal equations in the Laplace domain become

$$
\begin{aligned}
&[13.78S + .088]u(S) - .392\alpha(S) + .74\theta(S) = C_{x_{\delta e}}\,\delta e(S) \\
&1.48u(S) + [13.78S + 4.46]\alpha(S) - 13.78S\theta(S) = C_{z_{\delta e}}\,\delta e(S) \qquad (6.24) \\
&0 + [.0552S + .619]\alpha(S) + [.514S^2 + .192S]\theta(S) = C_{M_{\delta e}}\,\delta e
\end{aligned}
$$

These equations are of the form

$$
\left.\begin{aligned}
a\,u + b\,\alpha + c\,\theta &= d \\
e\,u + f\,\alpha + g\,\theta &= h \\
i\,u + j\,\alpha + k\,\theta &= \ell
\end{aligned}\right\} \qquad (6.25)
$$

and the set of equations can be readily solved for any of the variables. For example, from equation 6.25,

$$
\alpha(S) = \frac{\begin{vmatrix} a & d & c \\ e & h & g \\ i & \ell & k \end{vmatrix}}{\begin{vmatrix} a & b & c \\ e & f & g \\ i & j & k \end{vmatrix}} = \frac{\text{Numerator}(S)}{\text{Denominator}(S)}
$$

Recall that the denominator of the above equation in the S domain is the system characteristic equation and that the location of the roots of $\Delta(S)$ will immediately indicate the type of dynamic response. From equation 6.24

$$\Delta(S) = 97.5S^4 + 79S^3 + 128.9S^2 + .998S + .677 \qquad (6.26)$$

Factoring higher order equations such as 6.26 is not a simple thing, but some systematic approaches do exist. Blakelock refers to Lin's method and accomplishes the factoring of 6.26 into two quadratics.

$$\Delta(S) = (S^2 + .00466S + .0053)\ (S^2 + .806S + 1.211) \qquad (6.27)$$

Each of the quadratics listed in equation 6.27 will have a natural frequency and damping ratio associated with it, and the values can be rapidly computed by comparing the particular quadratic to the standard notation second order characteristic equation:

$$\zeta_1 = 0.352 \qquad \omega_{n_1} = 1.145 \text{ radians/sec} \tag{6.28}$$

$$\zeta_1 = 0.032 \qquad \omega_{n_2} = 0.073 \text{ radians/sec} \tag{6.29}$$

6.8.2 LONGITUDINAL MOTION MODES

Experience has shown that aircraft exhibit two different types of longitudinal oscillations:

1. One of short period with relatively heavy damping that is called the "short period" mode.

2. Another of long period with very light damping that is called the "phugoid" mode.

The periods and damping of these oscillations vary from aircraft to aircraft and with flight conditions.

The short period mode is characterized primarily by variations in angle of attack and pitch angle with very little change in forward speed. Relative to the phugoid, the short period has a high frequency and heavy damping.

Typical values for its damped period are in the range of 2 to 5 seconds. Generally, the short period motion is the more important longitudinal mode for handling qualities since it is contributing to the motion being observed and corrected by the pilot when the pilot is in the loop.

The phugoid mode is characterized mainly by variations in u and θ with α nearly constant. This long period oscillation can be thought of as a constant total energy problem with exchanges between potential and kinetic energy. The aircraft nose drops and airspeed increases as the aircraft descends below its initial altitude. Then the nose rotates up, causing the aircraft to climb above its initial altitude with airspeed decreasing until the nose lazily drops below the horizon at the top of the maneuver.

Because of light damping, many cycles are required for this motion to damp out. However, its long period combined with low damping results in an oscillation that is easily controlled by the pilot, even for a slightly divergent motion. When the pilot is in the loop, he is frequently not aware that the phugoid mode exists as he makes control inputs and obtains aircraft response before the phugoid can be seen. Typical values for its damped period range in the order of 45 to 90 seconds.

From the example problem in section 6.8.1 and from the descriptions of the longitudinal mode, it is possible to immediately specify which parameters are associated with the short period and which must be that aircraft's phugoid parameters.

6.8.3 SHORT PERIOD APPROXIMATION EQUATIONS

A logical approach to use when trying to get a simplified set of equations to describe the short period mode is to recall that the short period occurs at nearly constant airspeed and set $u = 0$ in the equations labeled 6.22. The result of this substitution is two unknowns appearing in three independent equations, and it is certainly desirable to select the correct set of two equations for solution. Note that the x force equation could be expected to contribute primarily to a change of velocity in the x direction; however the specification that $u = 0$ has been made for this approximation. Choosing to discard the x force equation and making the above substitutions results in a set of two equations with the unknowns α and θ. In the S domain these equations are

$$\left.\begin{aligned} &\left[\frac{mU_o}{Sq} S - C_{z_\alpha}\right]\alpha(S) + \left[\frac{-mU_o}{Sq} S\right]\theta(S) = C_{z_{\delta e}}\,\delta e(S) \\ &\left[-\frac{c}{2U_o} C_{M_{\dot\alpha}} S - C_{M_\alpha}\right]\alpha(S) + \left[\frac{I_y}{Sqc} S^2 - \frac{c}{2U_o} C_{M_q} S\right]\theta(S) = C_{M_{\delta e}}\,\delta e(S) \end{aligned}\right\} \quad (6.29)$$

where $C_{z_{\dot\alpha}}$ and C_{z_q} have been assumed to be negligible.

As for previous examples, recall that the characteristic equation can be found by expanding the determinant of the coefficients from the left side of the equations labeled 6.29. From these equations, the characteristic equation is

$$\Delta(S) = S\,(AS^2 + BS + C) \quad (6.30)$$

where

$$A = \left(\frac{I_y}{Sqc}\right)\left(\frac{mU_o}{Sq}\right)$$

$$B = \left(\frac{-c}{2U_o} C_{M_q}\right)\left(\frac{mU_o}{Sq}\right) - \left(\frac{I_y}{Sqc} C_{z_\alpha}\right) - \left(\frac{c}{2U_o} C_{M_{\dot\alpha}}\right)\left(\frac{mU_o}{Sq}\right)$$

$$C = \left(\frac{c}{2U_o} C_{M_q} C_{z_\alpha}\right) - \left(\frac{mU_o}{Sq} C_{M_\alpha}\right)$$

To find expressions for the short period damping ratio and natural frequency from the second order part of equation 6.30, it can be rewritten as

$$\Delta(S) = S\,(S^2 + \frac{B}{A} S + \frac{C}{A})$$

and compared to the standard notation second order characteristic equation.

Some rather involved expressions result for ζ and ω_n

$$\zeta = -\frac{1}{4}\left(C_{M_q} + C_{M_{\dot\alpha}} + \frac{2I_y}{mc^2} C_{z_\alpha}\right)\left[\frac{mc^2}{I_y\left(\frac{C_{M_q} C_{z_\alpha}}{2} - \frac{2m\, C_{M_\alpha}}{\rho Sc}\right)}\right]^{1/2}$$

$$\omega_n = \frac{U_o \rho Sc}{2}\left[\frac{\frac{C_{m_q} C_{z_\alpha}}{2} - \frac{2m}{\rho Sc} C_{M_\alpha}}{I_y m}\right]^{1/2} \qquad (6.31)$$

If the aircraft parameter values that were used in the example problem from section 6.8.1 are substituted in the equations labeled 6.31, the following values result for ζ and ω_n

$$\zeta = .35 \qquad \omega_n = 1.15 \qquad (6.32)$$

Comparison of the above to the values shown in the equations labeled 6.28 show good agreement for this problem.

The complicated expressions listed in the equations labeled 6.31 can be further simplified by discarding the terms that are usually the smallest contributors to the expressions. Stating that the $C_{M_{\dot\alpha}}$ and C_{z_α} terms in the numerator of the ζ expression are negligible when compared to C_{M_q} and that the $C_{M_q} C_{z_\alpha}$ term in the denominator is small compared to the C_{M_α} term results in a more simplified expression for ζ. This functional relationship can be used to predict trends in the short period damping ratio as flight conditions are changed.

$$\zeta \approx \zeta\left[\left(\sqrt{\frac{\rho}{I_y}}\right)\frac{\left(-C_{M_q}\right)}{\sqrt{-C_{M_\alpha}}}\right] \qquad (6.33)$$

Stating that the C_{M_α} term is the significant one in the numerator of the ω_n expression listed in 6.31 results in

$$\omega_n \approx \omega_n\left[U_o\left(\sqrt{\frac{\rho}{I_y}}\right)\left(\sqrt{\frac{Sc}{2}}\right)\left(\sqrt{-C_{M_\alpha}}\right)\right] \qquad (6.34)$$

Equation 6.34 can be used to predict the trends expected in the short period natural frequency as flight conditions change. Both equations 6.33 and 6.34 show the predominant stability derivatives which affect the short period damping ratio and natural frequency.

6.8.4 PHUGOID APPROXIMATION EQUATIONS

An approach similar to that used when obtaining the short period approximation will be used to obtain a set of equations to approximate the phugoid oscillation. Recalling that the phugoid motion occurs at nearly constant angle of attack, it is logical to substitute $\alpha = 0$ into the longitudinal motion equations. This results in a set of three equations with only two unknowns. Further reasoning that the phugoid motion is characterized primarily by altitude excursions and changes in aircraft speed implies that the z force and x force equations are the two equations which should be used. The resulting set of two equations for the phugoid approximation in the Laplace domain is

$$\left[\frac{mU_o}{Sq} S - C_{x_u}\right] u(S) + \left[\frac{mg}{Sq}\right] \theta(S) = C_{x_{\delta e}} \delta e \ (S)$$

$$\left[-C_{z_u}\right] u(S) + \left[\frac{-mU_o}{Sq}\right] S \ \theta \ (s) = C_{z_{\delta e}} \delta e \ (S)$$

where C_{x_q} and C_{z_q} have been assumed to be negligibly small.

The characteristic equation for the phugoid approximation can now be found using the above equations.

$$\Delta(S) = -\left[\frac{mU_o}{Sq}\right]^2 S^2 + \left[\frac{mU_o}{Sq} C_{x_u}\right] S + \left[\frac{mg}{Sq}\right] C_{z_u} \qquad (6.35)$$

Note that lift and weight are not equal during phugoid motion, but also realize that the net difference between lift and weight is quite small. If the approximation is made that

$$L = W$$

and then the substitution that

$$W = mg$$

it can be written that

$$\frac{mg}{Sq} = C_L$$

The phugoid characteristic equation can thus be rewritten as

$$\Delta(S) = S^2 - \frac{C_{x_u}}{\frac{mU_o}{Sq}} S - \frac{C_L}{\left(\frac{mU_o}{Sq}\right)^2} C_{z_u}$$

The phugoid natural frequency is then found to be

$$\omega_n = \frac{\rho S U_o}{2m} \sqrt{-C_{z_u} C_L} \qquad (6.36)$$

In an effort to further simplify equation 6.36, the C_{z_u} terms should be examined. Examining figure 6.22

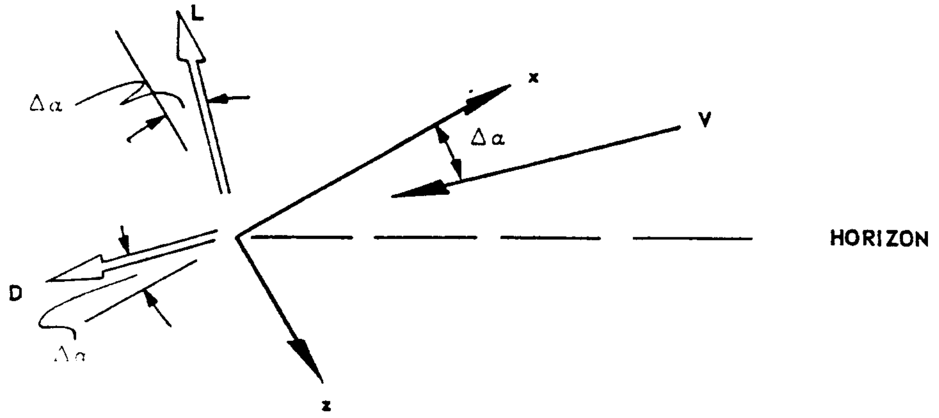


Figure 6.22

leads to

$$F_z = - L \cos \alpha - D \sin \alpha$$

then,

$$\frac{\partial F_z}{\partial u} = - \frac{\partial L}{\partial u} \cos \alpha + L \sin \alpha \frac{\partial \alpha}{\partial u} - \frac{\partial D}{\partial u} \sin \alpha - D \cos \alpha \frac{\partial \alpha}{\partial u}$$

which must be evaluated at the equilibrium conditions for use in the first order Taylor Series that was used in obtaining the linearized equations of motion

$$\left.\frac{\partial F_z}{\partial u}\right|_o = - \frac{\partial L}{\partial u}$$

And,

$$\frac{\partial F_z}{\partial u} = - \frac{\partial}{\partial u} C_L S \frac{1}{2} \rho V^2$$

When the approximation that $V = U$ is made in the above equation, the result is

$$\frac{\partial F_z}{\partial u} = - \frac{\partial C_L}{\partial u} S \frac{1}{2} \rho U^2 - C_L S \rho U$$

In order to have a nondimensional expression, the correct factor must be used

$$C_{z_u} = \frac{U_o}{Sq} \frac{\partial F_z}{\partial u}$$

So that

$$C_{z_u} = - U_o C_{L_u} - 2 C_L \qquad (6.37)$$

The variation of C_L with velocity is primarily due to Mach effects, and except for the transonic regime, C_{L_u}, is approximately zero in many cases. Figure 6.23 shows a typical plot.

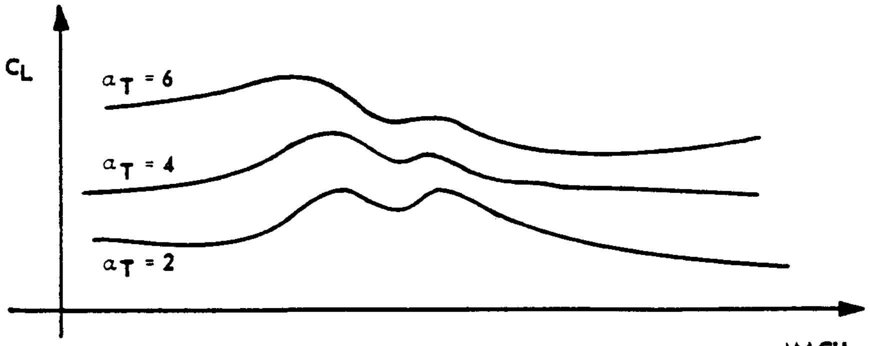


Figure 6.23 Typical Plot of C_L vs Mach

If the aircraft and flight conditions are such that $C_{L_u} \approx 0$, then

$$C_{z_u} = - 2 C_L$$

Substituting the above equation into the expression for the phugoid natural frequency results in

$$\omega_n = \frac{Sq}{mU_o} \sqrt{2} \; C_L$$

An equation showing that the phugoid natural frequency is inversely related to aircraft velocity results from substituting $C_L = \frac{mg}{Sq}$ into the above equation.

$$\omega_n = \frac{45.5}{U_o} \qquad (6.38)$$

Where U_o is true velocity in feet per second.

A simplified approximate expression for the phugoid damping ratio can also be obtained and is given by

$$\zeta = \left(\frac{1}{\sqrt{2}}\right)\left(\frac{C_D}{C_L}\right) \qquad (6.39)$$

Equations 6.38 and 6.39 can be used to understand some major contributors to the natural frequency and damping ratio of the phugoid motion.

● 6.8.5 EQUATION FOR n/α

Noting that the requirements of MIL-F-8785 for the short period natural frequency are stated as a function of n/α, it is desirable to develop a theoretical capability to predict n/α. Consider the z force equation for longitudinal motion.

$$\Sigma \Delta F_z = m\,(\dot{w} - U_o \dot{\theta})$$

and recall that Newton's Second Law is a directional relationship

$$\bar{F} = m\,\bar{a}$$

Thus

$$a_z = \dot{w} - U_o\,\dot{\theta} \tag{6.40}$$

where all the variables appearing in equation 6.40 are perturbations about the equilibrium condition. Rewriting the above equation and stating that $a_z \approx n$

$$n = U_o\,\left(\frac{\dot{w}}{U_o} - \dot{\theta}\right)$$

$$n = U_o\,(\dot{\alpha} - \dot{\theta}) \tag{6.41}$$

Of course expressions for $\dot{\alpha}$ and $\dot{\theta}$ can be obtained from the short period solutions for $\alpha(t)$ and $\theta(t)$, and an expression that gives $n(t)$ can be written from equation 6.41. The ratio of the magnitudes of the n and α envelopes can then be used to determine n/α.

Using a theoretically obtained n/α along with the short period natural frequency and damping ratio obtained from the equations of motion makes it possible to accomplish a design problem to check whether or not the aircraft is being designed to comply with the MIL-F-8785.

■ 6.9 LATERAL-DIRECTIONAL MOTION MODES

There are three typical asymmetric modes of motion exhibited by aircraft. These modes are the roll, spiral, and Dutch roll.

● 6.9.1 ROLL MODE

The roll mode is considered to be a first order response which describes the aircraft roll rate response to an aileron input. Figure 6.24 depicts an idealized roll rate time history to a step aileron input. The roll mode time constant is normally small, with a MIL-F-8785 requirement to be less than three seconds.

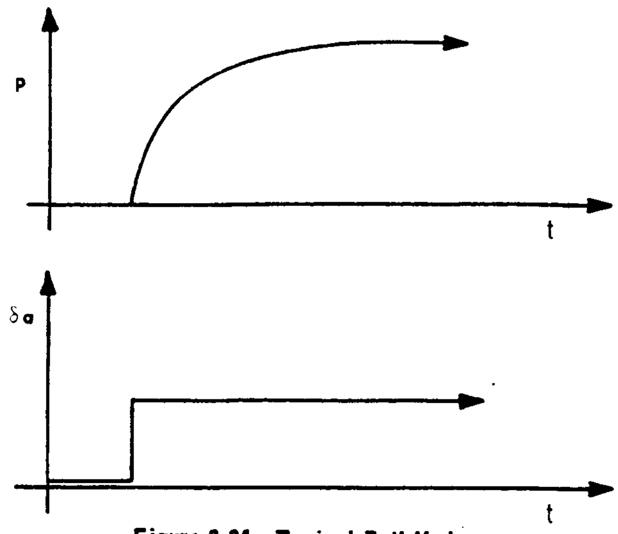


Figure 6.24 Typical Roll Mode

●6.9.2 SPIRAL MODE

The spiral mode is considered to be a first order response which describes the aircraft bank angle time history as ϕ tends to increase or decrease from a small, non-zero bank angle. After a wings level trim shot, the spiral mode can be observed by releasing the aircraft from bank angles as great as 20 degrees and allowing the spiral mode to occur without control inputs. If this mode is divergent, the aircraft nose continues to drop as the bank angle continues to increase, resulting in the name, "spiral mode." This mode, similar to the phugoid in that a pilot can easily control it even if it is dynamically unstable, has somewhat loose requirements in MIL-F-8785. A typical divergent time history as shown in figure 6.25 and might be characterized by T_2, the time to double amplitude.

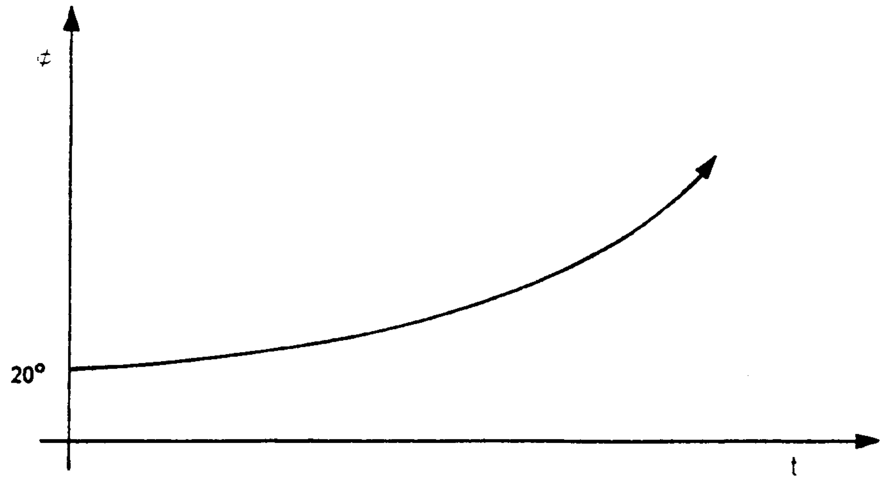


Figure 6.25 Typical Spiral Mode

6.9.3 DUTCH ROLL MODE

The Dutch roll mode is a tightly coupled yawing and rolling motion with a relatively high frequency. Some typical values for Dutch roll damped period at a cruise condition are 3 seconds for the A-7, and 3 seconds for the B-58. Typically, as the aircraft nose yaws to the right a right roll due to the yawing motion is generated. The combination of restoring forces and moments, damping, and aircraft inertia is generally such that after the motion peaks out to the right, a nose left yawing motion begins accompanied by a roll to the left. This coupled right - left - right - . . . motion often is lightly damped with a relatively high frequency.

One of the pertinent Dutch roll parameters is ϕ/β, the ratio of bank angle to yaw angle. A very low value for ϕ/β implies little bank action during the Dutch roll. In the limit when ϕ/β is zero, the Dutch roll motion consists of a pure yawing motion that most pilots consider less objectionable than a Dutch roll mode with a high value for ϕ/β.

Another parameter than can be used to characterize the Dutch roll or any other second order motion is the number of cycles required to damp to half amplitude, $C_{1/2}$.

A doublet rudder input is frequently used to excite the Dutch roll, and figure 6.26 shows a typical Dutch roll time history.

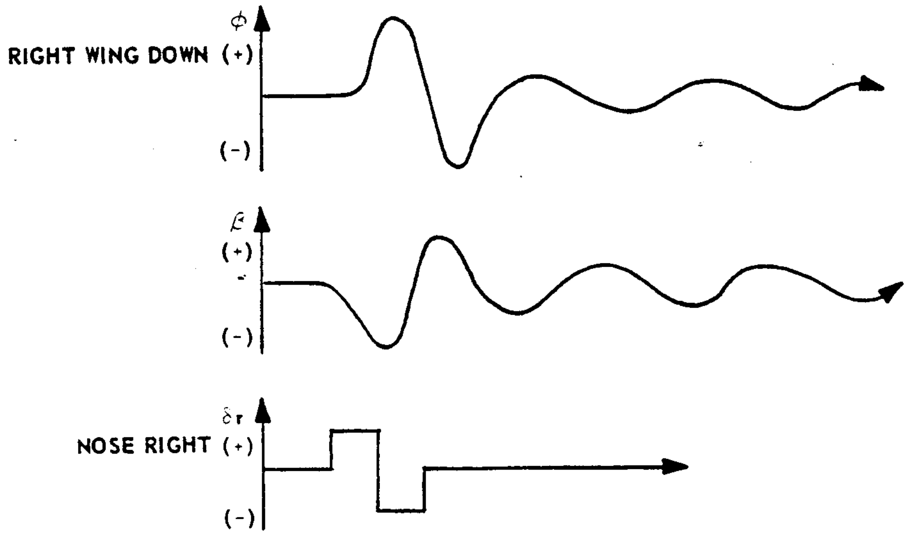


Figure 6.26 Typical Dutch Roll Mode

6.9.4 ASYMMETRIC EQUATIONS OF MOTION

Similarly to the separation of the longitudinal equations, the set of equations which describe lateral-directional motion can be separated from the six general equations of motion. Starting with equilibrium conditions and specifying that only asymmetric forcing functions, velocities, and accelerations exist results in the lateral-directional equations of motion. Assuming small perturbations and using a linear Taylor Series approximation for the forcing functions result in the linear, lateral-directional perturbation equations of motion

$$\left.\begin{aligned}
&\frac{-b}{2U_o} C_{Y_p} \dot{\phi} - C_{Y_\phi} \phi + \left(\frac{mU_o}{Sq} - \frac{b}{2U_o} C_{Y_r}\right)\dot{\psi} - C_{Y_\psi} \psi + \frac{mU_o}{Sq} \dot{\beta} - C_{Y_\beta} \beta = C_{Y_{\delta r}} \delta r + C_{Y_{\delta a}} \delta a \\
&\frac{I_x}{Sqb} \ddot{\phi} - \frac{b}{2U_o} C_{\ell_p} \dot{\phi} + \frac{I_{xz}}{Sqb} \ddot{\psi} - \frac{b}{2U_o} C_{\ell_r} \dot{\psi} - C_{\ell_\beta} \beta = C_{\ell_{\delta r}} \delta r + C_{\ell_{\delta a}} \delta a \\
&\frac{I_{xz}}{Sqb} \ddot{\phi} - \frac{b}{2U_o} C_{N_p} \dot{\phi} + \frac{I_z}{Sqb} \ddot{\psi} - \frac{b}{2U_o} C_{N_r} \dot{\psi} - C_{N_\beta} \beta = C_{N_{\delta r}} \delta r + C_{N_{\delta a}} \delta a
\end{aligned}\right\} \quad (6.42)$$

Note that the lateral-directional equations of motion have been non-dimensionalized by span, b, as opposed to chord. Also, recall that the stability derivative C_{ℓ_p} is not a lift-referenced stability derivative but that the script ℓ refers to rolling moment.

It is appropriate to point out that if the products of perturbation are not small, then the lateral-directional motion will couple directly into longitudinal motion. This can be readily seen by examination of the pitching moment equation and makes the point that asymmetric motion can couple into symmetric motion. Our analysis will assume that conditions are such that coupling does not exist.

● 6.9.5 ROOTS OF Δ (S) FOR ASYMMETRIC MOTION

Laplace transforming the equations labeled 6.42 puts them into a form that readily yields the characteristic equation for asymmetric motion or that can be used to find the time response for some specified input.

The roots of the lateral directional characteristic equation typically are comprised of a relatively large negative real root, a small real root that is either positive or negative, and a complex conjugate pair of roots.

The large real root is the one associated with the roll mode of motion. Note that a large negative value for this root implies a fast time constant.

The small real root that might be either positive or negative is associated with the spiral mode. A slowly changing time response results from this small root, and the motion is either stable for a negative root or divergent for a positive root.

The complex conjugate pair of roots corresponds to the Dutch roll mode which frequently exhibits high frequency and light damping for SAS off conditions. This second order motion is of great interest in handling qualities investigations.

Figure 6.24 shows typical characteristic equation root locations for the asymmetric motion modes.

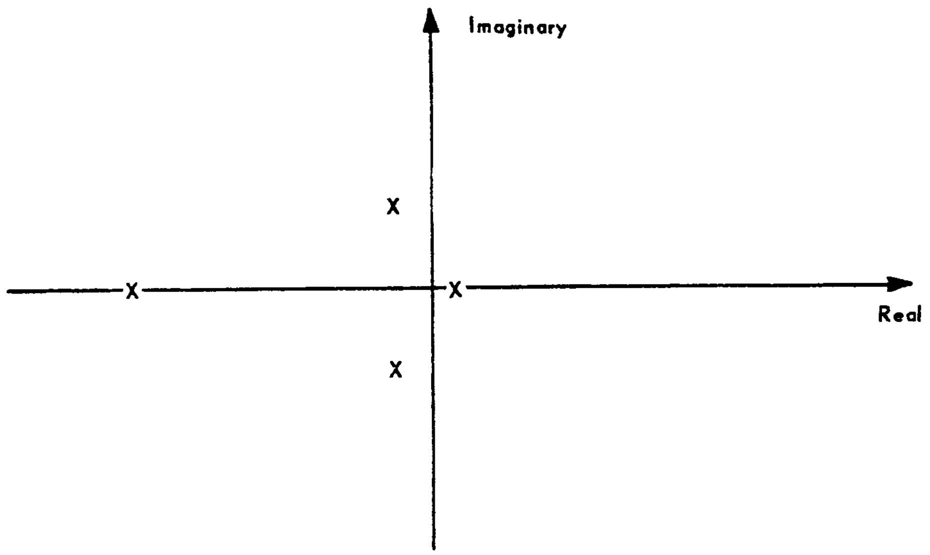


Figure 6.27 Typical Roots of Δ (S) for Asymmetric Motion

6.9.6 APPROXIMATE ROLL MODE EQUATION

This approximation results from the hypothesis that only rolling motion exists and use of the rolling moment equation as done in Blakelock (reference 2). The roll mode approximation equation is

$$\left[\frac{I_x}{Sqb} S - \frac{b}{2U_o} C_{\ell_p}\right] \dot{\phi}(S) = C_{\ell_{\delta a}} \delta a \qquad (6.43)$$

The roll mode characteristic equation root is

$$S_R = \frac{b^2 S \rho U_o}{4 I_x} C_{\ell_p} \qquad (6.44)$$

Note that C_{ℓ_p} less than zero implies stability for the roll mode and that a larger negative value of S_R implies an aircraft that approaches its steady state roll rate quickly. A functional analysis can be made using equation 6.44 to predict change trends in τ_R, the roll mode time constant, as flight conditions change.

6.9.7 SPIRAL MODE STABILITY

Blakelock lists the condition for dynamic spiral stability, namely that

$$C_{\ell_\beta} C_{N_r} > C_{N_\beta} C_{\ell_r}$$

and points out that increasing C_{ℓ_β} while decreasing C_{ℓ_r} is a reasonable design method if increasing spiral stability is desired. Also he lists an equation to calculate the spiral mode $\Delta(S)$ root

$$S_s = \frac{Sq}{mU_o} C_{Y_\phi} \left[\frac{C_{\ell_\beta} C_{N_r} - C_{N_\beta} C_{\ell_r}}{C_{\ell_p} C_{N_\beta}} \right] \tag{6.45}$$

● 6.9.8 DUTCH ROLL MODE APPROXIMATE EQUATIONS

The approximate equations for Dutch roll motion can be obtained by using the equations labeled 6.42 and specifying that pure sideslip exists ($\beta = -\psi$) and bank angle is zero. While this specification is generally not true, the result is a reasonable approximation for the Dutch roll damping ratio and natural frequency:

$$\left.\begin{aligned} \zeta &= \left(\frac{1}{8}\sqrt{2Sb^3}\right)\left(-C_{N_r}\right)\left(\frac{\rho}{I_z C_{N_\beta}}\right)^{\frac{1}{2}} \\ \omega_n &= \sqrt{\frac{Sb}{2}}\, U_o \left(\frac{C_{N_\beta}\,\rho}{I_z}\right)^{\frac{1}{2}} \end{aligned}\right\} \tag{6.46}$$

An approximate functional relationship can be found for the magnitude of ϕ to β

$$\left|\frac{\phi}{\beta}\right| = \frac{\phi}{\beta}\left[\left(\frac{C_{\ell_\beta}}{C_{N_\beta}}\right)\left(\frac{I_z}{I_x}\right)\left(\frac{1}{\rho\, U_o}\right)\right] \tag{6.47}$$

Equation 6.47 is of value in predicting trends in ϕ/β as flight conditions are changed.

● 6.9.9 COUPLED ROLL-SPIRAL MODE

This mode of lateral-directional motion has rarely been exhibited by aircraft, but the possibility exists that it can indeed happen. If this mode is present, the characteristic equation for asymmetric motion has two pairs of complex conjugate roots instead of the usual one complex conjugate pair along with two real roots. The phenomenon which occurs is the roll mode root decreases in absolute magnitude while the spiral mode root becomes more negative until they meet and split off the real axis to form a second complex conjugate pair of roots, as depicted in figure 6.28.

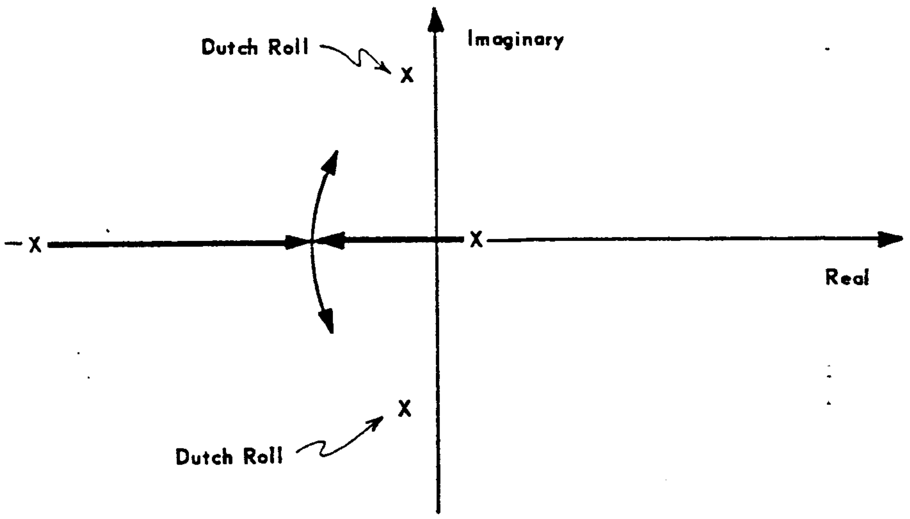


Figure 6.28 Coupled Roll Spiral Mode

At least two inflight experiences with this mode have been documented, and suffice to say that a coupled roll spiral mode causes significant piloting difficulties. One occurrence involved the M2-F2 lifting body, and a second involved the Flight Dynamics Lab variable-stability T-33. Some designs of V/STOL aircraft have indicated that these aircraft would exhibit a coupled roll spiral mode in a portion of their flight envelope[3]. Some pilot comments from simulator evaluations are "rolly," "requires tightly closed roll control loop," or "will roll on its back if you don't watch it."

A coupled roll spiral mode can result from a high value for C_{ℓ_β} and a low value for C_{ℓ_p}. The M2-F2 lifting body did in fact possess a high dihedral effect and a quite low roll damping. Examination of the equations for the roll mode and spiral mode characteristic equation roots shows how the root locus shown in figure 6.25 could result as C_{ℓ_p} decreases in absolute magnitude and C_{ℓ_β} increases.

6.10 STABILITY DERIVATIVES

6.10.1 INTRODUCTION

Some of the stability derivatives are particularly pertinent in the study of the dynamic modes of aircraft motion, and the more important ones appearing in the functional equations which characterize the dynamic modes of motion should be understood. C_{M_q}, C_{M_α}, C_{ℓ_p}, C_{ℓ_β}, C_{N_r}, and C_{N_β} are discussed in the following paragraphs.

[3] AFFDL-TR-65-39, Ground Simulator Elevations of Coupled Roll Spiral Mode Effects on Aircraft Handling Qualities, F. D. Newell, March 1965.

6.10.2 PARTICULAR STABILITY DERIVATIVES

6.10.2.1 C_{M_α}

This stability derivative is the change in pitching moment coefficient with varying angle of attack and is commonly referred to as the longitudinal static stability derivative. When the angle of attack of the airframe increases from the equilibrium condition, the increased lift on the horizontal tail causes a negative pitching moment about the center of gravity of the airframe. Simultaneously, the increased lift of the wing causes a positive or negative pitching moment, depending on the fore and aft location of the lift vector with respect to the center of gravity. These contributions together with the pitching moment contribution of the fuselage are combined to establish the derivative C_{M_α}. The magnitude and sign of the total C_{M_α} for a particular airframe configuration are thus a function of the center of gravity position, and this fact is very important in longitudinal stability and control. If the center of gravity is ahead of the neutral point, the value of C_{M_α} is negative, and the airframe is said to possess static longitudinal stability. Conversely, if the center of gravity is aft of the neutral point, the value of C_{M_α} is positive, and the airframe is then statically unstable. C_{M_α} is perhaps the most important derivative as far as longitudinal stability and control are concerned. It primarily establishes the natural frequency of the short period mode, and is a major factor in determining the response of the airframe to elevator motions and to gusts. In general, a large negative value of C_{M_α} (i.e., large static stability) is desirable for good flying qualities. However, if it is too large, the required elevator effectiveness for satisfactory control may become unreasonably high. A compromise is thus necessary in selecting a design range for C_{M_α}. Design values of static stability are usually expressed not in terms of C_{M_α} but rather in terms of the derivative C_{MC_L}, where the relation is: $C_{M_\alpha} = C_{MC_L} C_{L_\alpha}$. It should be pointed out that C_{MC_L} in the above expression is actually a partial derivative for which the forward speed remains constant.

6.10.2.2 C_{M_q}

The stability derivative C_{M_q} is the change in pitching moment coefficient with varying pitch velocity and is commonly referred to as the pitch damping derivative. As the airframe pitches about its center of gravity path, the angle of attack of the horizontal tail changes, and a lift force is developed on the horizontal tail producing a negative pitching moment on the airframe and hence a contribution to the derivative C_{M_q}. There is also a contribution to C_{M_q} because of various "deadweight" aeroelastic effects. Since the airframe is moving in a curved flight path due to its pitching, a centrifugal force is developed on all the components of the airframe. The force can cause the wing to twist as a result of the dead weight moment of overhanging nacelles, and can cause the horizontal tail angle of attack to change as a result of fuselage bending due to the weight of the tail section. In low speed flight, C_{M_q} comes mostly from the effect of the curved flight path on the horizontal tail and its sign is negative. In high speed flight the sign of C_{M_q} can be positive or negative, depending on the nature of the aeroelastic effects. The derivative C_{M_q} is very important in longitudinal dynamics because it contributes a major portion of the damping of the short period mode for conventional aircraft. As pointed out, this damping effect comes mostly from the horizontal tail. For tailless aircraft, the magnitude of C_{M_q} is consequently small; this is the main reason for the usually poor damping of this type of configuration. C_{M_q} is also involved to a certain extent in the damping of the phugoid mode. In almost all cases, high negative values of C_{M_q} are desired. In the light of the present design trend toward larger radii of gyration in pitch and high altitude flight, it is believed that consideration of C_{M_q} is necessary in the preliminary design stage.

● 6.10.2.3 C_{ℓ_β}

This stability derivative is the change in rolling moment coefficient with variation in sideslip angle and is usually referred to as the "effective dihedral derivative." When the airframe sideslips, a rolling moment is developed because of the dihedral effect of the wing and because of the usual high position of the vertical tail realtive to the equilibrium x-axis. No general statements can be made concerning the relative magnitudes of the contributions to C_{ℓ_β} from the vertical tail and from the wing since these contributions vary considerably from airframe to airframe and for different angles of attack of the same airframe. C_{ℓ_β} is nearly always negative in sign, signifying a negative rolling moment for a positive sideslip.

The derivative C_{ℓ_β} is very important in lateral stability and control, and it is therefore usually considered in the preliminary design of an airframe. It is involved in damping both the Dutch roll mode and the spiral mode. It is also involved in the maneuvering characteristics of an airframe, especially with regard to lateral control with the rudder alone near stall.

● 6.10.2.4 C_{ℓ_p}

The stability derivative, C_{ℓ_p}, is the change in rolling moment coefficient with change in rolling velocity and is usually known as the roll damping derivative. When the airframe rolls at an angular velocity p, a rolling moment is produced as a result of this velocity; this moment opposes the rotation. C_{ℓ_p} is composed of contributions, negative in sign, from the wing and the horizontal and vertical tails. However, unless the size of the tail is unusually large in comparison with the size of the wing, the major portion of the total C_{ℓ_p} comes from the wing.

The derivative C_{ℓ_p} is quite important in lateral dynamics because essentially C_{ℓ_p} alone determines the damping in roll characteristics of the aircraft. Normally, it appears that small negative values of C_{ℓ_p} are more desirable than large ones because the airframe will respond better to a given aileron input and will suffer fewer flight perturbations due to gust inputs.

6.10.2.5 C_{N_β}

The stability derivative, C_{N_β}, is the change in yawing moment coefficient with variation in sideslip angle. It is usually referred to as the static directional derivative or the "weathercock" derivative. When the airframe sideslips, the relative wind strikes the airframe obliquely, creating a yawing moment, N, about the center of gravity. The major portion of C_{N_β} comes from the vertical tail, which stabilizes the body of the airframe just as the tail feathers of an arrow stabilize the arrow shaft. The C_{N_β} contribution due to the vertical tail is positive, signifying static directional stability, whereas the C_{N_β} due to body is negative, signifying static directional instability. There is also a contribution to C_{N_β} from the wing, the value of which is usually positive but very small compared to the body and vertical tail contributions.

The derivative C_{N_β} is very important in determining the dynamic lateral stability and control characteristics. Most of the references concerning full-scale flight tests and free-flight wind tunnel model tests agree that C_{N_β} should be as high as possible for good flying qualities. A high value of C_{N_β} aids the pilot in effecting coordinated turns and prevents excessive sideslip and yawing motions in extreme flight maneuvers and in rough air. C_{N_β} primarily determines the natural frequency of the Dutch roll oscillatory mode of the airframe, and it is also a factor in determining the spiral stability characteristics.

6.10.2.6 C_{N_r}

The stability derivative C_{N_r} is the change in yawing moment coefficient with change of yawing velocity. It is known as the yaw damping derivative. When the airframe is yawing at an angular velocity r, a yawing moment is produced which opposes the rotation. C_{N_r} is made up of contributions from the wing, the fuselage, and the vertical tail, all of which are negative in sign. The contribution from the vertical tail is by far the largest, usually amounting to about 80 or 90 percent of the total C_{N_r} of the airframe.

The derivative C_{N_r} is very important in lateral dynamics because it is the main contributor to the damping of the Dutch roll oscillatory mode. It also is important to the spiral mode. For each mode, large negative values of C_{N_r} are desired.

6.11 PILOT EXTIMATION OF SECOND ORDER MOTION

Pilot-observed data can be used to obtain approximate values for the damped frequency and damping ratio for second order motion such as the short period or Dutch roll.

6.11.1 ESTIMATION OF ω_d

To obtain a value for ω_d, the pilot needs merely to observe the number of cycles that occur during a particular increment of time.

Then,

$$f_d = \frac{\text{Number of Cycles}}{\text{Time Increment}} = \text{cycles/sec} \qquad (6.47)$$

And

$$\omega_d = \left(f_d \frac{\text{cycles}}{\text{sec}}\right)\left(\frac{2\pi \text{ radians}}{\text{cycle}}\right) = \text{radians/sec}$$

The number of cycles can be estimated either by counting peaks or zeroes of the appropriate variable. For short period motion, perturbed θ is easily observed, and if counting zeroes is applied to the motion shown in figure 6.29 the result is

$$f_d = \frac{\frac{1}{2}(5 - 1)}{4} = .5 \text{ cycles/sec}$$

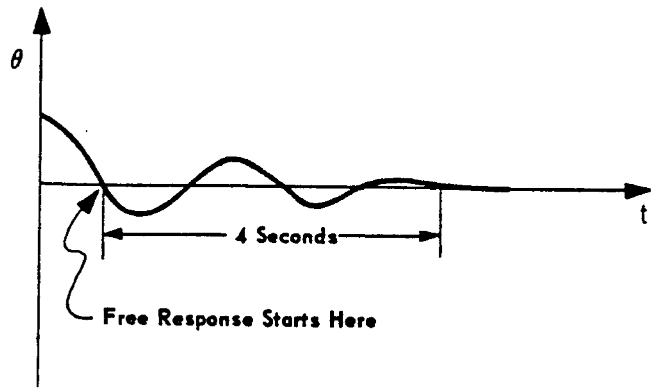


Figure 6.29 Second Order Motion

If zeroes are counted, then

$$f_d = \frac{\frac{1}{2}(\text{number of zeroes} - 1)}{(\text{Time Increment})} \text{ cycles/sec}$$

● 6.11.2 ESTIMATION OF ζ

The pilot can obtain an estimated value for ζ by noting the number of peaks that exist during second order motion and using the approximation

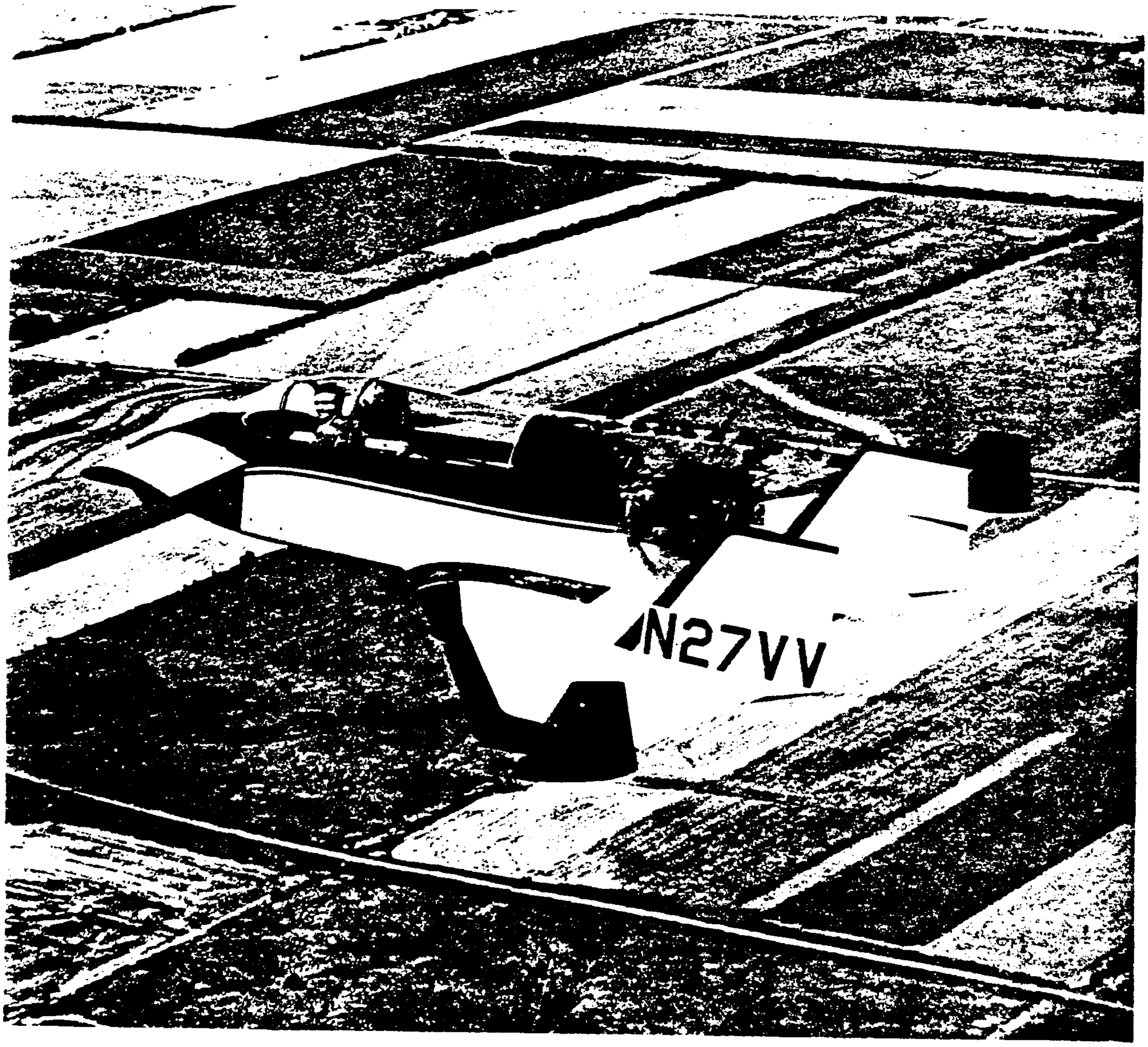
$$\zeta \approx \frac{1}{10}(7 - \text{Number of Peaks}) \qquad (6.48)$$

for $.1 < \zeta < .7$

The motion shown in figure 6.29 thus has an approximate value

$$\zeta \approx \frac{1}{10}(7 - 4) = .3$$

Note that the peaks which occur during aircraft free response are the ones to be used in equation 6.48. If zero observable peaks exist during a second order motion, the best estimate for the value of ζ is then "heavily damped, .7 or greater." If seven or more peaks are observed, the best estimate for the value of ζ is "lightly damped, .1 or less."



6.12 REFERENCES

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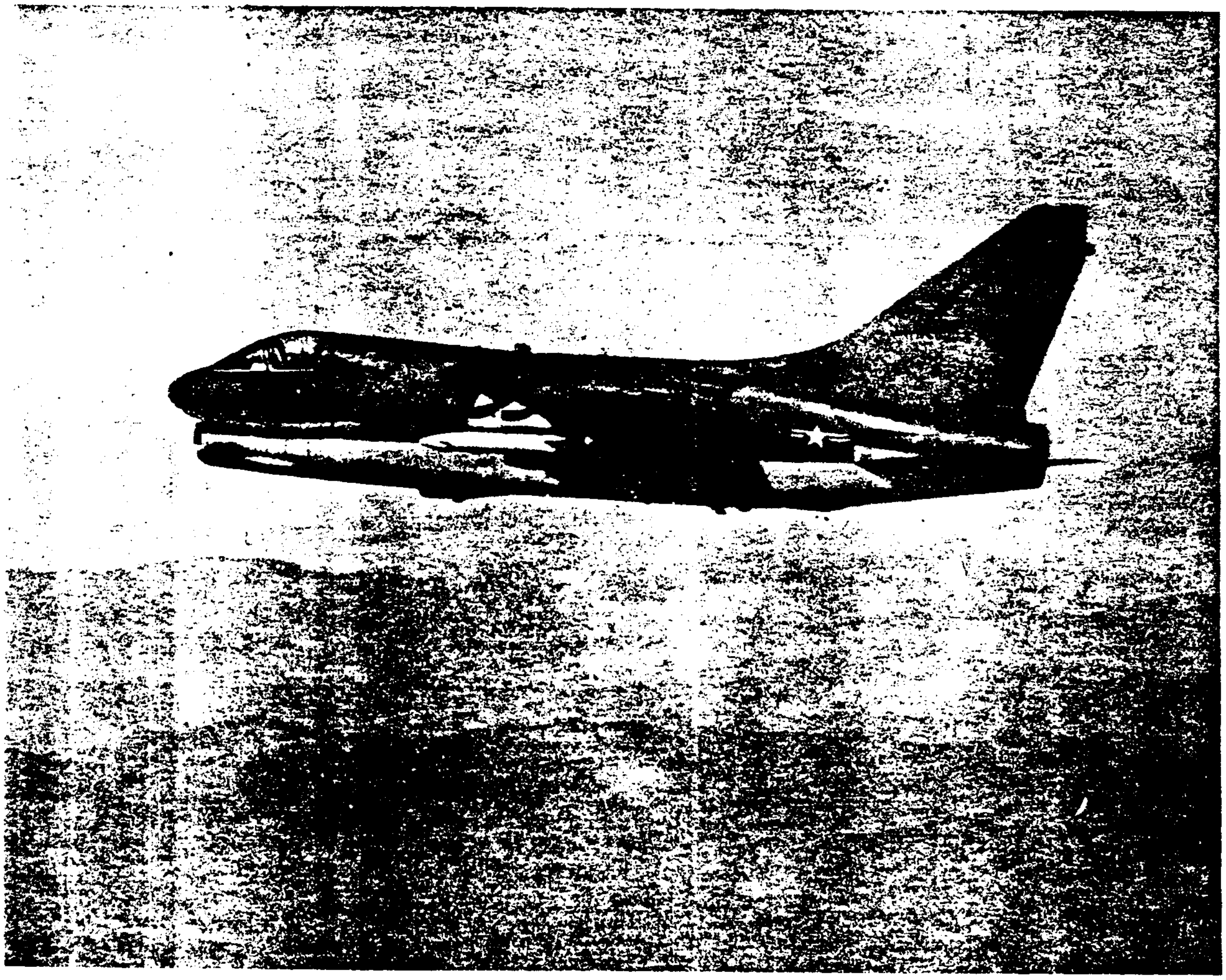


CHAPTER 7 POST-STALL GYRATIONS/SPINS

REVISED FEBRUARY 1977

7.1 INTRODUCTION

"Of the myriad of coupled motions which an airplane can perform, the spin stands out as being unique. When an airplane is stalled and left to itself, it will perform some sort of rolling, yawing and pitching motion which, if allowed to continue, may develop into a characteristic motion called a spin, in which the airplane descends rapidly toward the earth in a helical movement about the vertical axis at an angle of attack between the stall and 90°." (Reference 1, page 2)

From this classical description it is clear that an aircraft spin is an extremely complex maneuver simultaneously involving pitch, roll, and yaw rates along with extreme angles of attack and large angles of sideslip. It is indeed more complex than the description. In the initial stages, the aircraft will still have some of its translational velocity, and the aircraft can spin at angles of attack greater than 90 degrees.

Recently a renewed interest in the high angle of attack (AOA) flight regime has generated considerable interest in designing to avoid the spin entirely in tactical aircraft. Clearly such design goals are worthy, and a whole set of terms (not necessarily new) has been redefined to make more explicit the requirements which hopefully will make spin resistant tactical aircraft a reality. Words like "departure," stall," "post-stall gyration (PSG)," and "spin" itself have taken on different shades of meaning since the publication of reference 2. Given the complicated motions associated with a PSG or a spin and the explicit requirements now imposed by references 2 and 3, it is imperative that the test pilot clearly understand the precise terminology of the high AOA flight regime.

7.1.1 DEFINITIONS

7.1.1.1 Stall Versus Out-of-Control.

Stalls and associated aerodynamic phenomena have been described completely in chapter 2, but it is worth repeating the formal definition of a stall from page 67 of reference 3. In terms of angle of attack, the stall is defined as the lowest of the following:

a. Angle of attack for the highest steady load factor, normal to the flightpath, that can be attained at a given speed or Mach number.

b. Angle of attack, for a given speed or Mach number, at which abrupt or uncontrollable pitching, rolling, or yawing occurs. Angular limits of 20 degrees (Classes I, II, or III) or 30 degrees (Class IV) are specified in paragraph 3.4.2.1.2 of reference 3.

c. Angle of attack, for a given speed or Mach number, at which intolerable buffeting is encountered.

d. An arbitrary angle of attack, allowed by paragraph 3.1.9.2.1 of reference 3, which may be based on such considerations as ability to perform altitude corrections, excessive sinking speed, or ability to execute a go-around.

Reference 2 defines the stall angle of attack more simply: The angle of attack for maximum usable lift at a given flight condition. This latter definition is the one most useful in this course, but the student must understand that "maximum usable lift" is determined from one of the four conditions given above.

●7.1.1.2 Departure.

Departure is defined as that event in the post-stall flight regime which precipitates entry into a PSG, spin or deep stall condition (reference 2, paragraph 6.3.9). Notice two things about this definition. First, departure occurs in the post-stall flight regime; that is, the stall always precedes departure. It can be inferred then that the angle of attack for maximum usable lift is always less than the angle of attack at which departure occurs. The second point is that only one of three motions may result after departure - the aircraft enters either a PSG, spin or deep stall (of course, a PSG can progress into a spin or deep stall). Implicit in this definition is the implication that an immediate recovery cannot be attained. For example, a light aircraft whose stall is defined by a g-break, may recover immediately if the longitudinal control pressure is relaxed. However, note that movement or position of controls is not mentioned in the definition. The same light aircraft that would not depart if control pressures were relaxed at the stall may depart and enter a spin if pro-spin controls are applied at the stall. Hence, in discussing, susceptibility or resistance to departure one must specify control positions as well as loading and configuration.

The departure event is usually a large amplitude, uncommanded, and divergent motion. Such descriptive terms as nose slice or pitch-up are commonly used to describe the event. Large amplitude excursions imply changes in yaw, roll, or pitch greater than 20 degrees (class I, II, and III) or 30 degrees (class IV) (reference 3, paragraph 3.4.2.1.2). Uncommanded motions are motions not intended by the pilot, even though the control positions are legitimately causing the departure. The aircraft may not follow the pilot's commands for a number of reasons: the high angle of attack may render the control surface ineffective when moved to its desired position; or the pilot may be unable to position the stick to put the surface in the desired position due to lateral or transverse g loads. In either of these conditions the aircraft motion is "uncommanded." Finally, a divergent motion is one which either continuously or periodically increases in amplitude. The T-33 usually exhibits a "bucking" motion after the stall in which the nose periodically rises and falls. However, the motion is not divergent unless aggravated by full aft stick or some other pro-spin control. The T-38 will sometimes exhibit a non-divergent lateral oscillation near the stall angle of attack. Neither of these motions are normally counted as departures, though their occurrence does serve as warning of impending departure if further misapplications of controls are made. With this sort of background it is easy to see why a departure is so hard to define, yet is relatively easy for a pilot to recognize. Next one must examine the terms "post-stall gyration", "spin" and "deep stalls", used to define a departure.

● 7.1.1.3 Post-Stall Gyration.

A post-stall gyration is an uncontrolled motion about one or more axes following departure (reference 2, paragraph 6.3.10). PSG is a very difficult term to define concisely because it can occur in so many different ways. Frequently, the motions are completely random about all axes and no more descriptive term than PSG can be applied. On the other hand a snap roll or a tumble are post-stall gyrations. The main difficulty lies in distinguishing between a PSG and either the incipient phase of a spin or an oscillatory spin. The chief distinguishing characteristic is that a PSG may involve angles of attack that are intermittently below the airplane's stall angle of attack, whereas a spin always occurs at angles of attack greater than stall.

● 7.1.1.4 Spin.

A spin is a sustained yaw rotation at angles of attack above the aircraft's stall angle of attack (reference 2, paragraph 6.3.11). This definition bears a bit of explanation in that a spin is certainly not altogether a yaw rotation. Only the perfect flat spin (α = 90 degrees) could satisfy that constraint. The inference is, however, that the yaw rotation is dominant in characterizing a spin. Indeed, to a pilot, the recognition of a sustained (though not necessarily steady) yaw rate is probably the most important visual cue that a spin is occurring. Even though roll rate and yaw rate are often of nearly the same magnitude, the pilot still ordinarily recognizes the spin because of the yaw rate. In steep spins (with α relatively close to α_s) it is quite easy to confuse the roll rate and yaw rate and pilots sometimes have difficulty in recognizing this type of motion and treating it as a spin. The steep inverted spin is particularly confusing since the roll and yaw rates are in opposite directions. Once again though, the yaw rate determines the direction of the spin and the required control manipulations to recover. All in all, it is well to remember that the spin is truly a complicated maneuver involving simultaneous roll, pitch, and yaw rates and high angles of attack. And, even though the overall rotary motion in a spin will probably have oscillations in pitch, roll, and yaw superimposed upon it, it is still most easily recognized by its sustained yawing component.

● 7.1.1.5 Deep Stall.

A deep stall is an out-of-control flight condition in which the airplane is sustained at an angle of attack well beyond that for α_s while experiencing negligible rotational velocities (reference 2, paragraph 6.3.12). It may be distinguished from a PSG by the lack of significant motions other than a high rate of descent. The deep stall may be a fairly stable maneuver such as a falling leaf, or it can be characterized by large amplitude angle of attack oscillations. For an aircraft to stay in a deep stalled condition, significant oscillations must be limited to the longitudinal axis. Lateral and directional control surfaces are either stalled or blanked out. Depending on the pitching moment coefficient, recovery may or may not be possible.

7.1.2 SUSCEPTIBILITY AND RESISTANCE TO DEPARTURES AND SPINS:

Susceptibility/resistance to departures and spins has become an extremely important design goal for the generation of high performance aircraft presently in the design stage. Reference 4 offers convincing proof that such design emphasis is in fact overdue. But, for the designer to meet this requirement in an aircraft and for the test pilot to test against this requirement, it is essential that the words "susceptible" and "resistant" be understood alike by all concerned.

7.1.2.1 Extremely Susceptible to Departure (Spins). (PHASE A)

An aircraft is said to be extremely susceptible to departure (spins) if the uncontrolled motion occurs with the normal application of pitch control alone or with small roll and yaw control inputs. The only allowable roll and yaw control inputs are those normally associated with a given maneuver task. In short, an airplane that departs or enters a spin during Phase A of the flight test demonstration falls within this category (reference 2, paragraph 3.4.1.8).

7.1.2.2 Susceptible to Departure (Spins). (PHASE B)

An aircraft is said to be susceptible to departure (spins) when the application or brief misapplication of pitch and roll and yaw controls that may be anticipated in normal operational use cause departure (spin). The amount of misapplied controls to be used will be approved by the procuring activity for Phase B of the flight test demonstration. In other words each aircraft will be stalled and aggravated control inputs will be briefly applied to determine departure (spin) susceptibility.

7.1.2.3 Resistant to Departure (Spins). (PHASE C)

An aircraft is said to be departure (spin) resistant if only large and reasonably sustained misapplication of controls results in a departure (spin). "Reasonably sustained" means up to 3 seconds before recovery is initiated (reference 2, table I). This time delay may be increased for aircraft without positive indication of impending loss of control. This aircraft departs (spins) during Phase C of the flight test demonstration.

7.1.2.4 Extremely Resistant to Departure (Spins). (PHASE D)

An aircraft is said to be extremely resistant to departure (spins) if these motions occur only after abrupt, inordinately sustained application of gross, abnormal, pro-departure controls. Aircraft in this category will only depart (spin) in Phase D of the flight test demonstration when the controls are applied and held in the most critical manner to attain each possible mode of post-stall motion and held for various lengths of time up to 15 seconds or three spin turns, whichever is longer.

7.1.3 SPIN MODES:

Adjective descriptors are used to describe general characteristics of a given spin and these adjectives specify the spin mode. Average values of angle of attack, for example, would allow categorization of the spin as either upright (positive angle of attack) or inverted (negative angle of attack). An average value of angle of attack would also allow classification of a spin as either flat (high angle of attack) or steep (lower angle of attack). Finally, the average value of the rotational rate compared with the oscillations in angular rates about all three axes determines the oscillatory character of the spin. One descriptive modifier from each of these groups may be used to specify the spin mode.

Table I

SPIN MODE MODIFIERS

Group 1	Group 2	Group 3
Upright	Steep	Smooth Mildly Oscillatory
Inverted	Flat	Oscillatory Highly Oscillatory Violently Oscillatory

The most confusing thing about mode identification is the proper use of group 2 and group 3 modifiers. Perhaps the following tabulated data, extracted from reference 5, will provide insight for understanding how to use these terms.

Table II

F-4E SPIN MODES

Mode	Average AOA (deg)	AOA Oscillations (deg)	Yaw Rate (deg/sec)	Roll Rate (deg/sec)	Pitch Rate (deg/sec)
Steep-Smooth	42	±5	40-50	50	15
Steep-Mildly Oscillatory	45-60	±10	45-60	--	--
Steep-Oscillatory	50-60	±20	50-60 (with large oscillations)	Same as yaw rate	--
Flat-Smooth	77-80	Negligible	80-90	25	7

Note: One mode reported in reference 5 has been omitted from this table because the terminology did not fully conform to that of reference 2. It was called "highly oscillatory" with angle of attack excursions of ±30 degrees.

●7.1.4 SPIN PHASES:

A typical spin may be divided into the phases shown in figure 7.1

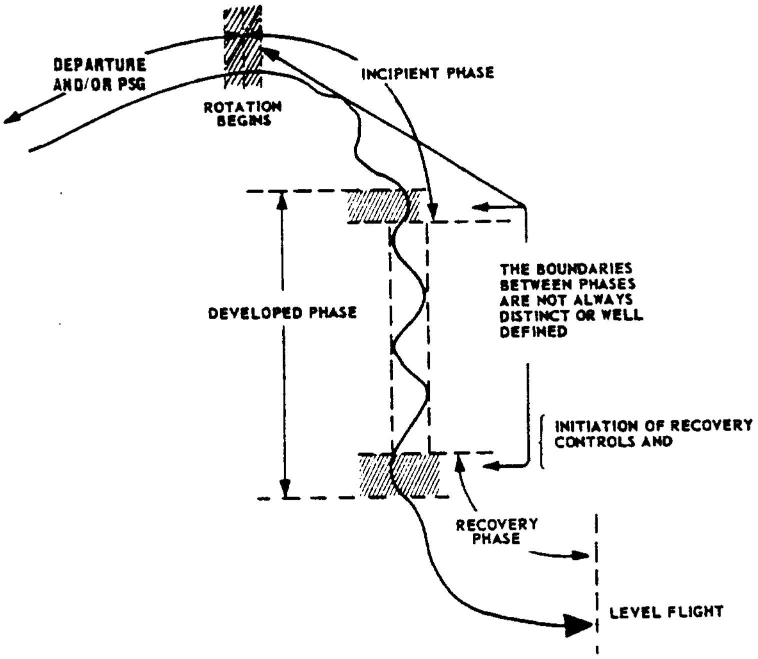


FIGURE 7.1 SPIN PHASES

●7.1.4.1 Incipient Phase.

The incipient phase of a spin is the initial, transitory part of the motion during which it is impossible to identify the spin mode. However, notice in figure 7.1 that the yaw rotation begins as the incipient phase begins; that is, the visual cue to the pilot is of a sustained (though by no means steady) yaw rotation. A further distinction between the PSG (if one occurs) and the incipient phase of the spin is that the angle of attack is continuously above the stalled angle of attack (α_S) for the aircraft, in the incipient phase of the spin. During a PSG the angle of attack may intermittently be less than α_S. This incipient phase continues until a recognizable spin mode develops, another boundary very difficult to establish precisely. In fact the test pilot may not recognize such a mode until he has seen it several times; but careful examination of data traces and film may reveal that a "recognizable" mode had occurred. In this case "recognizable" does not necessarily mean recognizable in flight, but distinguishable to the engineer from all available data. In short, the incipient phase of the spin is a transitory motion easily confused with a PSG, but distinctly different from either a PSG or the developed phase of the spin.

7.1.4.2 Developed Phase.

The developed phase of a spin is that stage of the motion in which it is possible to identify the spin mode. During this phase it is common for oscillations to be present, but the mean motion is still abundantly clear. The aerodynamic forces and moments are not usually completely balanced by the corresponding linear and angular accelerations, but at least equilibrium conditions are being approached. Generally it is evident in the cockpit that the developed phase is in progress, though the exact point at which it began may be quite fuzzy. Since the aircraft motion is approaching an equilibrium state, it is frequently advisable to initiate recovery before equilibrium is achieved. For example, during the T-38 test program warning lights were installed to signal a buildup in yaw rate. Test pilots initiated recovery attempts when these lights came on. Still, in the flat spin mode with recovery initiated at 85 degrees per second, a peak yaw rate of 165 degrees per second was achieved. The longitudinal acceleration at the pilot's station was approximately 3.5 g and the spin was terminated by deployment of the spin chute (reference 6, pages 10, 11). The developed spin, while it may be more comfortable due to less violent oscillations, can be deceptively dangerous, and the spin phase which follows can be disastrous.

7.1.4.3 Fully Developed Phase.

A fully developed spin is one in which the trajectory has become vertical and no significant change in the spin characteristics is noted from turn to turn. Many aircraft never reach this phase during a spin, but when they do, they are often very difficult to recover. The smooth, flat spin of the F-4 is a classic example in which this phase is attained and from which there is no known aerodynamic means of recovery. But a fully developed spin obviously requires time and altitude to be generated; how much time and how much attitude are strong fuctions of entry conditions. As a general rule, departures that occur at high airspeeds (high kinetic energy) require more time and altitude to reach the fully developed phase than departures which occur at low kinetic energy. Finally, the spin characteristics which remain essentially unchanged in the fully developed phase include such parameters as time per turn, body axis angular velocities, altitude loss per turn, and similar quantities. However, the definition does not prohibit a cyclic variation in any of these parameters. Hence, a fully developed spin can be oscillatory.

With this rather lengthy set of definitions in mind it is now appropriate to look more closely at spinning motions and at the aerodynamic and inertial factors which cause them and the PSG.

7.2 THE SPINNING MOTION

Because the PSG is a random and usually a highly irregular motion, it is very difficult to study. On the other hand, the spin can approach an equilibrium condition and is therefore much more easily understood. Further, since the PSG is affected by the same aerodynamic and mass loading characteristics as the spin, an understanding of the spin and the factors affecting it are appropriate to the purposes of this course.

7.2.1 DESCRIPTION OF FLIGHTPATH:

An aircraft spin is a coupled motion at extreme attitudes that requires all six equations of motion for a complete analysis. It is usually depicted with the aircraft center of gravity describing a helical path as the airplane rotates about an axis of rotation. Figures 7.2 shows such a motion. Notice that the spin axis of rotation may be curved and that the spin vector $\bar{\omega}$ is constantly changing. Such a motion is highly complex, but by making some approximations a simplification results which can be very useful in understanding the spin and its causes.

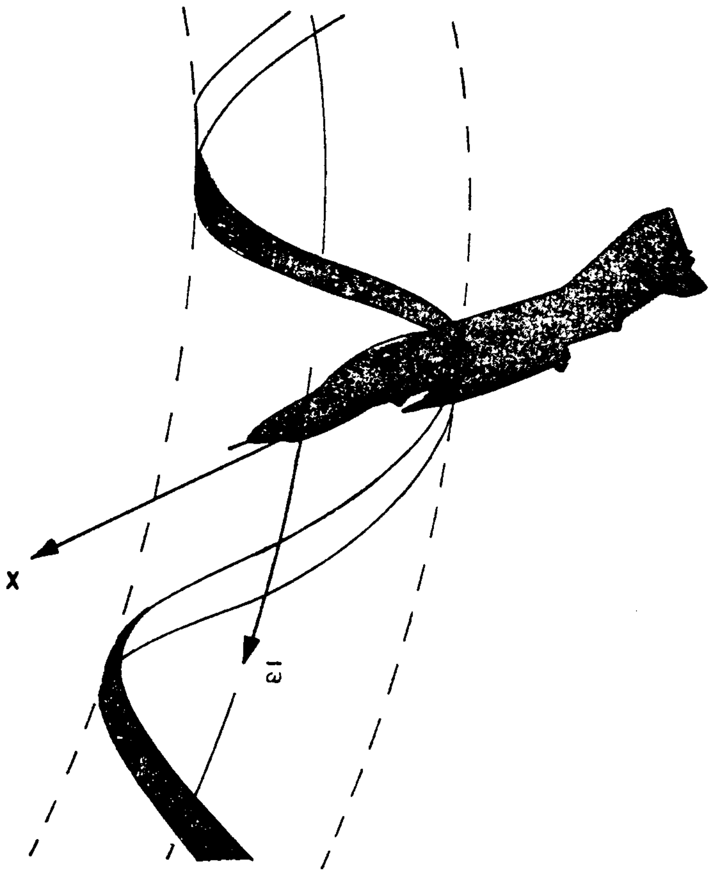


FIGURE 7.2 HELICAL SPIN MOTION

In a fully developed spin with no sideslip the spin axis is vertical as indicated in figure 7.3.

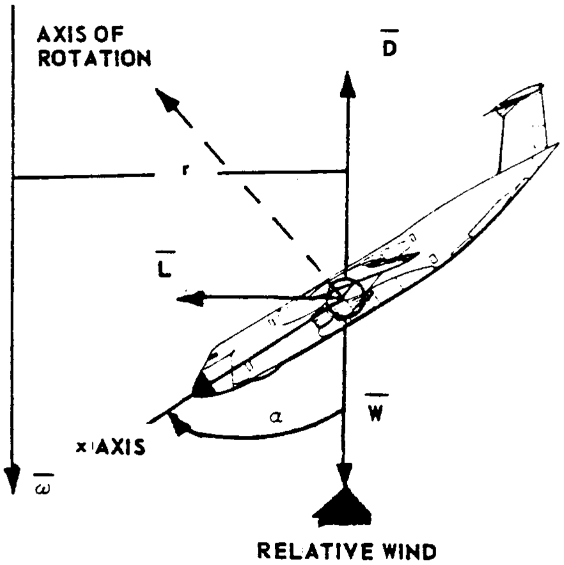
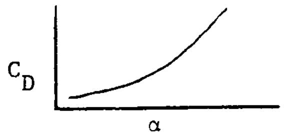


FIGURE 7.3 FORCES IN A STEADY SPIN WITHOUT SIDESLIP

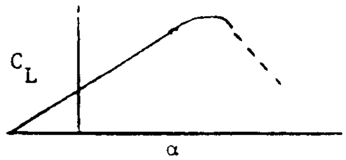
If one ignores the side force, the resultant aerodynamic force acts in the x-z plane and is approximately normal to the wing chord. Taking the relative wind to be nearly vertical, a summation of vertical forces gives:

$$W = D = \frac{1}{2} \rho V^2 S C_D \qquad (7.1)$$



A similar summation of horizontal forces suggests that the lift component balances the so-called centrifugal force so that

$$mr\omega^2 = L = \frac{1}{2} \rho V^2 S C_L \qquad (7.2)$$



Equation 7.1 suggests that as AOA increased (and C_D increased) the rate of descent (V) must decrease. Furthermore, at a stalled AOA, C_L decreases as AOA increases. With these two facts in mind it is clear that the left hand side of equation 7.2 must decrease as the AOA increases in a spin. The rotation rate, ω, as will be shown later, tends to increase as AOA increases; hence, the radius of turning r must decrease rapidly

as AOA increases. These observations point up the fact that in a fully developed spin $\bar{\omega}$ and the relative wind are parallel and become more nearly coincident as the AOA increases. In fact the inclination (η) of the flightpath (relative wind) to the vertical is given by

$$\tan \eta = \frac{r\omega}{V}$$

A typical variation of η with AOA is from about 5.5 degrees at α = 50 degrees to 1 degree at α = 80 degrees (reference 7, page 533). So, it is not farfetched to assume that $\bar{\omega}$ is approximately parallel to the relative wind in a fully developed spin.

All of these observations have been made under the assumption that the wings are horizontal and that sideslip is zero. These effects, while extremely important, are beyond the scope of this course, but references 7 and 8 offer some insight into them. It is also noteworthy that this simplified analysis is valid only for a fully developed spin. However, the trends to be noted and an understanding of the underlying physical phenomena will give the student a greater appreciation of the other phases of the spin and of the post-stall gyration.

7.2.2 AERODYNAMIC FACTORS:

In the post-stall flight regime the aircraft is affected by very different aerodynamic forces than those acting upon it during unstalled flight. Many aerodynamic derivatives change sign; others which are insignificant at low angles of attack become extremely important. Probably the most important of these changes is a phenomenon called autorotation which stems largely from the post-stall behavior of the wing.

7.2.2.1 Automtative Couple of the Wing.

If a wing is operating at α_1 (low angle of attack) and experiences a $\Delta\alpha$ due to wing drop, there is a restoring moment from the increased lift. If, on the other hand, a wing operating at α_2 ($\alpha_2 > \alpha_s$) experiences a sudden drop, there is a loss of lift and an increase in drag that tends to prolong the disturbance and sets up autorotation. These aerodynamic changes are illustrated in figure 7.4.

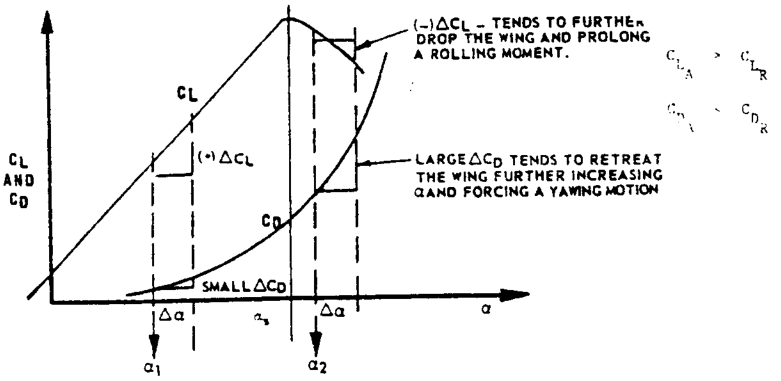


FIGURE 7.4 CHANGES IN C_L AND C_D WITH $\alpha < \alpha_s$ AND $\alpha > \alpha_s$

Consider now a wing flying in the post-stall region of figure 7.4 and assume that some disturbance has given that wing an increase in α which tends to set up a yawing and rolling motion to the right as shown in figure 7.5. The angle of attack of the advancing wing (section A) corresponds to α_2 in figure 7.4 while the angle of attack of the retreating wing (section R) corresponds to $\alpha_2 + \Delta\alpha$ in figure 7.4. Figure 7.6 shows these two sections and illustrates why the advancing wing is operating at a lesser angle of attack than the retreating wing. In each case the velocity vectors are drawn as they would be seen by an observer fixed to the respective wing section. The difference in resultant aerodynamic force $\bar{R}_A - \bar{R}_R$ acting at section A will, in general, be a force $\Delta\bar{F}$, depicted in figure 7.7. Notice that ΔF_x is in a positive x-direction, while ΔF_z is in a negative z-direction. ΔF_x forms a couple as depicted in figure 7.8 which tends to sustain the initial yawing moment to the right. Of course, ΔF_z contributes a similar rolling couple about the x-axis which tends to sustain the initial rolling moment to the right. Ordinarily, the autorotative couples generated by the wing are the most important aerodynamic factors causing and sustaining a spin. However, the other parts of the aircraft also have a part to play.

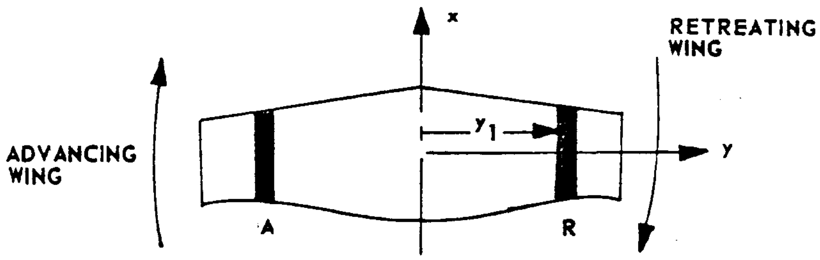


FIGURE 7.5 PLAN VIEW OF AUTOROTATING WING

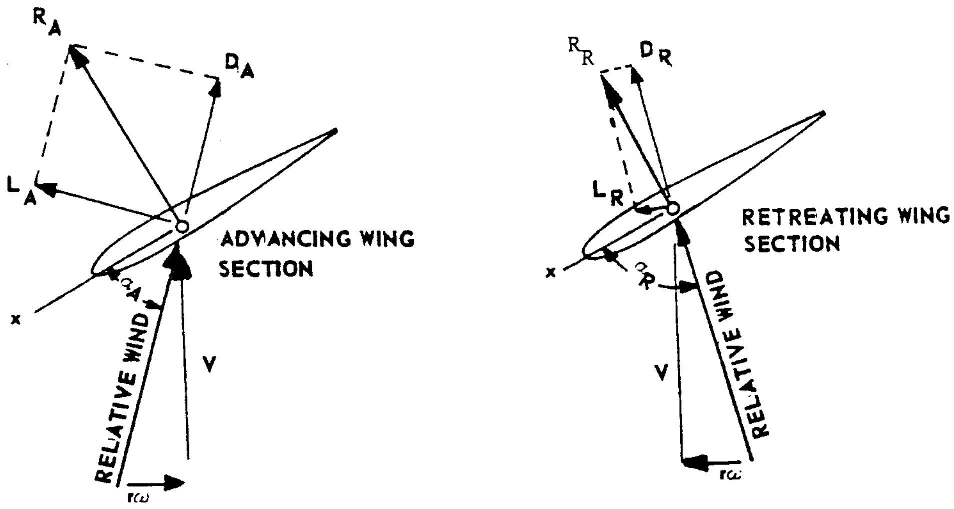


FIGURE 7.6 DIFFERENCE IN AOA FOR THE ADVANCING AND RETREATING WING IN AUTOROTATION

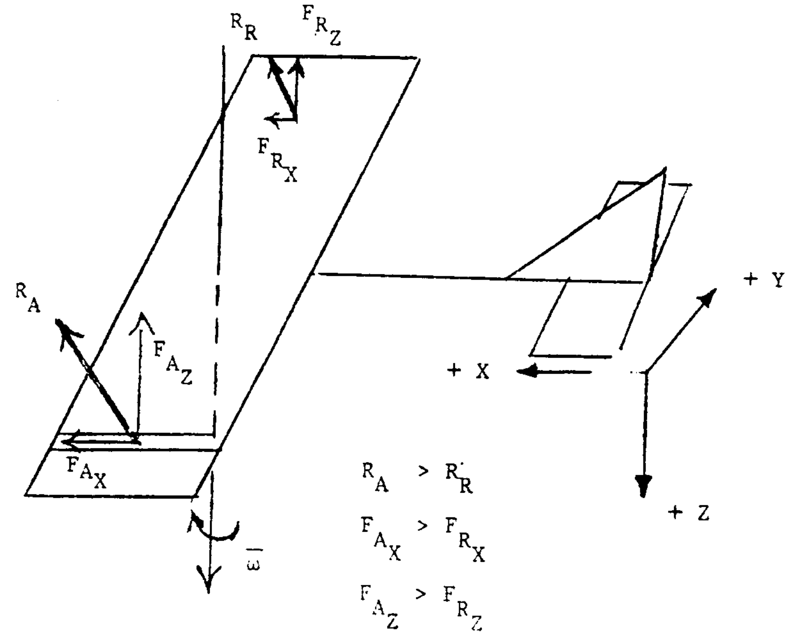
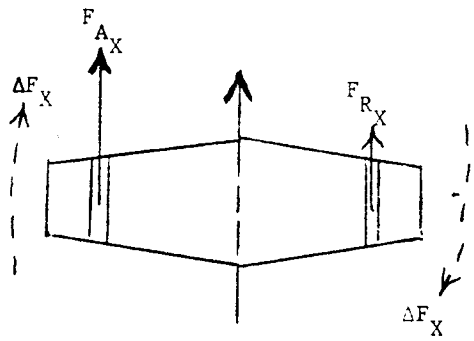


FIGURE 7.7 DIFFERENCE IN RESULTANT AERODYNAMIC FORCES



$F_{A_X} - F_{R_X} = \Delta F_X$ YAW SUSTAINING

SIMILARLY

$F_{A_Z} - F_{R_Z} = \Delta F_Z$ ROLL SUSTAINING

FIGURE 7.8 AUTOROTATIVE YAWING COUPLE

● 7.2.2.2 Fuselage Contribution.

The aerodynamic forces on the fuselage at stalled angle of attack are very complex, are highly dependent on fuselage shape, and may either oppose or increase the autorotative couples. Sidewash flow over the fuselage greatly affects the dihedral effect (C_{ℓ_β}) and may even increase it to values greater than those observed for unstalled flight (reference 7, page 529). Weathercock stability (C_{n_β}) will also be affected significantly by sidewash flow over the fuselage. As an example of the possible contributions of the fuselage to the autorotative couple consider the effects of fuselage shape as illustrated in figure 7.9. The fuselage in figure 7.9a acts much like an airfoil section and may well generate a resultant aerodynamic force which would contribute to the yawing autorotative couple. Of course the fuselage shape will determine the relative sizes of "lift" and "drag" contributed by the rotating nose section. A box-like fuselage cross-section will probably give a resultant aerodynamic force opposing the yaw autorotation. An extreme example of this type of fuselage cross-section reshaping is the strakes added to the nose of the T-37, as in figure 7.9b. Clearly the flow separation produced by the strakes in a flow field with considerable sidewash reorients the resultant aerodynamic force in such a way as to produce an anti-spin yawing moment. Such devices have also been proposed (and tested) for the F-100, F-106, and F-111.

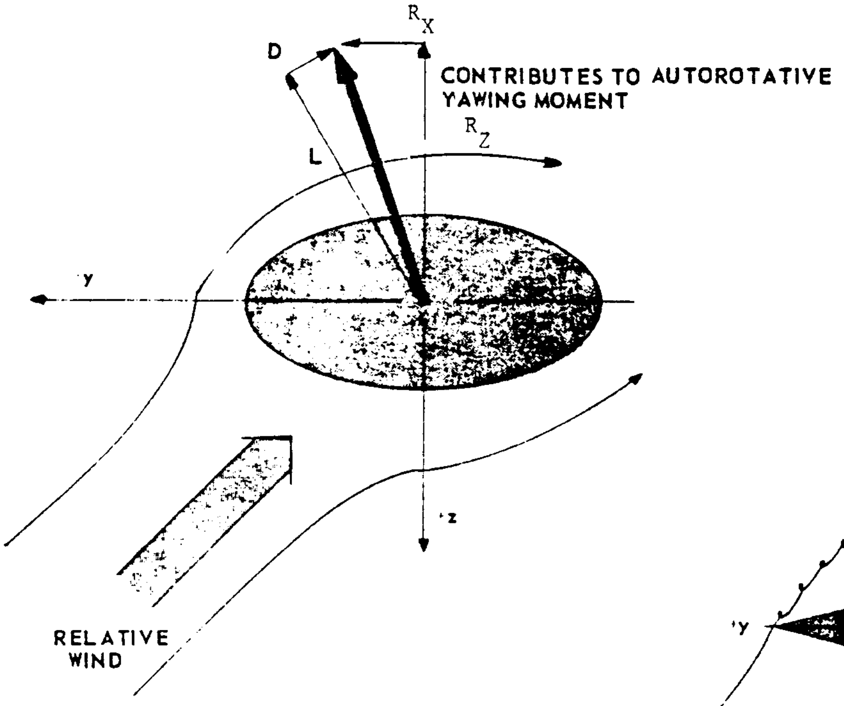


FIGURE 7.9A PLAIN FUSELAGE

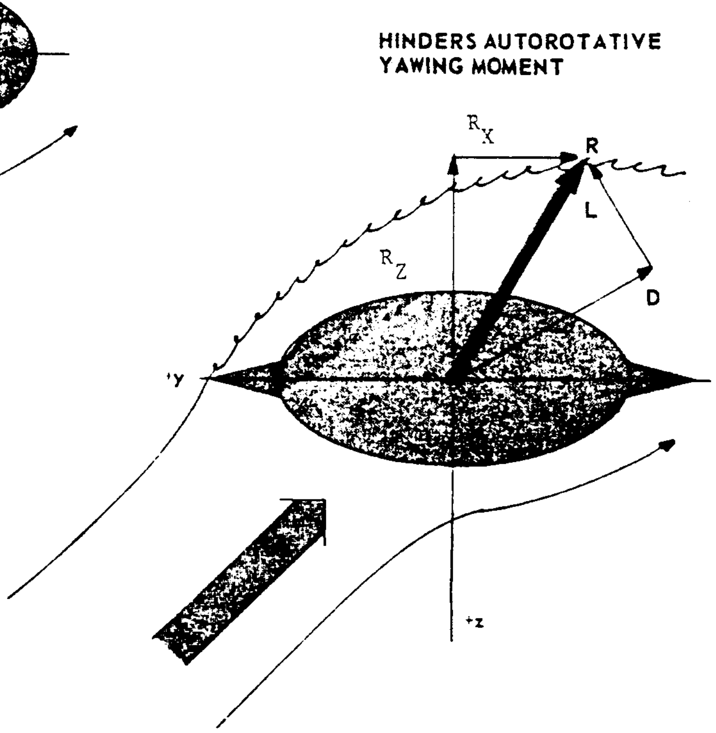


FIGURE 7.9B FUSELAGE WITH STRAKES

7.2.2.3 Changes in Other Stability Derivatives.

All of the other stability derivatives, especially those depending on the lift curve slope of the wing, behave in a different manner in the post-stall flight regime. However, a fuller discussion of the post-stall behavior of such derivatives as C_{ℓ_p}, C_{n_p}, C_{n_r}, and combinations of these derivatives is given in reference 7, page 529. For the purposes of this course it suffices to say that C_{ℓ_p} becomes positive and C_{n_p} may become positive in the post-stall flight regime; C_{n_r} may also become greater in stalled flight. Each of these changes contributes to autorotation, the aerodynamic phenomenon which initiates and sustains a spin. However, aerodynamic considerations are by no means the only factors affecting the post-stall motions of an aircraft. The inertia characteristics are equally important.

7.2.3 AIRCRAFT MASS DISTRIBUTION:

7.2.3.1 Principal Axes.

For every rigid body there exists a set of principal axes for which the products of inertia are zero and one of the moments of inertial is the maximum possible for the body. For a symmetrical aircraft, this principal axis system is frequently quite close to the body axis system. For the purpose of this course, the small difference in displacement is neglected, and the principal axes are assumed to lie along the body axes. Figure 7.10 illustrates what the actual difference might be.

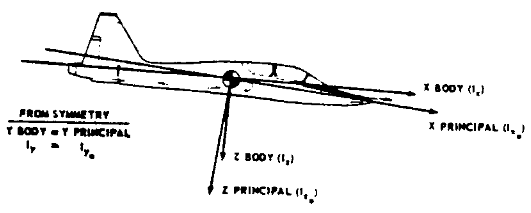


FIGURE 7.10 BODY AND PRINCIPAL AXES PROXIMITY

7.2.3.2 Radius of Gyration.

The center of gyration of a body with respect to an axis is a point at such a distance from the axis that, if the entire mass of the body were concentrated there, its moment of inertia would be the same as that of the body. The radius of gyration (K) of a body with respect to an axis is the distance from the center of gyration to the axis. In equation form

$$\int (y^2 + z^2)\, dm = I_x = K_x^2 m$$

$$\int (x^2 + z^2)\, dm = I_y = K_y^2 m$$

$$\int (x^2 + y^2)\, dm = I_z = K_z^2 m$$

or

$$K_i^2 = I_i/m, \quad i = x, y, \text{ or } z \qquad (7.3)$$

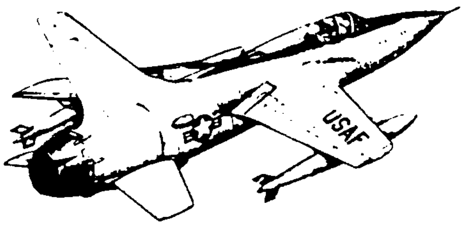
7.2.3.3 Relative Aircraft Density.

A nondimensional parameter called relative aircraft density (μ) is frequently used to compare aircraft density to air density.

$$\mu = \frac{m/Sb}{\rho} = \frac{m}{\rho Sb} \qquad (7.4)$$

7.2.3.4 Relative Magnitude of the Moments of Inertia.

The aircraft mass distribution is frequently used to classify the aircraft according to loading. Because aircraft are "flattened" into the XY plane, I_z is invariably the maximum moment of inertia. I_x is greater or less than I_y depending on the aircraft's mass distribution. The relative magnitudes of the moments of inertia are shown in figure 7.11. As will be seen in the next paragraph the relative magnitudes of I_x, I_y, and I_z are of utmost importance in interpreting the equations of motion.



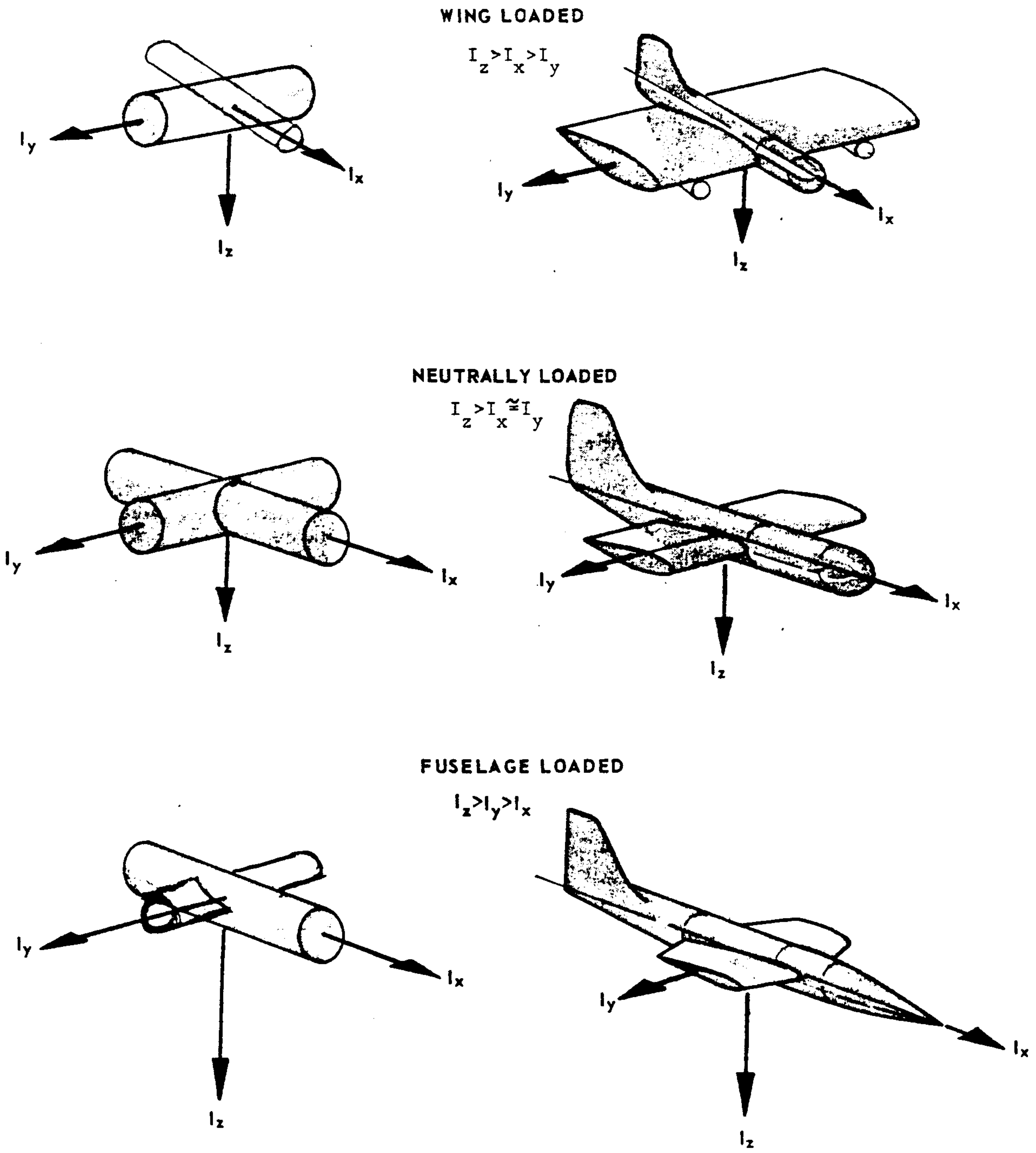


FIGURE 7.11 AIRCRAFT MASS DISTRIBUTION

7.3 EQUATIONS OF MOTION

Maneuvers within the post-stall flight regime can be analyzed by using all six equations of motion and integrating them numerically on a computer. From such studies, predictions of rate of rotation, angle of attack, magnitude of the oscillations, optimum recovery techniques, and other parameters can be made. However, such studies must use rather inaccurate theory to predict stability derivatives or else depend on wind tunnel data or free flight model tests to provide the aerodynamic data. Hence, many researchers prefer to rely almost completely on model tests for predictions prior to flight tests. Correlation between model tests and aircraft flight tests is generally good. But model tests also have limitations. Spin tunnel tests primarily examine developed or fully developed spins; there is no good way to investigate PSG's or the incipient phase of the spin in the spin tunnel. Reynolds number effects on both spin tunnel and free flight models make it very difficult to accurately extrapolate to the full scale aircraft. Engine gyroscopic effects are not often simulated in model tests. Finally, model tests are always done for a specific aircraft configuration, which is a distinct advantage for a flight test program even though it does not suit the purposes of this course. However, it would be foolish to ignore either computer analyses or model tests in preparing for a series of post-stall flight tests. For obvious reasons, this course will be restricted to a much simplified look at the equations of motion as applied to a fully developed spin.

7.3.1 ASSUMPTIONS:

The analytical treatment used in these notes is based on many simplifying assumptions, but even with these assumptions good qualitative information can be obtained. The most important assumption is that only a fully developed spin with the wings horizontal will be considered. The wings horizontal, fully developed spin involves a balance between applied and inertial forces and moments. Some of the ramifications of this assumption are:

a. Initially, it will also be assumed that the applied moments consist entirely of aerodynamic ones, although other factors will be considered in later paragraphs.

b. With the wings horizontal, $\bar{\omega}$ lies entirely within the xz plane. Also, with the aerodynamic and inertia forces balanced, $q = 0$, ie $\bar{\omega} = p\,\hat{i} + r\,\hat{k}$.

c. The rate of descent (V) is virtually constant, as is altitude loss per turn.

d. V and ω are parallel.

e. The time per turn is constant, or $\bar{\omega}$ is constant. Hence, $\dot{p} = \dot{q} = \dot{r} = 0$.

7.3.2 GOVERNING EQUATIONS:

The reference frame for expressing moments, forces, accelerations, etc., is the xyz body axis frame which rotates at the same rate as the spin rotation rate $\bar{\omega}$. The origin of the xyz axes is centered at the aircraft's cg and translates downward at a rate equal to the constant rate of descent $\bar{V}$. With this background the forces acting on the aircraft can be examined.

● 7.3.2.1 Forces.

The external forces applied to the aircraft and expressed in an inertial reference frame follow Newton's second law.

$$\bar{F} = m\,\dot{\bar{V}}$$

Expressing $\dot{\bar{V}}$ in the xyz reference frame,

$$\bar{F} = m\,(\dot{\bar{V}} + \bar{\omega} \times \bar{V})$$

But since $\bar{V}$ is constant in the fully developed spin and since $\bar{\omega}$ and $\bar{V}$ are parallel,

$$\bar{F} = 0$$

The elimination of the force equations in this fashion merely reinforces the idea that the rotary motion is the important motion in a spin and one would expect the significant equations to be the moment equations.

● 7.3.2.2 Moments.

The moment equations to be considered have already been developed in Chapter II and are repeated below.

$$G_x = \dot{p}\,I_x + qr\,(I_z - I_y) - (\dot{r} + pq)\,I_{xz} \qquad (7.5)$$

$$G_y = \dot{q}\,I_y - pr\,(I_z - I_x) + (p^2 - r^2)\,I_{xz} \qquad (7.6)$$

$$G_z = \dot{r}\,I_z + pq\,(I_y - I_x) + (qr - \dot{p})\,I_{xz} \qquad (7.7)$$

Utilizing the assumption that the body axes xyz are also principal axes and considering $\bar{G}$ to consist of aerodynamic moments only, these equations become:

$$\mathcal{L} = \dot{p}\, I_x + qr\,(I_z - I_y) \quad \longrightarrow \text{ROLLING MOMENT} \tag{7.8}$$

$$\mathcal{M} = \dot{q}\, I_y - pr\,(I_z - I_x) \quad \longrightarrow \text{PITCHING MOMENT} \tag{7.9}$$

$$\mathcal{N} = \dot{r}\, I_z + pq\,(I_y - I_x) \quad \longrightarrow \text{YAWING MOMENT} \tag{7.10}$$

Solving for the angular accelerations shows the contributions of each type of moment to that acceleration.

$$\dot{p} = \frac{\mathcal{L}}{I_x} + \frac{I_y - I_z}{I_x}\, q\, r \tag{7.11}$$

$$\dot{q} = \frac{\mathcal{M}}{I_y} + \frac{I_z - I_x}{I_y}\, p\, r \tag{7.12}$$

$$\dot{r} = \frac{\mathcal{N}}{I_z} + \frac{I_x - I_y}{I_z}\, p\, q \tag{7.13}$$

aerodynamic term   inertial term

The body axis angular accelerations can also be expressed in terms of aerodynamic coefficients and the relative aircraft density.

$$\frac{\mathcal{L}}{I_x} = \frac{\frac{1}{2}\rho V^2 S b}{K_x^2\, m} C_\ell = \frac{V^2}{\frac{2m}{\rho S b} K_x^2} C_\ell$$

$$= \frac{V^2}{2\mu K_x^2} C_\ell$$

In a similar manner,

$$\frac{\mathcal{N}}{I_z} = \frac{V^2}{2\mu K_z^2} C_n$$

It is common practice in post-stall/spin literature to define C_m on the basis of wingspan instead of on the basis of wing chord as is done in most other stability and control work. This change is made to allow a consistent definition of μ:

where $\mu = \frac{m}{\rho S b}$

and is indicated by a second subscript; that is, C_m becomes $C_{m,b}$. Then,

$$\frac{\mathcal{M}}{I_y} = \frac{V^2}{2\mu K_y^2} C_{m,b}$$

Equations 7.11 through 7.13 then become

$$\dot{p} = \frac{V^2 C_\ell}{2\mu K_x^2} + \frac{I_y - I_z}{I_x} qr \qquad (7.14)$$

$$\dot{q} = \frac{V^2 C_{m,b}}{2\mu K_y^2} + \frac{I_z - I_x}{I_y} pr \qquad (7.15)$$

$$\dot{r} = \frac{V^2 C_n}{2\mu K_z^2} + \frac{I_x - I_y}{I_z} pq \qquad (7.16)$$

With this brief mathematical background it is now appropriate to consider the aerodynamic prerequisites for a fully developed spin to occur.

● 7.3.3 AERODYNAMIC PREREQUISITES:

For a fully developed upright spin with the wings horizontal $\dot{p} = \dot{q} = \dot{r} = q = 0$ and equations 7.14, 7.15, and 7.16 yield

$$C_\ell = 0 \qquad (7.17)$$

$$-\frac{V^2 C_{m,b}}{2\mu K_y^2} = \frac{I_z - I_x}{I_y} pr \qquad (7.18)$$

$$C_n = 0 \qquad (7.19)$$

What do each of these results imply about a stable condition like the fully developed spin?

● 7.3.3.1 Pitching Moment Balance.

By examining equation 7.18 in conjunction with the $C_{m,b}$ versus α curve for an aircraft it is at least possible to identify regions where a fully developed spin can occur.

First, the angle of attack must be above the stall angle of attack. This condition is obvious, since the definition of a spin demands $\alpha > \alpha_s$.

Second, $C_{m,b}$ must be opposite in sign to the inertial term on the right hand side of equation 7.18. For an upright spin this requirement means that $C_{m,b}$ must be negative. This fact is clear if one observes that $I_z > I_x$ and that p and r are of the same sign in an upright spin (figure 7.12). In fact it is possible to express the rotation rate in a convenient form by slightly rearranging equation 7.18. Recall that

$$\frac{\mathcal{M}}{I_Y} = \frac{V^2 C_{m,b}}{2\mu K_y^{\,2}}$$

Figure 7.12 illustrates the fact that with wing levels

$p = \omega \cos \alpha$ and $r = \omega \sin \alpha$

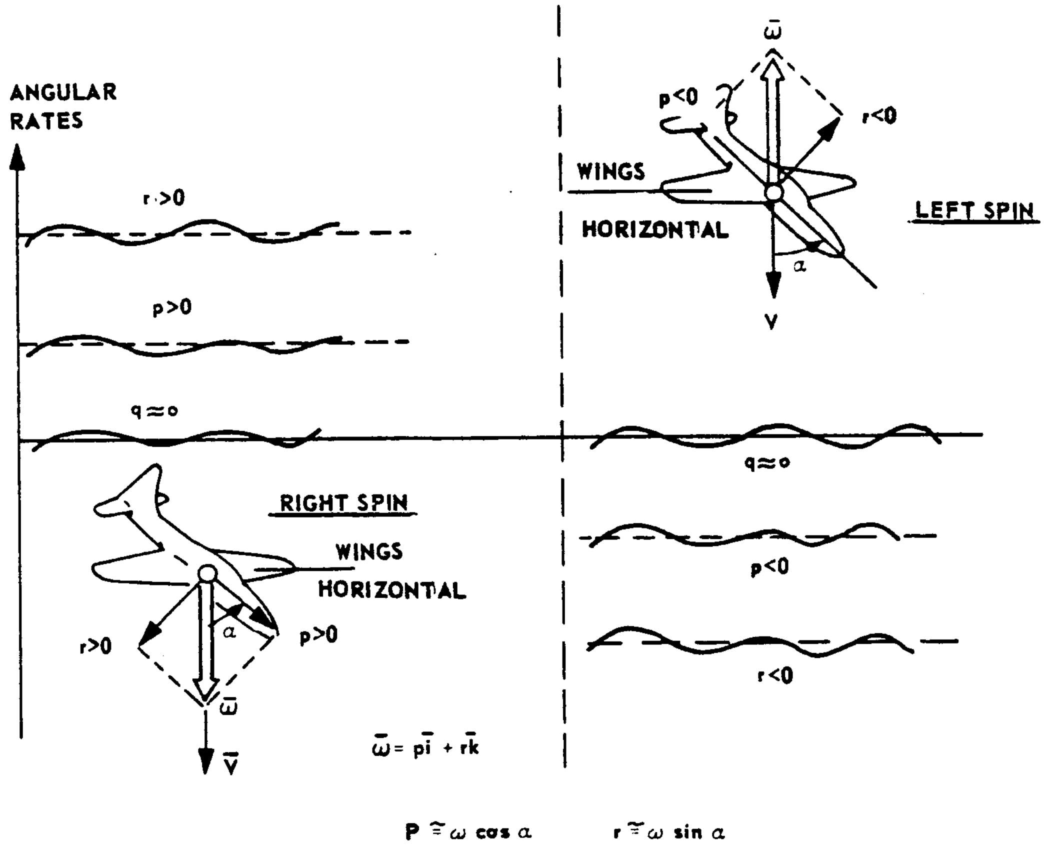


FIGURE 7.12 SPIN VECTOR COMPONENTS

Substituting into equation 7.18,

$$-\frac{\mathcal{M}}{I_y} = \frac{I_z - I_x}{I_y} \omega^2 \cos \alpha \sin \alpha$$

$$\omega^2 = \frac{-\mathcal{M}}{\frac{1}{2} (I_z - I_x) \sin 2\alpha} \qquad (7.20)$$

Equation 7.20 suggests that the minimum rotation rate occurs near an α of 45 degrees, although strong variations in M_{aero} may preclude this minimum. In fact, there is one additional prerequisite which must be satisfied before a fully developed spin can occur.

The slope of $C_{m,b}$ versus α must be negative or stabilizing and must be relatively constant. This is required simply because a positive $\frac{dC_{m,b}}{d\alpha}$ represents a divergent situation and would therefore require a pitching acceleration, $\dot{q} \neq 0$. But this angular acceleration would violate the assumption of a constant $\bar{\omega}$ in a fully developed spin. Said another way, any disturbance in angle of attack would produce a $\Delta C_{m,b}$ tending to restore $C_{m,b}$ to its initial value only so long as $\frac{dC_{m,b}}{d\alpha} < 0$. To summarize these constraints, consider figure 7.13. Aircraft B can enter a fully

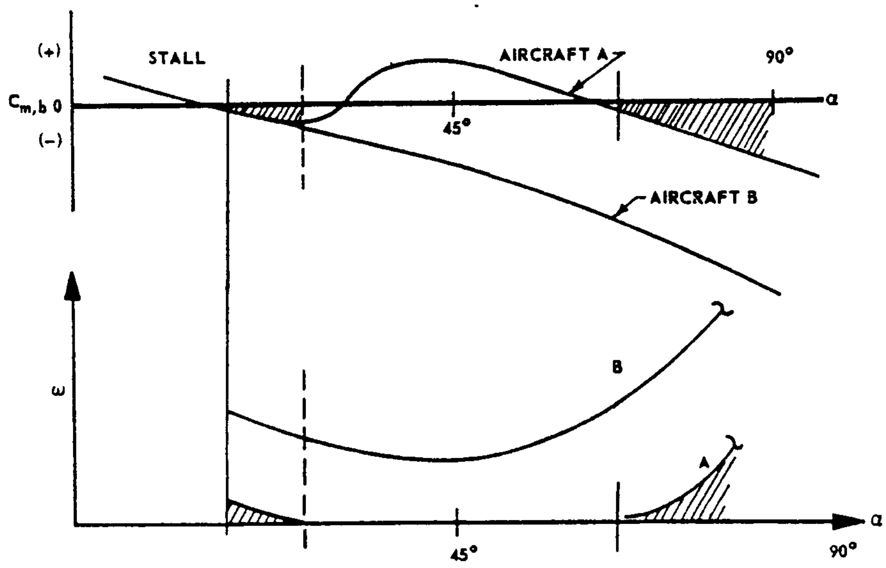


FIGURE 7.13 AERODYNAMIC PITCHING MOMENT PREREQUISITES

developed, upright spin at any AOA above α_s insofar as the pitching moment equation is concerned because its $C_{m,b}$ versus α is always negative and $\frac{dC_{m,b}}{d\alpha}$ is always negative. However, aircraft A can meet the three constraints imposed by the pitching equation only in the shaded areas. Of course, the pitching moment equation is not the sole criterion; the rolling and yawing moment equations must also be considered.

●7.3.3.2 Rolling and Yawing Moment Balance.

Equations 7.17 and 7.19 suggest at least four other conditions which must be satisfied to have a fully developed spin occur. Although not specifically pointed out in paragraph 7.3.3.1 all the aerodynamic derivatives, even $C_{m,b}$ are functions of both α, β, and the rotation rate ω (reference 9, page 6). Having considered $C_{m,b}$ as a function of α alone, it is convenient to consider C_n and C_ℓ as functions of ω alone. There is little justification for this choice other than the fact the lateral-directional derivatives are more directly linked to rotation rate while the longitudinal derivative is more directly linked to angle of attack. But it is well to keep in mind that all these variables do affect $C_{m,b}$, C_ℓ, and C_n.

The conditions imposed by both C_n and C_ℓ to allow a fully developed spin are that the derivatives must be equal to zero and the rate of change of the derivatives with respect to changes in ω must be negative. The first of these conditions is explicitly stated by equations 7.17 and 7.19. But the second requirement ($\frac{dC_\ell}{d\omega} < 0$ and $\frac{dC_n}{d\omega} < 0$) stems from the fact that a fully developed spin must be a stable condition. If an increase in ω will produce an increased C_ℓ or C_n, then any change in rotation rate will cause the autorotative moments to diverge away from the supposedly stable initial condition. Figure 7.14 illustrates this point.

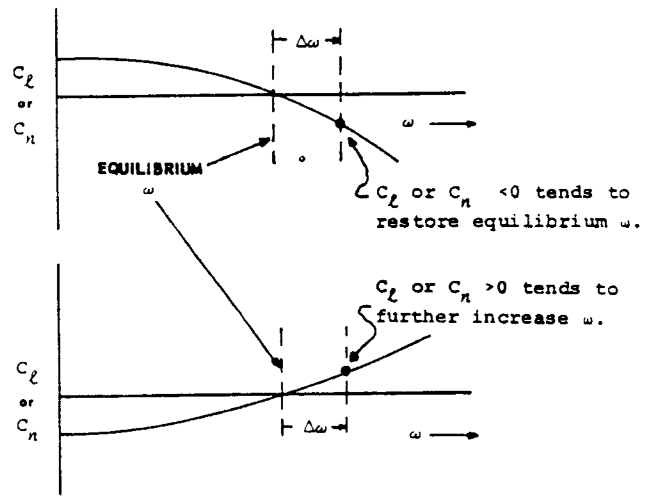


FIGURE 7.14 STABILIZING AND DESTABILIZING SLOPES FOR C_ℓ AND C_n VERSUS ω

Obviously, these aerodynamic prerequisites must all be met for a fully developed spin to exist in a true equilibrium form. Of course, oscillatory spins may occur with some relaxation of one or more of these conditions. It is extremely rare to observe an ideal case which would precisely meet all these conditions in an actual spin. So, while exactly satisfying all these conditions is essential for a fully developed spin to actually exist, it is common to estimate spin parameters with less than perfect fulfillment of these prerequisites. An example of how such estimations are made will be considered next.

7.3.4 ESTIMATION OF SPIN CHARACTERISTICS:

Reference 9, appendix B, describes in detail a method of estimating spin characteristics which was designed to estimate initial conditions for a computer study investigating possible steady state spin modes of the McDonnell F-3H Demon. Although this estimation method was only intended to help predict initial conditions for the numerical integration and thus save computer time, it serves as an excellent example of how model data and the aerodynamic prerequisites discussed in paragraph 7.3.3 can be combined to get a "first cut" at spin characteristics.

The aerodynamic data on which this example is based were measured by steadily rotating a model about an axis parallel to the relative wind in a wind tunnel. Hence, no oscillations in angular rates are taken into account. This limitation on the aerodynamic data is indicated by the subscript "r b" (rotation-balance tunnel measurements). In addition, the data are presented as a function of a nondimensional rotation rate, $\frac{\omega b}{2V}$. To help simplify the estimation process and partly because the rolling moment data were not as "well-behaved" as the yawing moment data, the rolling moment data were ignored. However, all the other prerequisites of paragraph 7.3.3 were observed. The estimation method is outlined below and the interested student is referred to reference 9, page 18, for a fuller description and a numerical example.

7.3.4.1 Determining $C_{m,rb}$ From Aerodynamic Data.

Use the $\frac{\omega b}{2V}$ and α for which $C_{n,rb} = 0$ and $\frac{dC_{n,rb}}{d(\frac{\omega b}{2V})} < 0$ to determine $C_{m,rb}$. This amounts to using the model data to determine aerodynamic pitching moments for which the aerodynamic yawing moment is zero.

7.3.4.2 Calculating Inertial Pitching Moment.

Using a modified form of equation 7.20, and recognizing that the inertial pitching moment is the negative of the aerodynamic pitching moment on a fully developed spin, $-C_{m,rb}$ is calculated.

$$\omega^2 = \frac{-\mathcal{M}}{\frac{1}{2}(I_z - I_x)\sin 2\alpha}$$

$$= \frac{-C_{m,rb}\frac{1}{2}\rho V^2 S b}{\frac{1}{2}(I_z - I_x)\sin 2\alpha}$$

Solving for $-C_{m,rb}$

$$-C_{m,rb} = \frac{I_z - I_x}{\rho S b}\left(\frac{\omega}{V}\right)^2 \sin 2\alpha \qquad (7.21)$$

● 7.3.4.3 Comparing Aerodynamic Pitching Moment and Inertial Pitching Moment.

Plot $C_{m,rb}$ versus α from the wind tunnel data (paragraph 7.3.4.1) and the results of equation 7.21 on the same plot, like figure 7.15.

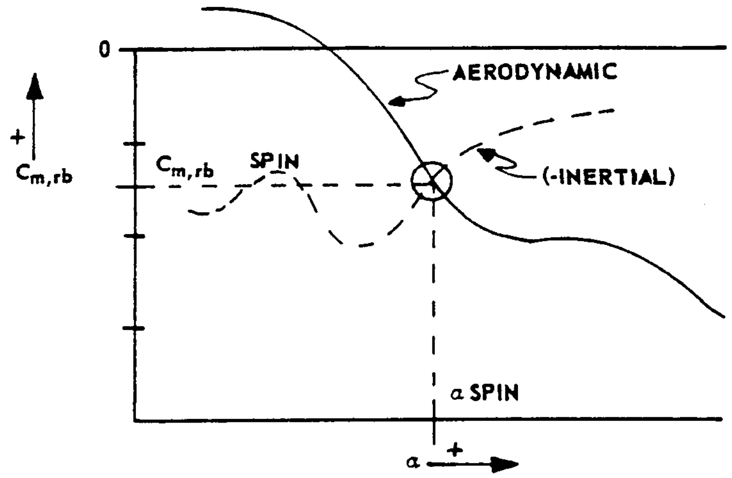


FIGURE 7.15 AERODYNAMIC PITCHING MOMENTS COMPARED TO -INERTIAL PITCHING MOMENTS

The intersection of the two curves indicates a possible fully developed spin. From this plot the angle of attack of the potential spin is read directly and the value of $C_{m,rb}$ is used to calculate the potential rotation rate.

7.3.4.4 Calculation of ω.

Rearranging equation 7.21,

$$\left(\frac{\omega}{V}\right)^2 = \frac{(-C_{m,rb})(\rho Sb)}{(I_z - I_x)\sin 2\alpha} \qquad (7.22)$$

the ratio $\frac{\omega}{V}$ can be calculated. But equation 7.1 allows calculation of V if C_D is known. The model force measurements provide C_D and then

$$V^2 = \frac{W}{\frac{1}{2}\rho S C_D} \qquad (7.23)$$

Then, of course,

$$\omega^2 = \frac{(-C_{m,rb})(\rho Sb)\ W}{(I_z - I_x)(\sin 2\alpha)\ \frac{1}{2}\rho S C_D}$$

$$\omega^2 = \frac{-2C_{m,rb}\ b\ W}{C_D\ (I_z - I_x)\ \sin 2\alpha} \qquad (7.24)$$

7.3.4.5 Results.

A typical set of results from the numerical integration of the six equations compared with the estimated parameters is given in table III below (extracted from reference 9, pages 26, 27).

Table III

TYPICAL COMPUTER RESULTS VERSUS ESTIMATION

Computer Results			Estimation		
α (deg)	ω (rad/sec)	V (ft/sec)	α (deg)	ω (rad/sec)	V (ft/sec)
36.0	1.88	294	38.2	1.90	285
37.0	1.92	372	45.1	1.83	327
Oscillated out of spin			48.2	1.89	453
51.8	2.18	619	50.5	2.18	620
80.0	4.72	494	70.0	3.50	515
36.5	2.80	380	37.4	2.69	365

7.3.5 GYROSCOPIC INFLUENCES:

Only aerodynamic moments have been considered so far in expanding the applied external moments. Ordinarily the aerodynamic moments are the dominant ones, but gyroscopic influences of rotating masses can also be important. The NF-104, for example has virtually no aerodynamic moments at the top of its rocket-powered zoom profile. There is convincing evidence that gyroscopic moments from the engine dominate the equations of motion at these extreme altitudes (reference 10, page 13). The external applied moments should be generalized to include gyroscopic influences and other miscellaneous terms (anti-spin rockets, anti-spin chutes, etc.). The applied external moments become

$$G_x = \mathcal{L} + L_{gyro} + L_{other}$$

$$G_y = \mathcal{M} + M_{gyro} + M_{other}$$

$$G_z = \mathcal{N} + N_{gyro} + N_{other}$$

The next paragraph will consider a simplified expansion of the gyroscopic terms.

7.3.5.1 Gyroscope Theory.

By virtue of its rotation, a gyroscope tends to maintain its spin axis aligned with respect to inertial space. That is, unless an external torque is applied, the gyro spin axis will remain stationary with respect to the fixed stars. If a torque is applied about an axis which is perpendicular to the spin axis, the rotor turns about a third axis which is orthogonal to the other two axes. On removing this torque the rotation (precession) ceases - unlike an ordinary wheel on an axle which keeps on rotating after the torque impulse is removed.

These phenomena, all somewhat surprising when first encountered, are consequences of Newton's laws of motion. The precessional behavior represents obedience of the gyro to Newton's second law expressed in rotation form, which states that torque is equal to the time rate of change of angular momentum.

$$\bar{T} = \frac{d\,\bar{H}}{dt} \qquad (7.25)$$

with $\bar{T}$ = external torque applied to the gyroscope

$\bar{H}$ = angular momentum of the rotating mass

$H = I\,\Omega$

with I = moment of inertia of the rotating mass

Ω = angular velocity of the rotating mass

Equation 7.25 applies, like all Newton's laws, only in an inertial frame of reference. If it assumed that $\bar{H}$ is to be expressed within a frame of reference rotating at the precession rate of the gyroscope, $\dot{\bar{H}}/_{inertial} = \dot{\bar{H}}/_{rotating} + \omega_p \times \bar{H}$. If the gyro spin rate is unchanged, then $\dot{\bar{H}}$ measured in the rotating frame will be zero and equation 7.25 becomes

$$\bar{T} = \bar{\omega}_p \times \bar{H} \tag{7.26}$$

The direction of precession for a gyro when a torque is applied is given by equation 7.26. This direction is such that the gyro spin axis tends to align itself with the total angular momentum vector, which in this case is the vector sum of the angular moment due to the spinning rotor and the angular momentum change due the applied torque, $\Delta\bar{H}$ as shown in figure 7.16. The law of precession is a reversible one. Just as a torque input results in an angular velocity output (precession), an angular velocity input results in a torque output along the corresponding axis.

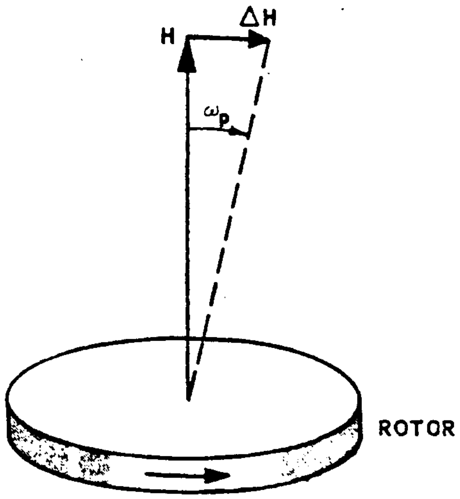


FIGURE 7.16 DIRECTION OF PRECESSION

Three gyro axes are significant in describing gyro operation; the torque axis, the spin axis and the precession axis. These are commonly referred to as input (torque), spin, and output (precession). The directions of these axes are shown in figure 7.17; they are such that the spin axis rotated into the input axis gives the output axis direction by the right hand rule. The direction of rotational vectors such as spin, torque, and precession can be shown by means of the right hand rule. If the curve of the fingers of the closed right hand point in the direction of rotation, the thumb extended will point along the axis of rotation. For gyro work, it is convenient to let the thumb, forefinger, and middle finger represent the spin, torque, and precession axes respectively. Figure 7.18 illustrates this handy memory device.

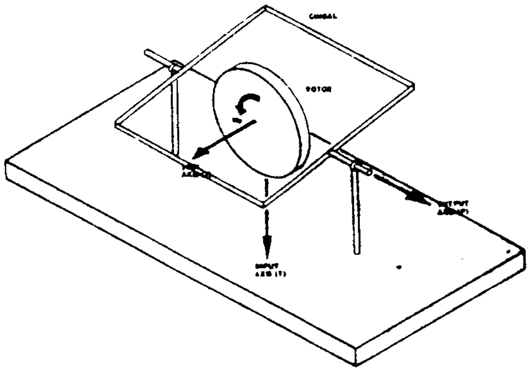


FIGURE 7.17 GYRO AXES

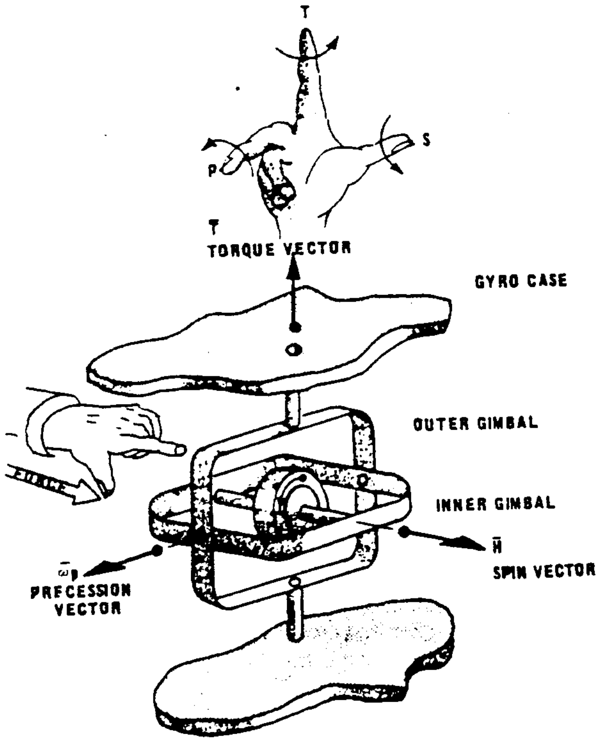


FIGURE 7.18 SPIN, TORQUE AND PRECESSION VECTORS

● 7.3.5.2 Engine Gyroscopic Moments.

In figure 7.19 consider the rotating mass of the engine as a gyroscope and analyze the external torque applied to the engine by the engine mounts of an aircraft in a spin. Then the total angular velocity of the rotating mass is the vector sum of $\bar{\omega}_E + \bar{\omega}$, with $\bar{\omega}_E$ being the engine rpm (assumed constant) and $\bar{\omega}$ being the aircraft's spin rotation rate.

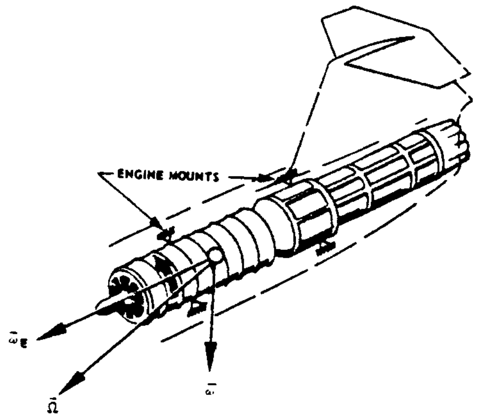


FIGURE 7.19 ANGULAR VELOCITIES OF THE ENGINE'S ROTATING MASS

$$\bar{\Omega} = \bar{\omega}_E + \bar{\omega}$$

But $\omega << \omega_E$

$$\bar{\Omega} \doteq \bar{\omega}_E$$

If one also assumes that the rotational axis of the engine is parallel to the x-axis,

$$\bar{\Omega} \doteq \omega_E \hat{i}$$

Then the angular momentum of the engine is

$$\bar{H}_E = I_E \omega_E \hat{i}$$

with I_E = moment of inertia of the engine about the x-axis.

Considering figure 7.19 again and applying equation 7.26, the external torque applied to the engine must be the precession rate of the aircraft ($\bar{\omega}$) crossed into the engine's angular momentum.

$$\bar{T} = \bar{\omega} \times \bar{H}_E$$

But the moment applied by the engine through the engine mounts to the spinning aircraft is equal but opposite in sign (Newton's Third Law).

$$\bar{G}_{gyro} = -\bar{\omega} \times \bar{H}_E$$

$$\begin{bmatrix} L_{gyro} \\ M_{gyro} \\ N_{gyro} \end{bmatrix} = - \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ p & q & r \\ I_E\omega_E & 0 & 0 \end{bmatrix}$$

$$L_{gyro} = 0 \tag{7.27}$$

$$M_{gyro} = -I_E\omega_E\, r \tag{7.28}$$

$$N_{gyro} = I_E\omega_E\, q \tag{7.29}$$

Then equations 7.11, 7.12, and 7.13 can be expanded to

	AERO	INERTIAL COUPLING (sometimes called gyrodynamic term)	GYROSCOPIC TERM (an engine effect)	MISC (rockets, spin chutes, etc.)	
$\dot{p} =$	$\dfrac{\mathcal{L}}{I_x}$ +	$\dfrac{I_y - I_z}{I_x}\, q\, r$ +	$\dfrac{L_{gyro}}{I_x}$ +	$\dfrac{L_{other}}{I_x}$	(7.30)
$\dot{q} =$	$\dfrac{\mathcal{M}}{I_y}$ +	$\dfrac{I_z - I_x}{I_y}\, p\, r$ +	$\dfrac{M_{gyro}}{I_y}$ +	$\dfrac{M_{other}}{I_y}$	(7.31)
$\dot{r} =$	$\dfrac{\mathcal{N}}{I_z}$ +	$\dfrac{I_x - I_y}{I_z}\, p\, q$ +	$\dfrac{N_{gyro}}{I_z}$ +	$\dfrac{N_{other}}{I_z}$	(7.32)

Equation 7.20 becomes

$$\omega^2 = \frac{-\mathcal{M} + I_E\omega_E\, r}{\frac{1}{2}(I_z - I_x)\sin 2\alpha} \tag{7.33}$$

Equation 7.33 shows that the effect of the engine gyroscopic moment is to shift the curves of figure 7.13 as shown below. An engine that rotates in a counter clockwise direction (as viewed from the tailpipe) will cause an aircraft to spin faster in a right upright spin and slower in a left upright spin. Generally speaking, however, this engine gyroscopic moment is negligible in comparison to the other external moments.

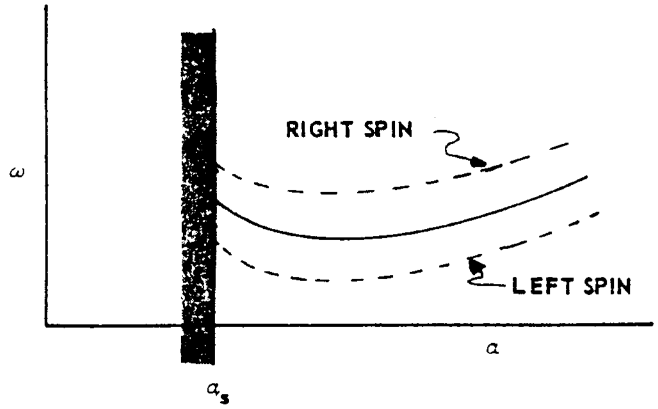


FIGURE 7.20 EFFECT OF M_{gyro} ON SPIN ROTATION RATE
(FOR COUNTER CLOCKWISE ENGINE DUE TO NOSE DOWN PITCH)

7.3.6 SPIN CHARACTERISTICS OF FUSELAGE-LOADED AIRCRAFT:

It is appropriate to consider briefly some of the spin characteristics peculiar to modern high performance aircraft in which the mass is generally concentrated within the fuselage (I_y larger than I_x and almost as large as I_z). It can be shown that a system which has no external moments or forces tends to rotate about its largest principal axis, which in the case of an aircraft, is the Z axis. In an actual spinning aircraft, the external moments are not zero and thus the aircraft spins about some intermediate axis. For the idealized spin thus far considered, the pitching moment equation leads one to the observation that fuselage-loaded aircraft will probably spin flatter than their wing-loaded counterparts.

7.3.6.1 Fuselage-Loaded Aircraft Tend to Spin Flatter Than Wing-Loaded Aircraft.

For a fully developed spin

$$G_y = -pr(I_z - I_x) \tag{7.34}$$

In an aircraft, $(I_z - I_x)$ can never be zero. Hence, if $G_y = 0$ then p must be zero, in which case $\bar{\omega} = r\bar{k}$ and the spin is flat ($\bar{\omega} = p\bar{i}$ is excluded by the definition of a spin). If the spin is not flat, then both p and r exist and, in an upright spin, have the same algebraic sign. Because $(I_z - I_x)$ is always positive, examination of equation 7.34 shows that G_y must always be negative (or zero) for an upright spin.

The smaller the pitch attitude (θ in figure 7.21) the flatter the spin, and θ can be defined as $\sin^{-1}\frac{p}{\omega}$ for the spin depicted in figure 7.21. θ varies with the relative magnitude of (I_z and I_x), as can readily be seen by rearranging equation 7.34.

$$p = \frac{|G_y|}{r(I_z - I_x)}$$

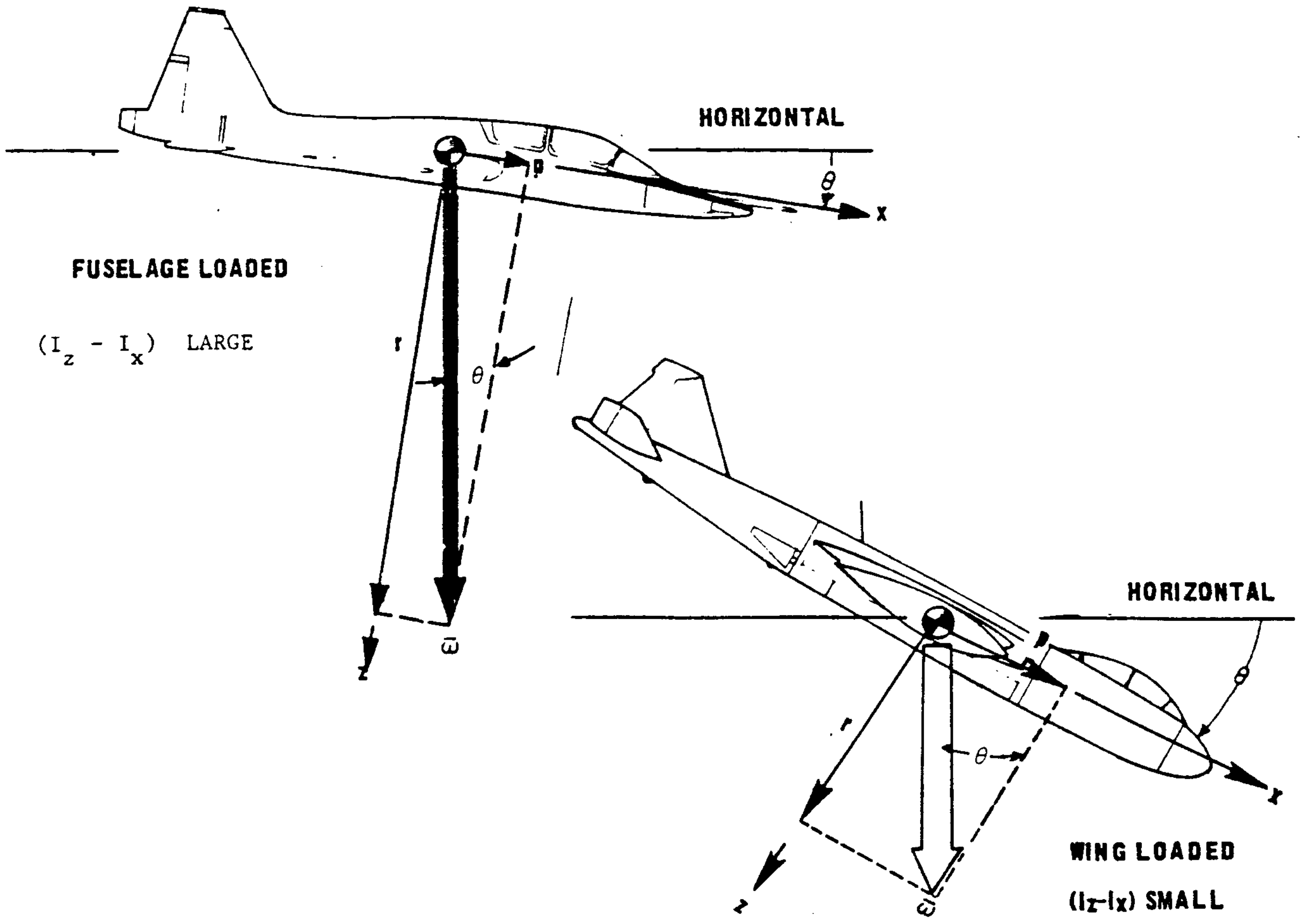


FIGURE 7.21 EFFECT OF MAGNITUDES OF I_z AND I_x ON SPIN ATTITUDE

Since p becomes smaller as $(I_z - I_x)$ increases, it is clear that fuselage-loaded aircraft tend to spin flatter than wing-loaded aircraft. But what about the effect of increasing I_y upon the roll equation?

●7.3.6.2 Fuselage-Loaded Aircraft Tend to Exhibit More Oscillations.

On aircraft where I_y is approximately equal to I_z in magnitude, the fully developed spin is more likely to be oscillatory. In the limit, if $I_y = I_z$, the reference spin could be wing down, since any axis in the YZ plane would be a maximum principal axis. Although these facts suggest that the bank angle is easily disturbed and that a developed spin often occurs with the bank angle not zero, a restoring tendency does exist which leads to periodic oscillations in bank angle. Consider again the rolling moment equation,

$$G_x = \dot{p} I_x + q r (I_z - I_y) \tag{7.35}$$

If an "o" subscript is used to represent the reference or steady-state conditions,

$$G_{x_o} = \dot{p}_o I_x + q_o r_o (I_z - I_y)$$

If instantaneous values are represented by equation 7.35, the change in external moments due to the perturbations of the angular acceleration and angular velocities is

$$(G_x - G_{x_o}) = (\dot{p} - \dot{p}_o) I_x + (q r - q_o r_o)(I_z - I_y)$$

Assuming perturbations in roll will not significantly change r_o, $r \approx r_o$ and

$$\Delta G_x = \Delta\dot{p} I_x + \Delta q (I_z - I_y) r_o$$

$$\Delta\dot{p} = \frac{\Delta G_x}{I_x} - \Delta q \frac{I_z - I_y}{I_x} r_o \tag{7.36}$$

The second term on the right side of equation 7.36 serves to damp oscillations in that it reduces the ability of perturbations in rolling moment (ΔG_x) to produce perturbations in roll acceleration ($\Delta\dot{p}$). For fuselage-loaded aircraft, in which $(I_z - I_y)$ is small, the damping is much reduced. Thus, any perturbations in the motion tend to persist longer in fuselage-loaded aircraft than they do in wing-loaded aircraft.

● 7.3.7 SIDESLIP:

It is beyond the scope of this course to deal with the effects of sideslip in any detail. However, it is noteworthy that sideslip need not be zero in a developed spin; in fact it usually is not. Reference 7, page 535, shows that sideslip in a spin arises from two sources: wing tilt with respect to the horizontal (ϕ) and the inclination of the flight path to the vertical (η).

$$\beta \doteq \phi - \eta \tag{7.37}$$

If then, one considers a spin with a helical flight path as opposed to a vertical flight path, the inclination of the flight path to the vertical is positive and equal to the helix angle. Then, in order to maintain zero sideslip, the retreating wing must be inclined downwards by an amount equal to the helix angle in order to have zero sideslip. However, it is quite common to have fully developed spins (with the spin axis vertical, not the flight path) with varying amounts of sideslip. Sideslip on a stalled wing will generally increase the lift on the wing toward which the sideslip occurs and reduce the lift on the opposite wing. It is easy to understand that a small amount of sideslip can produce a large rolling moment and thereby significantly alter the balance of rolling moments. These qualitative comments are quite cursory and the inquisitive student may wish to pursue these effects further. Reference 7 offers an expanded discussion, but to adequately discuss sideslip effects in any detail one must consider all three moment equations and their coupling effects. The consideration of sideslip leads to the general conclusion that the rolling couple can be balanced over a wide range of angles of attack and spin rotation rates.

7.4 INVERTED SPINS

Since PSG's are definitely uncontrolled aircraft motions, there is absolutely no guarantee that all spins will be of an upright variety, as has so far been assumed. The test pilot particularly (and operational pilots as well) will continue to experience inverted spins and PSG's which may be mainly inverted aircraft motions. As reference 11, page 1, points out,

> "...inverted spins cannot be prevented by handbook entries that 'the airplane resists inverted spins'."

It is, therefore, essential that the test pilot have some appreciation of the nature of the inverted PSG/spin. As usual, the analytical emphasis will necessarily be restricted to the fully developed spin, but the qualitative comments which follow also apply in a general way to other types of post-stall motion.

The most common pilot reaction to an inverted post-stall maneuver is, "I have no idea what happened! The cockpit was full of surprise, dirt, and confusion." Why? First, negative g flight is disconcerting in and of itself, particularly when it is entered inadvertently. But even experienced test pilots can be upset and their powers of observation reduced in an anticipated inverted spin. This disorientation usually takes one of two forms: (1) inability to distinguish whether the motion is inverted or upright or (2) inability to determine the direction of the spin. Each of these problems will be considered separately.

7.4.1 ANGLE OF ATTACK IN AN INVERTED SPIN:

The angle of attack in an inverted spin is always negative (figure 7.22). It might appear that it would be easy to determine the difference in an upright or inverted spin; if the pilot is "hanging in the straps," it is an inverted spin. Such an "analysis" is accurate in some spin modes (the Hawker Hunter has an easily recognized smooth, flat mode such as this); however, if the motion is highly oscillatory, not fully developed, or a PSG, the pilot's tactile senses are just not good enough. If the aircraft has an angle of attack indicator, this is probably the most reliable means of determining whether the maneuver is erect or inverted. Lacking an angle of attack system, the pilot must rely on the accelerometer or his sensory cues, neither of which are easy to interpret. But what about determining spin direction?

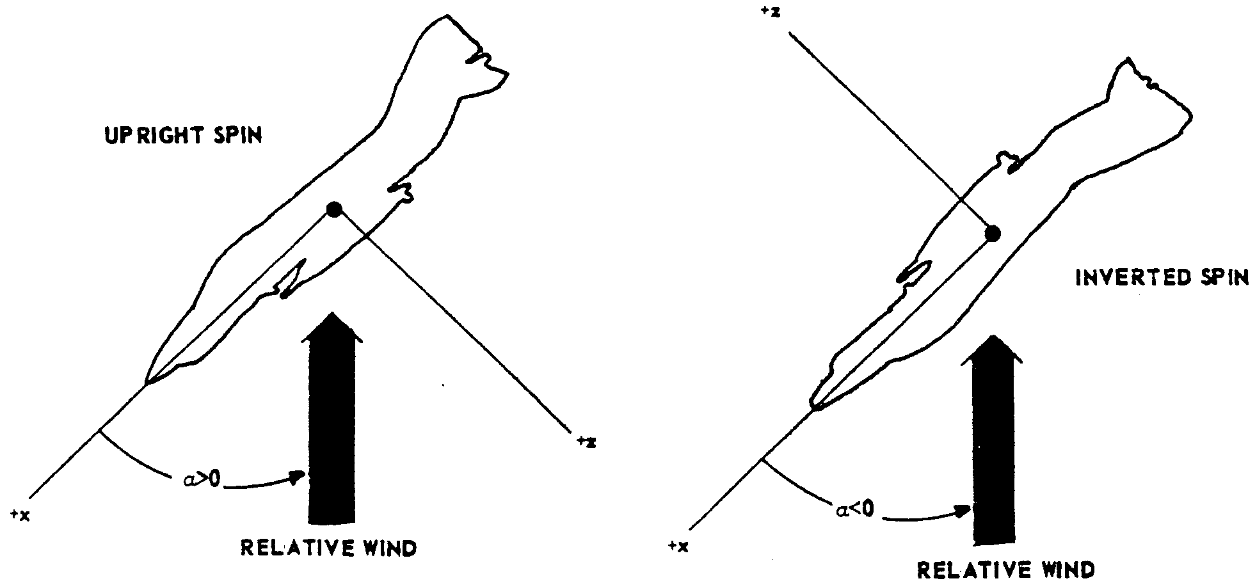
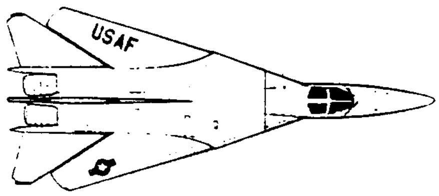


FIGURE 7.22 ANGLE OF ATTACK IN AN INVERTED SPIN

7.4.2 ROLL AND YAW DIRECTIONS IN AN INVERTED SPIN:

Consider two identical aircraft, one in an upright spin and the other in an inverted spin as shown in figure 7.23. Notice that the spin direction in either an upright or an inverted spin is determined by the sense of the yaw rate. Notice also that in an inverted spin the sense of the roll rate is always opposite to that of the yaw rate. It is common for pilots to mistakenly take the direction of roll as the spin direction. The chances of making this error are considerably enhanced during a PSG or the incipient phase of the spin when oscillations are extreme. In steep inverted spins ($|\alpha|$ nearly equals $|\alpha_s|$) the rolling motion is the largest rotation rate and further adds to the confusion. However, there is a reliable cockpit instrument, the turn needle, which always indicates the direction of yaw. With such confusion possible, what about the previously obtained equations of motion? Is it necessary to modify them for the inverted spin?



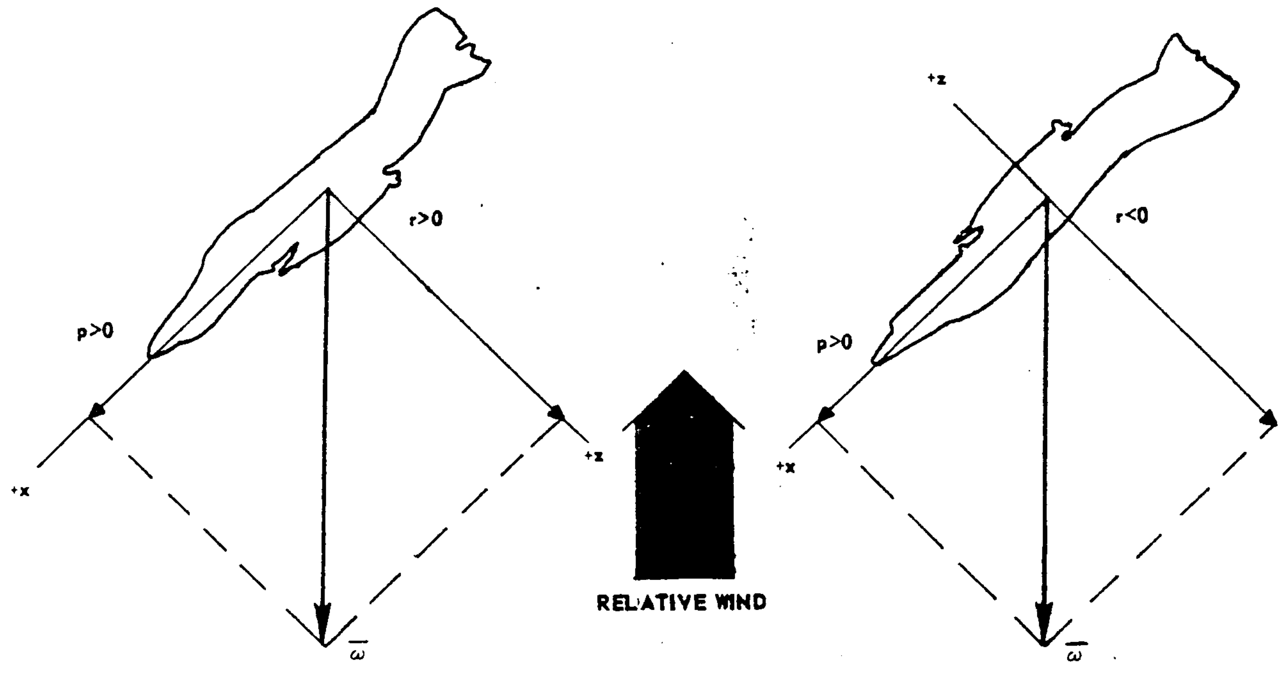


FIGURE 7.23 ROLL AND YAW RATES IN AN INVERTED SPIN

7.4.3 APPLICABILITY OF EQUATIONS OF MOTION:

All the equations previously described are directly applicable to the inverted spin. Of course, the differences in sign for angle of attack and the dearth of aerodynamic data collected at negative angle of attack pose a significant practical problem in trying to do detailed analyses of the inverted spin. But for the qualitative purposes of this course, the equations of motion are usable. However, it is instructive to note the difference in the sense of the pitching moments between an upright and an inverted spin. Recall that in an upright spin the applied external pitching moment (dominated by the aerodynamic pitching moment) had to be negative to balance the inertia couple, as equation 7.34 for a fully developed spin shows.

$$G_y = -pr(I_z - I_x) \qquad (7.34)$$

But when p and r are of opposite, as in the inverted spin, the applied external moment must be positive. This fact is illustrated in figure 7.24, where the mass of the aircraft is represented as a rotating dumb-bell.

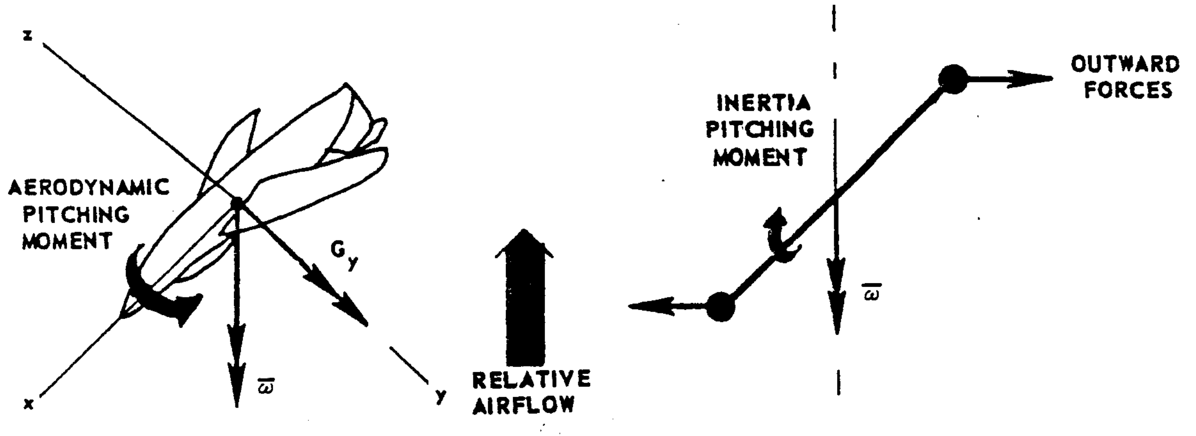


FIGURE 7.24 PITCHING MOMENTS IN AN INVERTED SPIN

It is apparent that in the inverted spin the external pitching moment is positive; that is, expressed as a vector, it lies along the positive y-axis. As a final point, the recovery from PSG's/spins, both erect and inverted, must be examined in some detail.

■ 7.5 RECOVERY

Obtaining developed spins today is generally difficult, but when obtained, the factors that make it difficult to obtain this type of spin may also make it difficult to recover from the spin. Current and future aircraft designs may be compromised too much for their intended uses to provide adequate aerodynamic control for termination of the developed spin; also, there is a problem of pilot disorientation associated with developed spins. As a result, the PSG and the incipient phase of the spin must be given more attention than they have received in the past, and preventing the developed spin through good design and/or proper control techniques has become a primary consideration.

Current aircraft have weights which are appreciably larger and have moments of inertia about the Y and Z axes which may be ten times as large as those of World War II aircraft. With the resulting high angular momentum, it is difficult for a spin to be terminated as effectively as a spin in earlier airplanes by aerodynamic controls which are generally of similar size. Furthermore, controls which are effective in normal flight may be inadequate for recovery from the spin unless sufficient consideration has been given to this problem in the design phase.

●7.5.1 TERMINOLOGY:

The recovery phase terminology was purposely omitted from paragraph 7.1.4 for inclusion here. Referring to figure 7.1, the whole of the recovery phase begins when the pilot initiates recovery controls and ends when the aircraft is in straight flight; however, there are several terms used to differentiate between the subparts of this whole phase.

● 7.5.1.1 Recovery.

Recovery is defined as the transitional event from out-of-control conditions to controlled flight. In more usable terms, this period of time normally is counted from the time the pilot initiates recovery controls and that point at which the angle of attack is below α_s and no significant uncommanded angular motions remain. The key phrase in this expanded definition is "angle of attack below α_s;" once this objective is attained the aircraft can be brought back under control provided there are sufficient altitude and airspeed margins to maneuver out of whatever unusual attitude ensues.

● 7.5.1.2 Dive Pullout and Total Recovery Altitude.

The dive pullout is the transition from the termination of recovery to level flight. Total recovery altitude is the sum of the altitude losses during the recovery and dive pullout. Notice that reference 3, paragraph 3.4.2.2.2, specifies altitude loss during recovery - not total recovery altitude.

● 7.5.2 ALTERATION OF AERODYNAMIC MOMENTS:

The balanced condition of the developed spin must be disturbed in order to effect a recovery, and prolonged angular accelerations in the proper direction are needed. Several methods for obtaining these accelerations are available but not all are predictable. Also, the accompanying effects of some methods are adverse or potentially hazardous. The general methods available for generating anti-spin moments are presented with the applicable terms of the general equations given below. Alteration of the aerodynamic moments (C_ℓ, $C_{m,b}$, and C_n) through the use of flight controls is the conventional means of spin recovery; seldom are configuration changes presently used to accomplish spin recovery. The all-important question is "How should the flight controls be used to recover from a PSG or a spin?"

1. Modify aerodynamic moments
   a. With flight controls
   b. Configuration changes (gear, flaps, strakes)

2. Reposition the aircraft attitude on the spin axis

$$\dot{p} = \frac{V^2}{2\mu K_x^2} C_\ell + \frac{I_y - I_z}{I_x} q r + \frac{L_{other}}{I_x}$$

$$\dot{q} = \frac{V^2}{2\mu K_y^2} C_{m,b} + \frac{I_z - I_x}{I_y} p r - \frac{I_E \phi_E}{I_y} r + \frac{M_{other}}{I_y}$$

$$\dot{r} = \frac{V^2}{2\mu K_z^2} C_n + \frac{I_x - I_y}{I_z} p q + \frac{I_E \phi_E}{I_z} q + \frac{N_{other}}{I_z}$$

3. Variations in Engine Power

4. Spin chutes
   Spin Rockets

7.5.2.1 Use of Longitudinal Control.

The longitudinal control surface can only be effective if it can drive the angle of attack below α_s. Rarely is the elevator capable of producing this much change in pitching moment in a fully developed spin, but its use during a PSG or the incipient phase of a spin may well reduce angle of attack sufficiently. However, forward stick during a fully developed upright spin will merely cause many spin modes to progress to a higher rotation rate, which is also usually flatter. Model tests and computer studies should thoroughly investigate this control movement before it is recommended to the test pilot. Then a thorough flight test program must be conducted to confirm these predictions before such a recommendation is passed on to operational users.

7.5.2.2 Use of Rudder.

Considering only the alteration of C_ℓ, $C_{m,b}$, or C_n by deflection of the appropriate control surfaces, the use of rudder to change C_n has proven to be the most effective in recovering from a developed spin. Rudder deflection, if the rudder is not blanked out, produces a reduction in yaw rate which persists. The reduction in yaw rate reduces the inertia pitching couple and the angle of attack consequently decreases. Once the rotation rate has been reduced sufficiently, the longitudinal control can be used to reduce angle of attack below α_s.

The student will notice that the use of ailerons to produce an anti-spin rolling moment has not been discussed. Generally, in stalled flight the ailerons are not effective in producing moments of any significance, though they can still be the primary anti-spin control by causing a small change in bank angle and thereby reorienting the aircraft attitude on the spin axis so that the inertial terms operate to cause recovery.

7.5.3 USE OF INERTIAL MOMENTS:

Examination of the inertial terms listed below reveals that the relative magnitudes of I_x and I_y determine how the ailerons should be used to reorient the aircraft attitude.

$$\dot{p} = \ldots + \frac{I_y - I_z}{I_x} qr \ldots$$

$$\dot{q} = \ldots + \frac{I_z - I_x}{I_y} pr \ldots$$

$$\dot{r} = \ldots + \frac{I_x - I_y}{I_z} pq \ldots$$

For a fuselage-loaded aircraft the pitch rate must be positive in an upright spin to develop anti-spin yawing and rolling accelerations from these terms. In both cases aileron applied in the direction of the spin causes the aircraft body axes to tilt so as to produce a positive component of ω along the y-axis (see figure 7.25). Another way to help achieve a positive pitch rate is to hold aft stick until the rotation rate begins to drop. This procedure is common in some fuselage-loaded aircraft, although it is unacceptable in others (F-104) for example). However, the most important factor is the relative sizes of I_x and I_y. Considering that $\frac{I_x - I_y}{I_z}$ is approximately six times greater for the F-104 than for the T-28, it is little wonder that aileron is a more important spin recovery control in the F-104 than is the rudder.

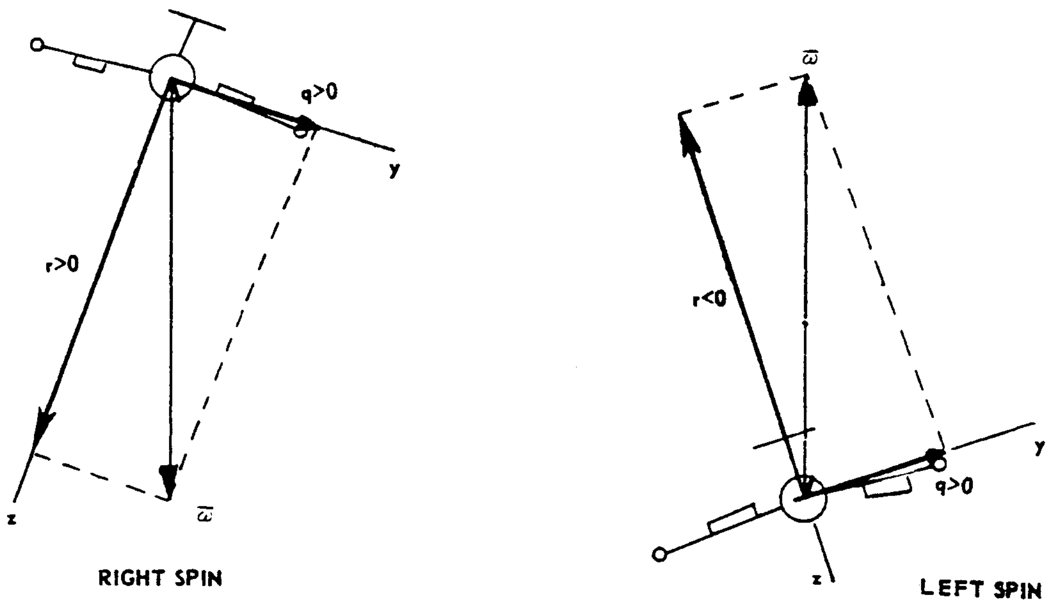


FIGURE 7.25 AILERON WITH RECOVERY PROCEDURE

A similar analysis shows that aileron against the upright spin in a wing-loaded aircraft will produce an anti-spin yaw acceleration, but a pro-spin roll acceleration. Since wing-loaded aircraft generally spin more nose low than fuselage-loaded aircraft (with $p \approx r$), and since they generally are recoverable with rudder and elevator, aileron-against recovery procedures are rarely recommended.

● 7.5.4 OTHER RECOVERY MEANS:

The other two terms which can produce anti-spin accelerations include engine gyroscopic terms and emergency recovery devices.

● 7.5.4.1 Variations in Engine Power.

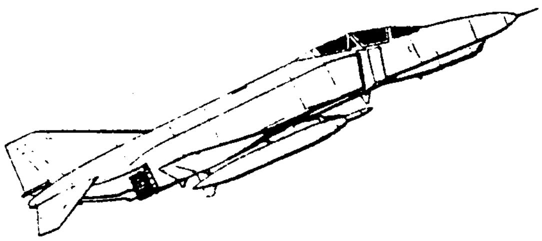
The gyroscopic terms are usually so small that they have little effect on recovery characteristics. Furthermore, jet engines often flame out during PSG or spin motions, particularly if the throttle is not at idle. So, although there are potential pitch and yaw accelerations available from the gyroscopic terms, NASA experience indicates that changes in engine power are generally detrimental to recovery.

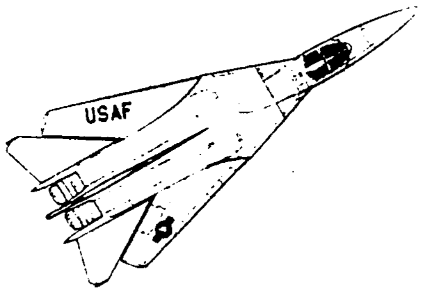
●7.5.4.2 Emergency Recovery Devices.

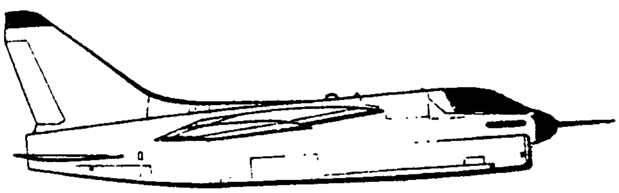
Emergency recovery devices may take many forms - anti-spin parachutes attached to the aft fuselage, anti-spin parachutes attached to the wing tip, anti-spin rockets, strakes, etc. The design of such devices is a complex subject worthy of careful engineering in its own right. Certainly such design considerations are not the concern of test pilots; but the reliability of the device, its attachments, and its jettison mechanism are of vital concern to him. He is also likely to be concerned with tests to validate this reliability.

●7 5.5 RECOVERY FROM INVERTED SPINS:

Recovery from inverted spins is generally easier than recovery from upright spins, particularly if the rudder is in undisturbed airflow. In fact many aircraft will recover from an inverted spin as soon as the controls are neutralized. In any case rudder opposite to the turn needle may be recommended, often in conjunction with aft stick. Some fuselage-loaded T-tailed aircraft may require anti-spin aileron. An analysis similar to that in paragraph 7.5.3 shows that in an inverted spin aileron against the spin is the correct anti-spin control for a fuselage-loaded aircraft.







Item	Definition
β	angle of sideslip
Δ	small increment
η	angle of inclination of the flight-path from the vertical
θ	pitch angle
μ	relative aircraft density
ρ	density
ρ	air density
ϕ	engine rotation rate
ϕ	bank angle
Ω	angular velocity
ω	angular velocity

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LIST OF ABBREVIATIONS AND SYMBOLS USED IN THIS CHAPTER

Item	Definition
AOA	angle of attack
b	wing span
C_D	drag coefficient
cg	center of gravity
C_L	coefficient of lift
C_ℓ	rolling moment coefficient
C_m	pitching moment coefficient
$C_{m,b}$	pitching moment coefficient non-dimensionalizing with b
C_n	yawing moment coefficient
D	drag
g	acceleration equal to that of gravity
H	angular momentum
$\hat{i}$	unit vector
$\hat{j}$	unit vector
$\hat{k}$	unit vector
K	radius of gyration
L	rolling moment
M	pitching moment
m	mass
N	yawing moment
p	roll rate
PSG	post-stall gyration
q	pitch rate
r	yaw rate
r	radius
S	wing area
sec	second
T	torque
V	true velocity
W	weight

Symbols

α	angle of attack
α_s	stall angle of attack

CHAPTER 8

ROLL COUPLING

REVISED FEBRUARY 1977

8.1 INTRODUCTION

Divergencies experienced during rolling maneuvers have frequently been referred to as "**Inertia** Coupling." This leads to a misconception of the problems involved. The divergence experienced during rolling maneuvers is complex because it involves not only inertia properties, but aerodynamic ones as well. It is the intention of this chapter to offer a physical explanation of the more important causes of roll coupling and to also introduce a brief mathematical tool to aid in predicting roll coupling divergencies.

Coupling results when a disturbance about one aircraft axis causes a disturbance about another axis. An example of uncoupled motion is the disturbance created by an elevator deflection. The resulting motion is restricted to pitching motion and no disturbance occurs in yaw or roll. An example of coupled motion is the disturbance created by a rudder deflection. The ensuing motion will be some combination of both yawing and rolling motion. Although all lateral-directional motions are coupled, the only motion that ever results in coupling problems large enough to threaten the structural integrity of the aircraft is coupling as a result of rolling motion. Thus our study of "roll coupling."

Although there are numerous contributions to the roll coupling characteristics of an aircraft, aeroelastic effects, etc., this chapter will only consider three: (1) inertia coupling, (2) the I_{xz} parameter, (3) aerodynamic coupling. These effects occur simultaneously in a very complicated fashion. Therefore, the resulting aircraft motion cannot be predicted by analyzing these effects separately. The complicated interrelationship of these parameters can best be seen by analyzing the aircraft equations of motion.

Roll $$\dot{p} = \frac{\sum L}{I_x} - qr\left(\frac{I_z - I_y}{I_x}\right) + (\dot{r} + qp)\frac{I_{xz}}{I_x}$$

Pitch $$\dot{q} = \frac{\sum M}{I_y} - pr\left(\frac{I_x - I_z}{I_y}\right) - (p^2 - r^2)\frac{I_{xz}}{I_y} \qquad (8.2)$$

Yaw $$\dot{r} = \frac{\sum N}{I_z} - pq\left(\frac{I_y - I_x}{I_z}\right) - (qr - \dot{p})\frac{I_{xz}}{I_z} \qquad (8.3)$$

Drag $$\dot{u} = \frac{\sum F_x}{m} - qw + rv \qquad (8.4)$$

Lift $$\dot{w} = \frac{\sum F_z}{m} - pv + qu \qquad (8.5)$$

Side $$\dot{v} = \frac{\sum F_y}{m} - ru + pw \qquad (8.6)$$

Consider equations 8.1 - 8.3. In each case, the first term on the right hand side of the equations represents the aerodynamic contribution. the second term the inertial contribution, and the third term the I_{xz} parameter. It can be seen that the relationships involved could never occur singularly and that they actually occur in conjunction with one another to either become additive and aggravating or opposing and thus alleviate the tendency to diverge.

"Divergence" in roll coupling is manifested by a departure from the intended flight path that will result in either loss of control or structural failure. As defined, this "divergence" is what we are concerned with in roll coupling. Smaller roll coupling effects that do not result in divergence will not be considered. It should be noted that divergence about any one axis will be closely followed by divergence about the others.

In attempting to explain the subject of roll coupling, this chapter will first explain the physical aspects of inertia coupling, aerodynamic coupling, and the I_{xz} parameter. Next, a mathematical model for roll coupling will be developed that will permit determination of the approximate roll rate that will drive an aircraft to divergence.

8.2 INERTIAL COUPLING

The problem of inertial coupling did not manifest itself until the introduction of the century series aircraft. As the modern fighter plane evolved from the conventional fighter, such as the F-51 and F-47, through the first jet fighter, the F-80, and then to the F-100 and other century series aircraft, there was a slow but steady change in the weight distribution. During this evolution, more and more weight became concentrated in the fuselage as the aircraft's wings became thinner and shorter. This shift of weight caused relations between the moments of inertia to change. As more weight is concentrated along the longitudinal axis, the moment of inertia about the x-axis decreases relative to the moments of inertia about the y and z axes. This phenomena increases the coupling between the lateral and longitudinal equations. This can be seen by examining equation 8.2.

$$\dot{q} = \frac{\sum M}{I_y} - pr\left(\frac{I_x - I_z}{I_y}\right) - (p^2 - r^2)\frac{I_{xz}}{I_y} \tag{8.2}$$

As I_x becomes much smaller than I_z, the moment of inertia difference term $(I_x - I_z)/I_y$ becomes large. If a rolling moment is introduced, the term $pr\,(I_x - I_z)/I_y$ may become large enough to cause an uncontrollable pitching moment.

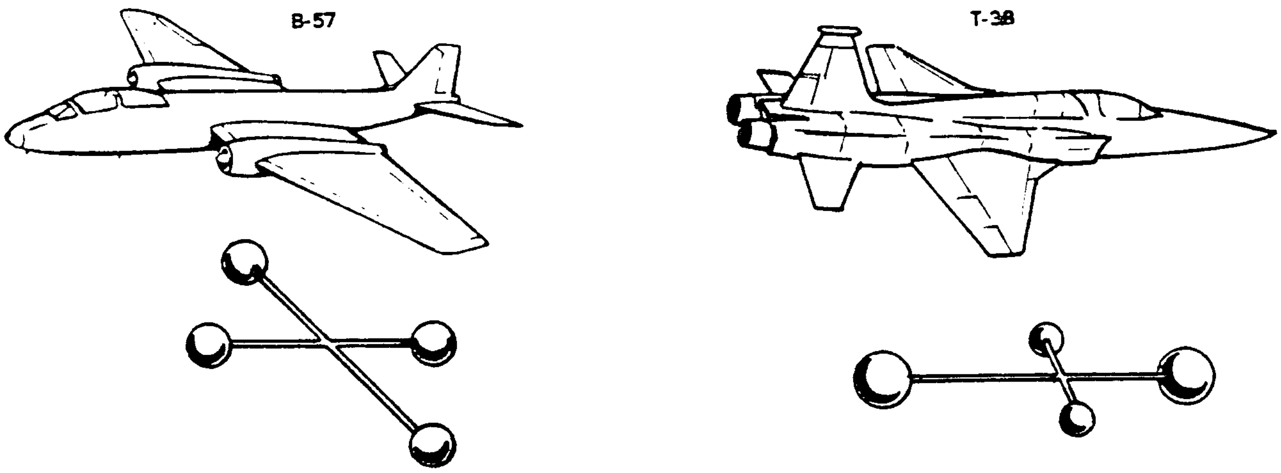
Modern fighter design is characterized by a long slender, high-density fuselage with short, thin wings. This results in a roll inertia which is quite small in comparison to the pitch and yaw inertia. The more conventional low speed airplane may have a wingspan greater than the fuselage length, and a great deal of weight concentrated in the wings. A comparison of these configurations is presented in figure 8.1.

The mass distribution of these aircraft can be represented by a pair of dumbells. The axis with the larger dumbells will tend to align itself with the plane that is perpendicular to the roll axis. Therefore, inertia coupling for the B-57 is opposite that of the T-38. A roll will tend to decrease angle of attack for the B-57 and increase it for the T-38.

It can be seen that the conventional design presents considerable resistance to rotation about the x-axis. Thus, with this design, one would not expect high roll rates.

Figure 8.1

CONVENTIONAL AND MODERN AIRCRAFT DESIGNS

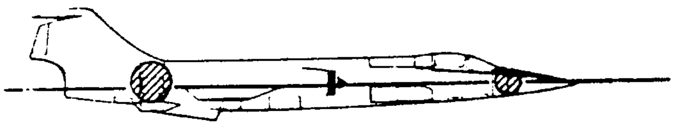


On the other hand, it can be seen that the modern design presents a relatively small resistance to rotation about the x-axis. Thus, with this design, one could expect to attain high rates of roll. It has been shown that high roll rates enhance the tendency toward inertial coupling.

This analysis of inertia coupling will consider rolls about two different axes: The inertia axis, and the aerodynamic axis. The inertia axis is formed by a line connecting the aircraft's two "centers of inertia." Refer to figure 8.2.

Figure 8.2

AIRCRAFT INERTIA AXIS

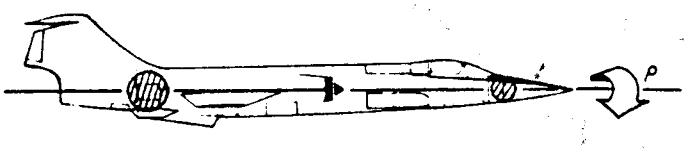


The aerodynamic axis is simply the stability x-axis first introduced in the investigation of the left hand side of the equations of motion. It is merely the line of the relative wind. Aircraft rotation in a roll is generally assumed to be about this axis. To visualize this, recall that to produce a rolling moment, a differential in lift must be created on the wings. For the time being let us assume that the aircraft will roll about the relative wind, or aerodynamic axis.

First, consider a roll when the aerodynamic and inertia axes are coincident. Figure 8.3.

Figure 8.3

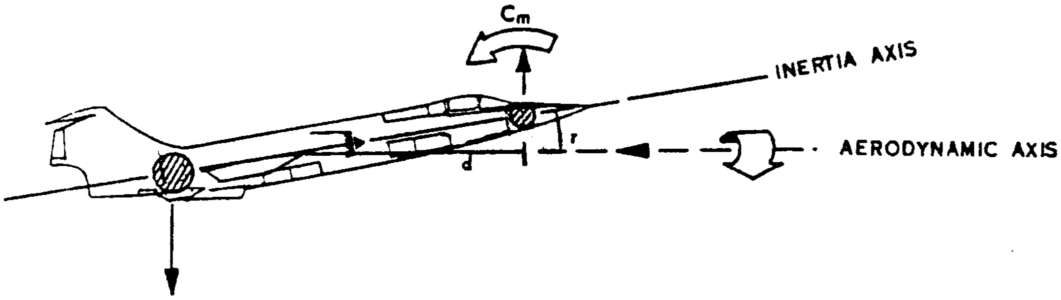
AERODYNAMIC AND INERTIA AXIS COINCIDENT



In this case, there is no force created by the centers of inertia that will cause the aircraft to be diverted from its intended flight path, and no inertia coupling results. Now, observe what happens when the inertia axis is displaced from the aerodynamic axis.

Figure 8.4

AERODYNAMIC AND INERTIA AXIS NON-COINCIDENT



As the aircraft is rotated about the aerodynamic axis, centrifugal force will act on the centers of inertia. Remembering that centrifugal force acts perpendicular to the axis of rotation, it can be seen that a moment will be created by this centrifugal force. For the case depicted in figure 8.4, where the aerodynamic axis is depressed below the inertia axis, a pitch up will result. Conversely, if the aerodynamic axis is above the inertia axis, a pitch down will result.

To appreciate the magnitude of the moment thus developed, refer to figure 8.4 and consider the following:

$$\text{Centrifugal Force} = \frac{mV_{\text{Tangential}}^2}{r} \qquad (8.7)$$

$$V_{\text{Tangential}} = r\omega = rp \qquad (8.8)$$

Therefore,

$$C.F. = mrp^2$$

the moment created by this centrifugal force is

$$M = (C.F.)(d) = mrp^2d \qquad (8.9)$$

For modern designs, m is large. Also, r will be larger for a low aspect ratio wing. (The aircraft will operate at a higher angle of attack.) As previously discussed, p will be large. Also, for long, dense fuselages, d will be large. Thus, the moment created by inertia coupling will be large.

8.3 THE I_{xz} PARAMETER

Three products of inertia I_{xy}, I_{yz} and I_{xz} appear in the equations of motion for a rigid aircraft. By virtue of symmetry, I_{xy} and I_{yz} are both equal to zero. However, the product of inertia I_{xz} can be of an appreciable magnitude and can have a significant effect on the roll characteristics of an aircraft.

The parameter, I_{xz}, can be thought of as a measure of the uniformity of the mass distribution about the x-axis. The axis about which I_{xz} is equal to zero is defined as the principle inertia axis, and the mass of the aircraft can be considered to be concentrated on this axis.

If the I_{xz} parameter is not equal to zero, then the principle inertia axis is not aligned with the aircraft x-axis. A typical modern aircraft design can be represented by two centers of mass in the xz plane designated m_1 and m_2 in figure 8.5.

It can be seen that if the aircraft is rolled about the x-axis, a pitch down will result. This phenomena is produced by exactly the same centrifugal force effects that produced inertial coupling. However, it should be noted that in this case the x-axis is aligned with the aerodynamic axis and that the pitching moment is a result of the inclination of the principle inertia axis. Thus, when an aircraft is rolled about any axis which differs from its principle inertia axis, pitching moments develcp which tend to cause the aircraft to depart from its intended flight path. Depending on its orientation, the I_{xz} parameter can either aggravate or oppose inertial coupling.

To appreciate the magnitude of this parameter, consider figure 8.5.

From equation 8.9, the moment produced by the forward center of mass is,

$$M_1 = (C.F.)(x_1) = m_1 x_1 p^2 z_1 \quad (8.10)$$

Similarly, the moment produced by the aft center of mass is,

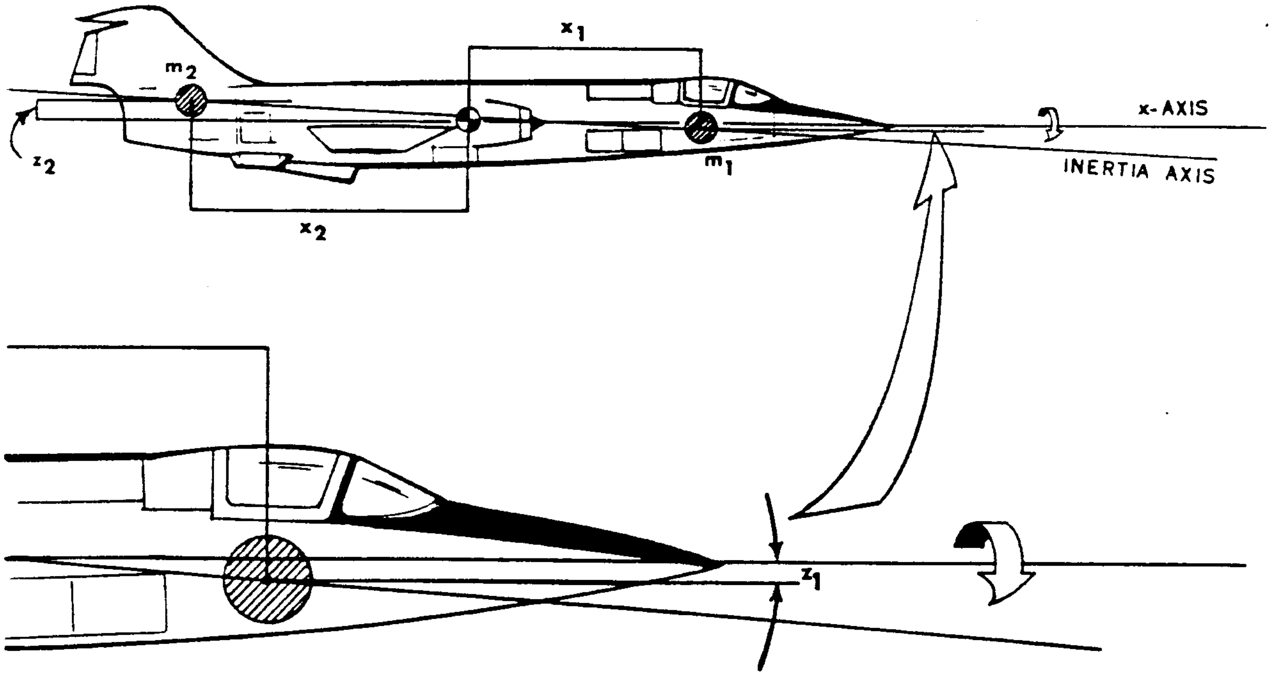
$$M_2 = m_2 x_2 p^2 z_2 \quad (8.11)$$

The total pitch moment is therefore,

$$M_T = m_1 + m_2 = m_1 x_1 p^2 z_1 + m_2 x_2 p^2 z_2 \quad (8.12)$$

$$M_T = p^2 (m_1 x_1 z_1 + m_2 x_2 z_2) \quad (8.13)$$

Figure 8.5
PRINCIPLE INERTIA AXIS BELOW AERODYNAMIC AXIS



But for a simplified system,

$$I_{xz} = m_1 x_1 z_1 + m_2 x_2 z_2 \qquad (8.14)$$

Therefore,

$$M_T = p^2 I_{xz} \qquad (8.15)$$

Thus, it can be seen that the magnitude of the pitching moment thus developed depends on the roll rate and the magnitude of the I_{xz} parameter relative to the roll axis.

8.4 AERODYNAMIC COUPLING

This analysis of roll coupling is not concerned with all aerodynamic coupling terms (C_{n_p}, $C_{n_{\delta_a}}$, C_{ℓ_r}, $C_{\ell_{\delta_r}}$, etc.). Only the "kinematic coupling" aspects of aerodynamic coupling will be considered.

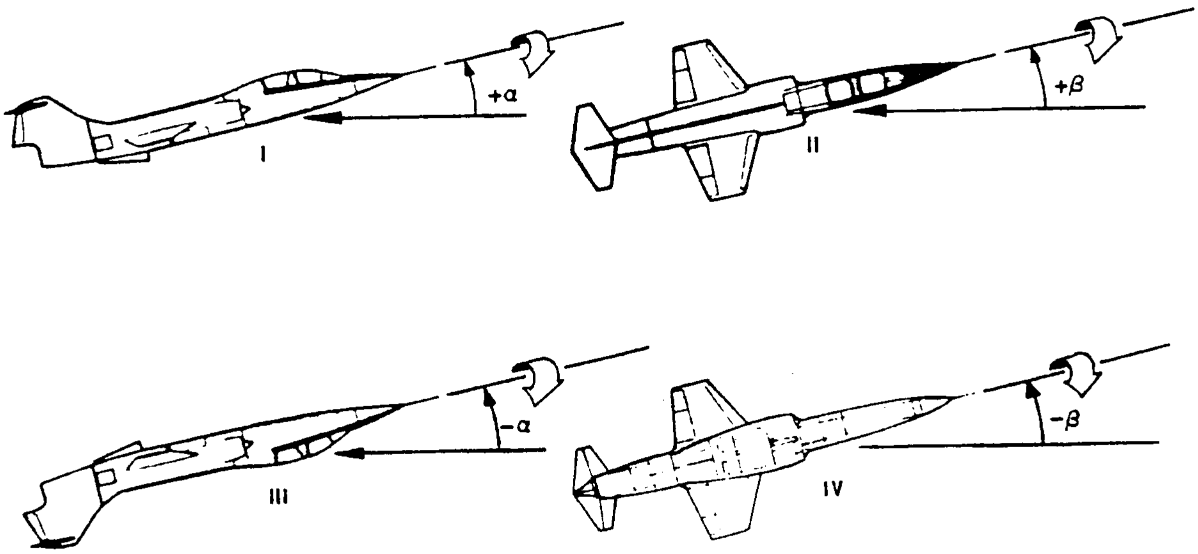
Kinematic coupling may be considered as an actual interchange of α and β during a rolling maneuver. This interchange is an important means by which the longitudinal and lateral motions are capable of influencing each other during a rapid roll.

To understand how this interchange of α for β occurs, consider figure 8.6.

In this figure the aircraft is assumed to have either infinitely large inertia or negligible stability. Thus, it will roll about its principle inertia axis. In (I) the aircraft initiates a roll from a positive angle of attack. In (II) the initial angle of attack is converted to a positive sideslip angle of equal magnitude after 90° of roll. In (III) the aircraft has again exchanged β and α and after 180° of roll has an angle of attack

Figure 8.6

KINEMATIC COUPLING. ROLLING OF AN AIRCRAFT WITH INFINITELY LARGE INERTIA OR NEGLIGIBLE STABILITY IN PITCH AND YAW



equal in magnitude but opposite in sign to the original α. The interchange continues and in (IV) thus −α is converted to −β.

Next, consider an aircraft with infinitely large stability in pitch and yaw or negligible inertia. Refer to figure 8.7.

In this case, the aircraft will roll about its aerodynamic axis, and no interchange of α or β will occur.

Since aircraft do not have infinitely large inertia or stability neither of these extremes can occur. Some combination of these effects will always result during a roll. The amount of kinematic coupling will depend upon the relative values of C_{n_β} and C_{m_α} and roll rates. This can be shown with two emperical relationships:

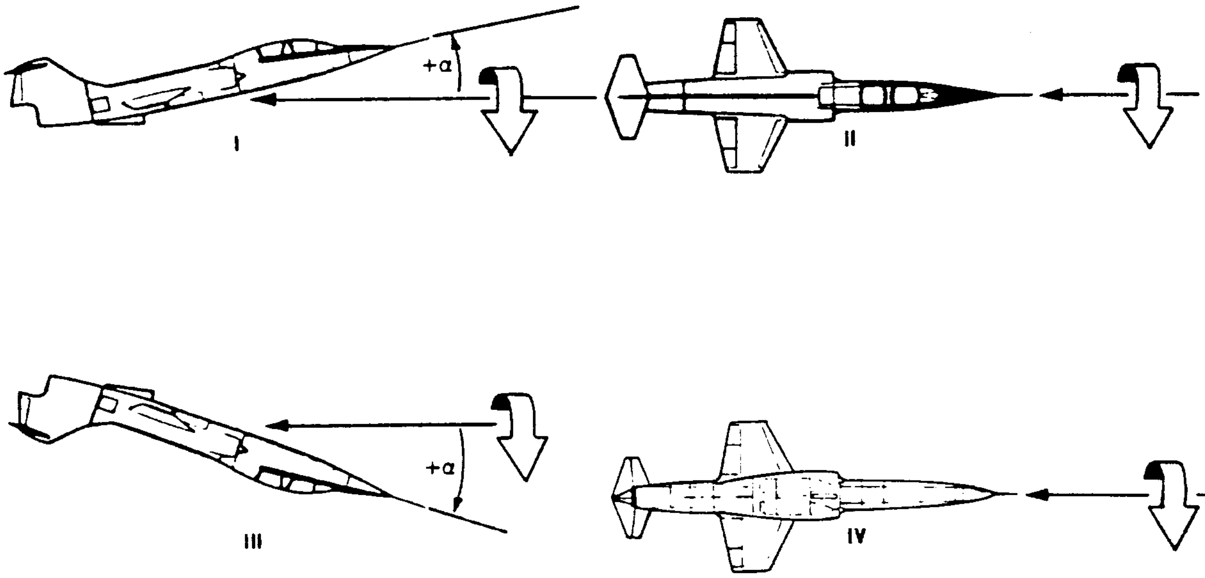
$$\dot{\alpha} = -Kp\beta \qquad (8.16)$$

$$\dot{\beta} = Kp\alpha \qquad (8.17)$$

These relationships show that any roll rate will cause an interchange of α and β. The exact amount depends on the magnitude of K which is determined by the relative values of the moments of inertia and C_{m_α} and C_{n_β}. It can also be seen that for a given aircraft, the rate of interchange of α and β depends on the roll rate. The higher the roll rate, the greater the kinematic coupling. As roll rate increases a point is reached where the stability of the aircraft is insufficient to counter the α and β build up. This divergence could ultimately result in departure from controlled flight. This point is of special interest to designers and is often the subject of an indepth mathematical analysis. Although little can be determined from equations 8.16 and 8.17, they provide a basis for showing how an aircraft's dynamic response can be used to make some rough predictions about the kinematic coupling characteristics. It has been shown in dynamics that the natural frequency of the short period mode is a function of C_{m_α}. Likewise ω_n of the Dutch Roll mode is a function of C_{n_β}.

Figure 8.7

KINEMATIC COUPLING. ROLLING OF AN AIRCRAFT WITH INFINITELY LARGE STABILITY OR NEGLIGIBLE INERTIA IN PITCH AND YAW



Assume that an aircraft is rolled at a rate that creates a disturbance in β at a rate equal to the maximum rate that the natural aircraft stability can damp out the disturbance. Thus,

$$\dot{\beta} = Kp\alpha = f\left[\omega_{n(\text{Dutch Roll})}\right] \quad (8.18)$$

In this case there would be no buildup of β, and a condition of neutral stability in yaw would result. However, if the roll rate were increased slightly above this value then successively larger increases in β would occur and divergence would result. This analysis can also be followed through for an initial disturbance in α. It is not important which diverges first, α or β, since any divergence about one axis will quickly drive the other divergent. As a matter of interest however, supersonically C_{n_β} decreases more rapidly than C_{m_α} and therefore, most modern aircraft will diverge in yaw first.

It can be shown on an analog computer that when $C_{m_\alpha} = C_{n_\beta}$ a stable condition will exist at all roll rates. This is often referred to as a "tuned condition," and is a possible dodge for an aircraft designer to utilize in a critical flight area. However, it is difficult to capitalize on this occurrence because of the wide variation of the stability derivatives with Mach number.

It may be that an aircraft will possess stability parameters such that a roll coupling problem exists at a given roll rate. However, if a relatively long time is required before large values of α and β are generated, then the aircraft may be rolled at the maximum value by restricting the aircraft to one 360 degree roll. In this situation, the aircraft is diverging during the roll, but at such a slow rate that by the time the aircraft has rolled 360 degrees, the maximum allowable α or β of the aircraft has not been exceeded.

8.5 AUTOROTATIONAL ROLLING

It has been shown that during rolling maneuvers large angles of attack and sideslip may occur as a result of inertial and kinematic coupling. For some aircraft certain conditions of α and β will produce a rolling moment that is in the same direction of the roll. If this moment is equal or greater than the moment created by roll damping the airplane will continue an uncommanded roll. In some cases, it may not be possible to stop the aircraft from rolling, although the lateral control is held against the roll direction. This is known as autorotational rolling or "auto-roll." There are various conditions that can cause auto roll. It can occur at positive or negative angle of attack with any combination of sideslip angle. It is highly dependent on aerodynamic design, however flight control and stability augmentation systems can also have a large effect. Auto-roll is normally caused by the development of sideslip due to kinematic or inertial coupling and the effect of C_{ℓ_β} once this sideslip has developed. On some aircraft with highly augmented flight control systems an auto-roll may result from control inputs commaned by the system itself.

A good example of auto-roll occurs in the F-104 at negative angles of attack. For analysis sake let us assume the aircraft is rolled to the right. In this case the negative α is converted into negative β (refer to Figure 8.6, III and IV). The vertical stabilizer for the F-104 is highly effective, therefore the -β develops a significant rolling moment to the right which reinforces the rolling motion. Since the F-104 is a fuselage loaded aircraft, the rolling motion causes the airplane to pitch down. This increases the -α and further complicates the problem. If allowed to continue this motion it could diverge until the aircraft departs from controlled flight. If an auto-roll of this type were to begin, the pilot should pull back on the stick to make α positive. With +α, kinematic coupling will tend to decrease the roll rate.

There are conditions in which auto-roll will occur at positive α's. For example, a combination of a negative C_{n_β}

with negative C_{ℓ_β} can cause an auto-roll. The negative Cn_β causes the aircraft to diverge in yaw, the ensuing Beta buildup causes the aircraft to roll in the same direction as yaw. Kinematic coupling will reduce the β and therefore compensates for the yaw. Under certain conditions a stabilized roll could result, however it is more probable that this motion would either diverge to departure or oscillate back and forth in a wing rocking motion.

Although no analysis of the effects of augmented flight control systems (SAS, CAS, etc.) will be presented here, it should be noted that these types of systems are prone to cause auto-roll tendencies. Rate feedbacks are hard to tailor to improve handling qualities throughout the flight regime without adversely affecting roll coupling tendencies somewhere in that regime. It is up to the flight test pilot and engineer to accurately predict where problems may exist and thoroughly investigate these areas.

8.6 A MATHEMATICAL ANALYSIS OF ROLL DIVERGENCE

The roll coupling characteristics of high performance modern fighters are thoroughly investigated by computer simulation prior to flight testing. However, smaller test programs may not have this capability. It is the intent of this section to provide the test pilot and engineer with a practical mathematical tool to aid in determining the roll rate at which an aircraft will start to diverge due to roll coupling. If the critical roll rate thus determined is attainable in the aircraft, then the test pilot should avoid higher rates of roll until a complete analysis can be conducted. Thus, this mathematic tool will enable the test pilot to identify potentially hazardous areas.

This mathematical analysis is based on certain simplifying assumptions. They are:

1. Velocity remains constant during the roll maneuver, $\dot{u} = 0$.
2. Roll rate is constant, $\dot{p} = 0$.
3. v, w, q, r are small therefore their products are negligible.
4. Engine gyroscopic effects are negligible.
5. The rudder and elevator are fixed in their initial trim position.
6. Aerodynamic parameters are negligible with the exception of M_α, M_q, N_β, N_r.

When these assumptions are applied to the equations of aircraft motion, the following results are obtained:

$$\frac{\sum F_x}{m} = \cancel{\dot{u}} + \cancel{qw} - \cancel{rv} = 0 \qquad (8.4)$$

$$\frac{\sum L}{I_x} = \cancel{\dot{p}} + \cancel{qr}\left(\frac{I_z - I_y}{I_x}\right) - (\dot{r} + qp)\frac{I_{xz}}{I_x} \qquad (8.1)$$

It can be shown that:

$$(\dot{r} + qp) = -\ddot{\theta}\,\text{SIN}\,\phi \approx 0 \qquad (8.19)$$

Therefore, in light of the assumptions made, equation 8.1 will be approximately equal to zero.

$$\frac{\sum M}{I_y} = \dot{q} + pr\left(\frac{I_x - I_z}{I_y}\right) + (p^2 - \cancel{r^2})\frac{I_{xz}}{I_y} \qquad (8.2)$$

Thus,

$$M_\alpha \cdot \alpha + M_q \cdot q = \dot{q} + pr\left(\frac{I_x - I_z}{I_y}\right) + p^2 \frac{I_{xz}}{I_y}$$

(8.20)

$$\frac{\sum N}{I_z} = \dot{r} + pq\left(\frac{I_y - I_x}{I_z}\right) + \cancel{(qr - \dot{p})} \frac{I_{xz}}{I_z}$$

(8.3)

Thus,

$$N_\beta \cdot \beta + N_r \cdot r = \dot{r} + pq\left(\frac{I_y - I_x}{I_z}\right)$$

(8.21)

The lift and side force equations will average zero throughout a roll.

$$\frac{\sum F_z}{m} = \dot{w} + pv - qu = 0 \qquad (8.5)$$

$$\frac{\sum F_y}{m} = \dot{v} + ru - pw = 0 \qquad (8.6)$$

To get equations 8.5 and 8.6 into a more suitable form, recall that for small α and β,

$$\alpha = \frac{w}{u} \qquad (8.22)$$

$$\beta = \frac{v}{u} \qquad (8.23)$$

Assuming $\dot{u} = 0$

$$\dot{\alpha} = \frac{\dot{w}}{u} \qquad (8.24)$$

$$\dot{\beta} = \frac{\dot{v}}{u} \qquad (8.25)$$

Thus, equation 8.5 becomes,

$$\frac{\dot{w}}{u} + p\frac{v}{u} - q = \dot{\alpha} + p\beta - q = 0 \qquad (8.26)$$

Equation 8.6 becomes,

$$\frac{\dot{v}}{u} + r - p\frac{w}{u} = \dot{\beta} + r - p\alpha = 0 \qquad (8.27)$$

To further streamline equations 8.20 and 8.21, the following substitutions are made,

$$A = \frac{I_y - I_x}{I_z}$$

$$B = \frac{I_z - I_x}{I_y}$$

To recap, simplifying assumptions have reduced the aircraft equations of motion to the following for rolling maneuvers:

$$\dot{\alpha} + p\beta - q = 0 \qquad (8.28)$$

$$\dot{\beta} + r - p\alpha = 0 \qquad (8.29)$$

$$M_\alpha \cdot \alpha + M_q \cdot q - \dot{q} + prB = p^2 \frac{I_{xz}}{I_y} \qquad (8.30)$$

$$N_\beta \cdot \beta + N_r \cdot r - \dot{r} - pqA = 0 \quad (8.31)$$

These equations can be recognized as four linear differential equations expressed in terms of the variables α, β, q, r. To examine the stability of these equations, it is necessary to find the transient solution. To do this, first express the equations in Laplace notation:

$$s\alpha + p\beta - q = 0 \quad (8.32)$$

$$s\beta + r - p\alpha = 0 \quad (8.33)$$

$$M_\alpha \cdot \alpha + (M_q - s)\, q + prB = 0 \quad (8.34)$$

$$N_\beta \cdot \beta + (N_r - s)\, r - pqA = 0 \quad (8.35)$$

The characteristic equation of this set of simultaneous, non-homogeneous, differential equations is identical to the determinate of the coefficients. The expanded determinate yields:

$$\begin{aligned} & s^4 \quad (8.36) \\ & + (-M_q - N_r)\, s^3 \\ & + (-M_\alpha + M_q N_r + N_\beta + p^2 + p^2 AB)\, s^2 \\ & + (M_\alpha N_r - M_q N_\beta - M_q p^2 - N_r p^2)\, s \\ & + M_\alpha p^2 A + M_q N_r p^2 - N_\beta B p^2 + ABp^4 - M_\alpha N_\beta \\ & = 0 \end{aligned}$$

For convenience, the following substitutions will be made:

$$a_3 = -M_q - N_r \quad (8.37)$$

$$a_2 = -M_\alpha + M_q N_r + N_\beta + p^2 (1 + AB) \quad (8.38)$$

$$a_1 = M_\alpha N_r - M_q N_\beta - p^2 (M_q + N_r) \quad (8.39)$$

$$a_o = p^2 (M_\alpha A + M_q N_r - N_\beta B + ABp^2) - M_\alpha N_\beta \quad (8.40)$$

Thus, the transient solution becomes

$$s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_o = 0 \quad (8.41)$$

If any roots of equation 8.41 have positive real parts, then the motion will be unstable and the aircraft will diverge. In order that there be no roots with positive real parts, it is necessary but not sufficient that:

1. All coefficients have the same sign.
2. None of the coefficients vanish.

If both of these conditions are met, then the equation must be examined further.

The coefficients of the characteristic equation will be examined in light of the following assumptions:

1. The aircraft possesses positive static stability in pitch and yaw, thus M_α = (-), N_β = (+).
2. The aircraft possesses positive damping in pitch and yaw, thus, M_q = (-), N_r = (-).
3. The aircraft mass distribution is such that $I_z > I_x$ and $I_y > I_x$, thus A = (+), B = (+). (This mass distribution is typical of a modern fuselage loaded aircraft.)

In view of these assumptions, it can be seen that the coefficients a_1, a_2, a_3 will be positive. However, a_o may be either positive or negative. If a_o is examined and found to be negative, the resulting rolling maneuver will be unstable.

A simple plot can be generated to aid in determining if a_o is positive for all roll rates. From equation 8.40, for a_o to be positive it is necessary that

$$p^2 \left[M_\alpha A + M_q N_r - N_\beta B + p^2 A B \right] - N_\beta M_\alpha > 0 \qquad (8.42)$$

This equation may be factored to:

$$(-M_\alpha - p^2 B)(N_\beta - p^2 A) + N_r M_q p^2 > 0 \qquad (8.43)$$

by further dividing by p^4 and transposing

$$\left(-\frac{M_\alpha}{p^2} - B \right) \left(\frac{N_\beta}{p^2} - A \right) > \frac{-N_r M_q}{p^2} \qquad (8.44)$$

which falls in the form of a rectangular hyperbo[la]

$$(y - B) \quad (x - A) = C$$

A plot of the hyperbola shows the regions of stability and instability (Figure 8.8).

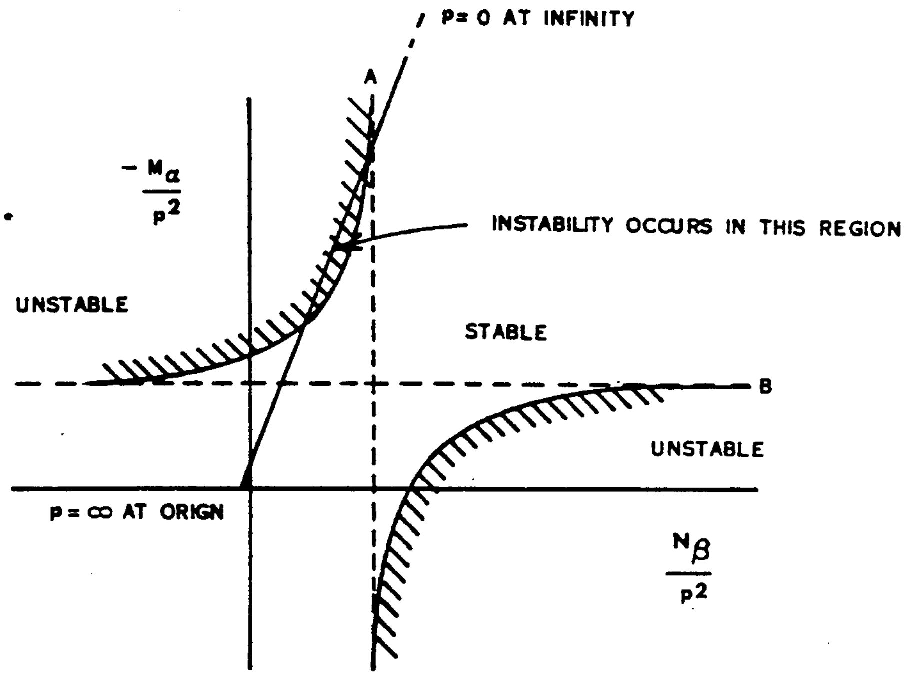
By assuming that M_α and N_β do not vary as a function of roll rate, a straight line can be plotted that represents aircraft roll coupling stability for all roll rates. The slope of the line is

$$m = \frac{\frac{-M_\alpha}{p^2}}{\frac{N_\beta}{p^2}} = \frac{-M_\alpha}{N_\beta}$$

It can be seen that the line intercepts the origin where $p = \infty$ and extends to infinity at $p = 0$. If this line crosses over into the region of instability at any point, the roll rate at the crossover point can be determined. If the aircraft were capable of attaining this roll rate a roll coupling problem would be suspected.

If a_o is found to be positive for all roll rates, the analysis is still not complete. This condition is necessary but not sufficient. To be complete a further analysis can be performed using the Routh Hurwitz Stability Criterion.

Figure 8.8



A brief description of the mechanics involved in the Routh-Horwitz method follows:

The first step is to arrange the polynomial coefficients into two rows: The first row will consist of the first, third, fifth coefficients, etc., and the second row will consist of the second, fourth, sixth coefficients, etc. The following example is presented:

a_o	a_2	a_4	a_6	a_8
a_1	a_3	a_5	a_7	a_9

The next step is to form the following array of numbers obtained by the indicated operations. The example shown is for a sixth-order system.

$$F(s) = a_o s^6 + a_1 s^5 + a_2 s^4 + a_3 s^3 + a_4 s^2 + a_5 s^1 + a_6$$

a_o	a_2	a_4	a_6
a_1	a_3	a_5	0
$\frac{a_1 a_2 - a_3 a_o}{a_1} = A$	$\frac{a_1 a_4 - a_o a_5}{a_1} = B$	$\frac{a_1 a_6 - a_o \times 0}{a_1} = a_6$	0
$\frac{A a_3 - a_1 B}{A} = C$	$\frac{A a_5 - a_1 a_6}{A} = D$	$\frac{A \times 0 - a_1 \times 0}{A} = 0$	0
$\frac{CB - AD}{C} = E$	$\frac{C a_6 - A \times 0}{C} = a_6$	$\frac{C \times 0 - A \times 0}{C} = 0$	0
$\frac{ED - C a_6}{E} = F$	0	0	0
$\frac{F a_6 - E \times 0}{F} = a_6$	0	0	0

The last step is to investigate the signs of the numbers in the first column in this array. If all of the elements in the first column are positive, the system is stable. If one or more of the elements is negative, the system is unstable.

When the Routh-Hurwitz Criterion is applied to equation 8.41, the following equations result. They must both be positive if the system is to be stable.

$$a_3 a_2 - a_1 > 0 \qquad (8.45)$$

$$a_1 a_2 a_3 - a_o a_3^2 - a_1^2 > 0 \qquad (8.46)$$

In summary, to determine if a given roll rate will result in a stable aircraft motion:

1. Determine the value of a_o (equation 8.40). If it is negative the system is unstable.

2. Determine the values of a_1, a_2, a_3 (equations 37 - 39).

3. Solve equation 8.45. If negative the system is unstable.

4. Solve equation 8.46. If negative the system is unstable.

5. If steps 1-4 yield no negative results, the system is stable for the roll rate investigated.

Although somewhat unwieldly, in the absence of adequate analog simulation, the foregoing system will yield fairly accurate results. It can serve to warn the test pilot or engineer of a perilous situation.

8.7 CONCLUSIONS

As an aircraft's inertias are disproportionately increased in relation to its aerodynamic stabilities in pitch and yaw, the aircraft will be liable to pitching and yawing motions during rolling maneuvers. The more typical case is a divergence in yaw by virtue of an inadequate value of $C_{n\beta}$.

The peak loads resulting from roll coupling generally increase in proportion to the initial incidence of the principle inertial axis and progressively with the duration of the roll and the rapidity of aileron application at the beginning and the end of the maneuver. The most severe cases naturally should be expected in a flight regime of low $C_{n\beta}$ and high dynamic pressures.

The rolling pull-out maneuver in a high performance aircraft is especially dangerous. It combines many unfavorable features: High speed hence high roll rate capability; high acceleration which favors poor coordination and inadvertent excitation of transients by the pilot; and high dynamic pressures which at large values of α and β may break the aircraft.

Most high performance aircraft incorporate roll rate limiters in addition to angular damping augmenters. In these aircraft a lateral control with enough power for low speed is almost certain to be too powerful for high speeds. Fortunately, limiters of various kinds are not too difficult to incorporate in a fully powered control system.

It is obvious that flight testing in suspected regions of roll coupling warrant a cautious methodical approach and must be accompanied by thorough computer studies that stay current with the flight test data. The only way that the pilot can discover the exact critical roll limit in flight is when he exceeds it, which is obviously not the approach to take. Because of this, flight tests are generally discontinued when computer studies indicate that the next data point may be "over the line."

The following example is cited. The Bell X-2 rocket ship in 1956 was launched from its mother ship at Edwards. The pilot flew a perfect profile but the rocket engine burned a few critical seconds longer than the engineers predicted, resulting in a greater speed (Mach 3.2) and greater altitude (119,800 feet) than planned. Unknown to the pilot, he was progressively running out of directional stability. When he was over the point at which he had preplanned to start his turn toward Roger's Dry Lake he actuated his controls. The X-2 went divergent with a resultant loss of control. The accident investigation revealed the cause to be a greater loss in directional stability than planned, resulting in divergent roll coupling.

A combination of reasonable piloting restrictions coupled with increased directional stability has provided the solution to roll coupling problems in the present generation of aircraft. The problem is one of understanding since a thinking pilot would no more exceed the roll limitations imposed on an aircraft than he would the structural "G" limitations.

Besides pilot education, some other schemes to eliminate roll coupling divergencies are:

1. Roll rate limiters.
2. Angular damping augmenters.
3. Placarded roll limits such as
    a. "G" limits.
    b. Total allowable roll at maximum rate.
    c. Altitude limits.
    d. Mach limits.
    e. Flap position limits.

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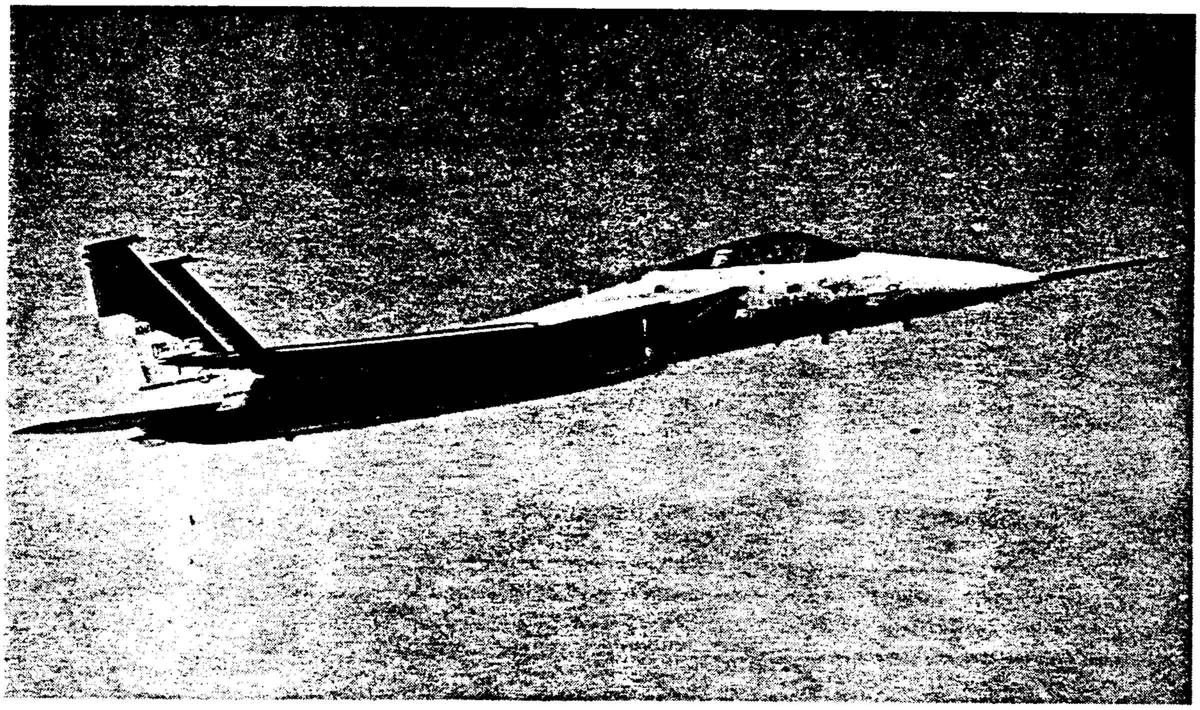
STABILITY AND CONTROL FLIGHT TEST TECHNIQUES

VOLUME II OF II



USAF TEST PILOT SCHOOL,

EDWARDS AIR FORCE BASE, CALIFORNIA

This handbook was submitted by the USAF Test Pilot School of the Air Force Flight Test Center under Job Order Number SC4000.

This report has been reviewed and cleared for open publication and/or public release by the AFFTC Office of Information in accordance with AFR 190-17 and DODD 5230.9. There is no objection to unlimited distribution of this report to the public at large, or by DDC to the National Technical Information Service.

This handbook has been reviewed and is approved for publication:

WARWICK H. GLASGOW
Colonel, USAF
Commandant, USAF Test Pilot School

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This handbook has been compiled by the instructors of the USAF Test Pilot School for use in the Stability and Control portion of the School's course. Most of the material in Volume I of this handbook has been extracted from several reference books and is oriented towards the test pilot. The flight test techniques and data reduction methods in Volume II have been developed at the Air Force Flight Test Center, Edwards Air Force Base, California.

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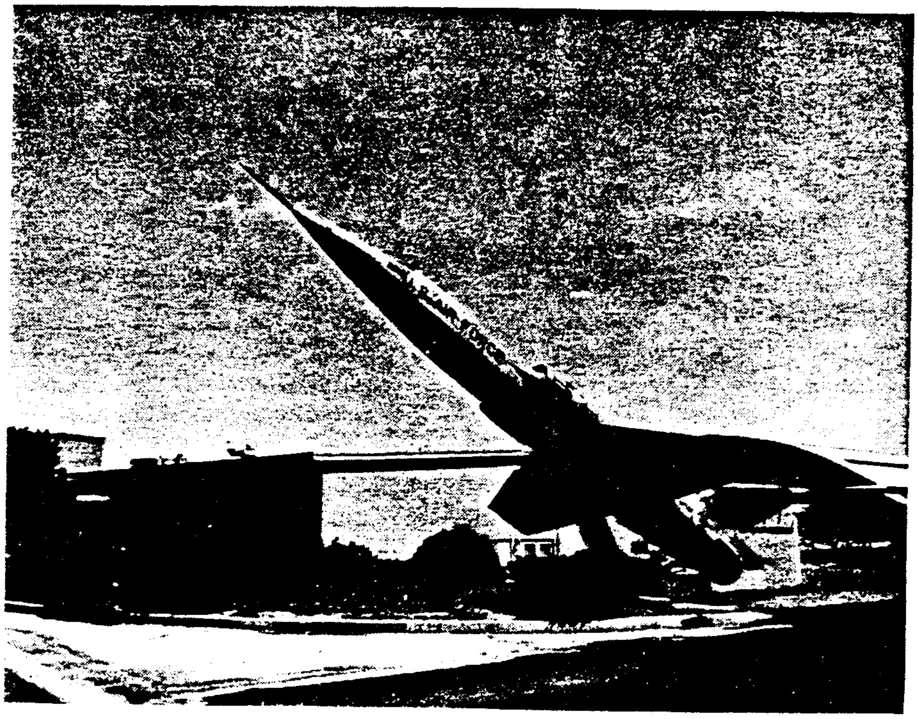

PREFACE

Stability and Control is that branch of the aeronautical sciences that is concerned with giving the pilot an aircraft with good handling qualities. As aircraft have been designed to meet greater performance specifications, new problems in Stability and Control have been encountered. The solving of these problems has advanced the science of Stability and Control to the point it is today.

This handbook has been compiled by the instructors of the USAF Test Pilot School for use in the Stability and Control portion of the School's course. Most of the material in Volume I of this handbook has been extracted from several reference books and is oriented towards the test pilot. The flight test techniques and data reduction methods in Volume II have been developed at the Air Force Flight Test Center, Edwards Air Force Base, California. This handbook is primarily intended to be used as an academic text in our School, but if it can be helpful to anyone in the conduct of Stability and Control testing, be our guest.

WARWICK H. GLASGOW
Colonel, USAF
Commandant, USAF Test Pilot School

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CHAPTER 1

INTRODUCTION TO STABILITY FLIGHT TEST TECHNIQUES

1.1 ATTITUDE FLYING

In stability flight testing, attitude flying is absolutely essential. Under a given set of conditions (altitude, power setting, center of gravity location, etc.) the aircraft's speed is entirely dependent upon the attitude. This being the case, the pilot's ability to fly the aircraft accurately depends upon his ability to see and interpret small attitude changes. This can best be done by reference to the outside horizon. Any change in aircraft attitude will be noticed by reference to the distant horizon long before the aircraft instruments (airspeed, etc.) show a change. Thus, it is often possible to change the attitude of the aircraft from a disturbed position back to the required position before the airspeed has a chance to change. The outside horizon is also very useful as a rate instrument. If a stabilized point is required, hold zero rate of change of pitch; i.e., hold aircraft attitude fixed in relation to some outside reference which calls for one particular speed. If, as in acceleration run, the airspeed is continuously increasing or decreasing, one should look for a steady, smooth, and extremely slow rate of change of the aircraft's attitude.

It is suggested that the method of lining up a particular spot on the aircraft with some outside reference can be useful at times but is often wasteful of time. A general impression is often all that is necessary; i.e., it is possible to see that the aircraft rate of pitch is zero by use of the pilot's peripheral vision while also glancing at the airspeed indicator or some other cockpit instrument. As soon as the pilot notes a rate of change of pitch, he can make proper control movements to correct the attitude of the aircraft. The pilot should always be aware of the outside view even while reading the instruments.

In flight tests involving turning flight, this overall view of the horizon is of utmost importance in order to be able to hold constant velocity or Mach number. If the airspeed is high the nose should be raised and then stabilized at the new position required, using the horizon as a displacement and a rate instrument. If a change of aircraft attitude is necessary, this change should be made relative to the present picture until the rate of change of aircraft attitude again goes to zero at the new stabilized condition.

If it is necessary to stabilize on an airspeed several knots from the existing airspeed, time can be saved by overshooting the required pitch attitude and using the rate of change of airspeed as an indication as to when one should raise or lower the nose to the required position. A little practice will allow the pilot to stabilize on a new airspeed with a minimum amount of airspeed overshoot in the least time.

1.2 TRIM SHOTS

Prior to each stability flight test requiring photopanel or oscillograph data, a trim shot will be taken near the test pressure altitude (±100 feet). The trim shot will be made using the remote camera and oscillograph switch (not the stick trigger) so that no control forces will be inadvertently fed to the system. The trim shot is used primarily to make any necessary corrections to the force-measuring equipment readings. For example if the stick force gage reads +0.1 with no force applied, it is apparent that 0.1 must be subtracted from all stick force readings for this particular test.

The means of obtaining the different force information will be covered in detail in class; however, it should be kept in mind that it is possible to get erroneous rudder force information if the foot is placed improperly on the rudder bar. The strain gages are located on the lower rectangular pad and the foot should be placed centrally on this pad. Care should be taken not to apply any force to the toe pads since force applied here will not register on the force-measuring equipment. When making force measurements using the instrumented stick grip, it is very important that no extraneous force inputs are made by torquing the stick grip. The force measurements should be taken by using straight fore and aft or left and right force inputs on the center of the stick grip.

The importance of proper trim in stability flight testing cannot be overemphasized. Most of the tests involve force information and therefore it is essential that the aircraft be properly trimmed at the desired speed since one is interested in forces necessary to fly in conditions differing from the trim condition.

In order to stabilize and trim at a particular speed or Mach number at a constant altitude, the speed should first be established by placing the aircraft in the required attitude to give this speed. While obtaining this approximate attitude by reference to the outside horizon, the throttle setting should be changed to give zero rate of climb at the proper test altitude. Minute changes in attitude may be necessary in order to hold the exact airspeed as the power is changed. Once the proper attitude and power setting have been established, the force should be trimmed to zero while holding the required control position to give the required attitude. Release the stick and check for a change in pitch attitude. If the nose starts up or down put the nose back at the trim position with the stick and retrim. Then repeat the procedure. The lateral and directional controls (aileron and rudder) should be used in the proper manner to hold the wings level, maintain a constant heading, and keep the ball in the center of the turn and bank indicator. The necessary forces should be held in order to accomplish this and then the forces should be relieved by proper trim actuation. As in all flying, the pilot who can get the aircraft trimmed most accurately and quickly is the pilot who can do the most things simultaneously. For example, the pilot who can make the required attitude correction while adjusting the power will become trimmed before the pilot who flies strictly by the numbers. The often-used method of moving the trim device and allowing the aircraft to seek a new speed "hands off" is very time-consuming and inaccurate. Hold the aircraft attitude fixed and then relieve the existing control forces. If the aircraft gains or loses a little altitude during the trimming process the parameters of interest in stability testing will not have changed significantly. Therefore if the altitude is within

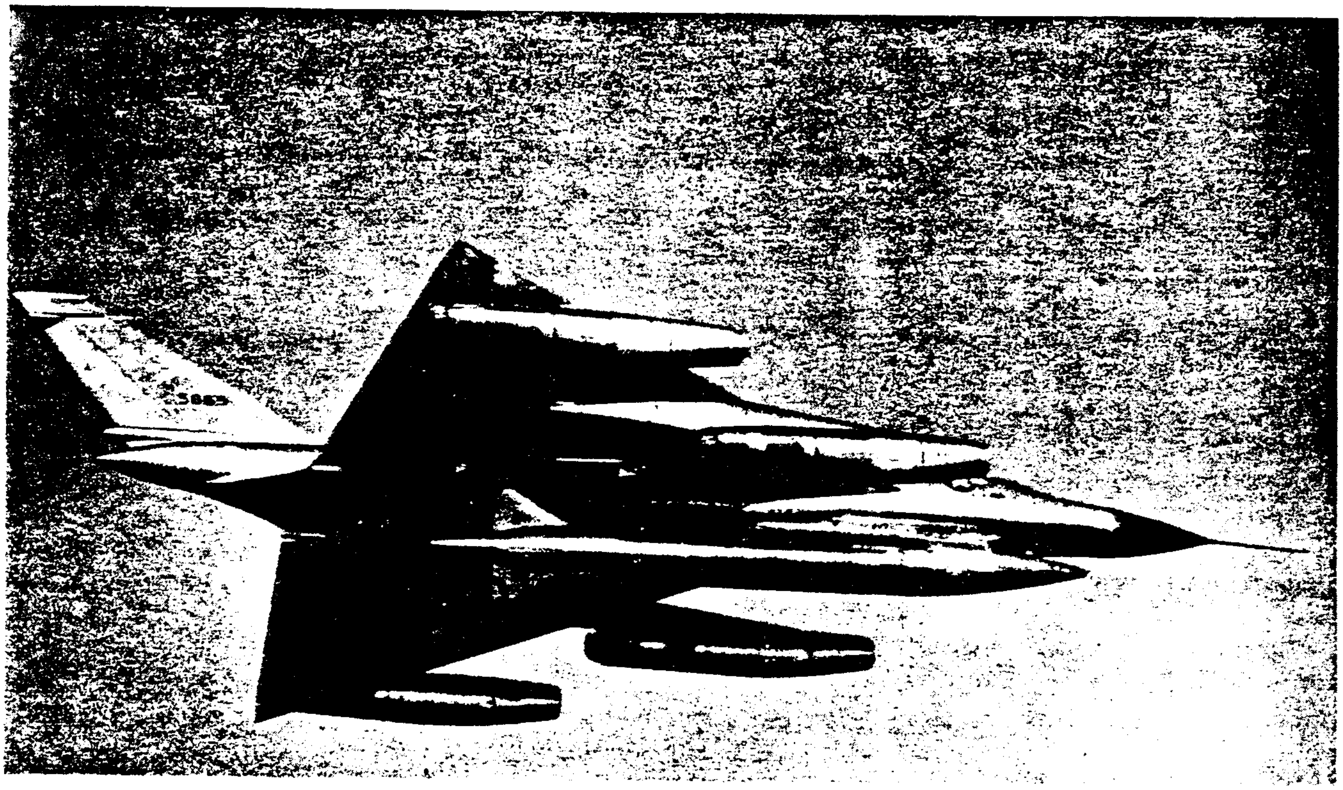
100 feet of the test altitude, the pilot should then take his trim shot.

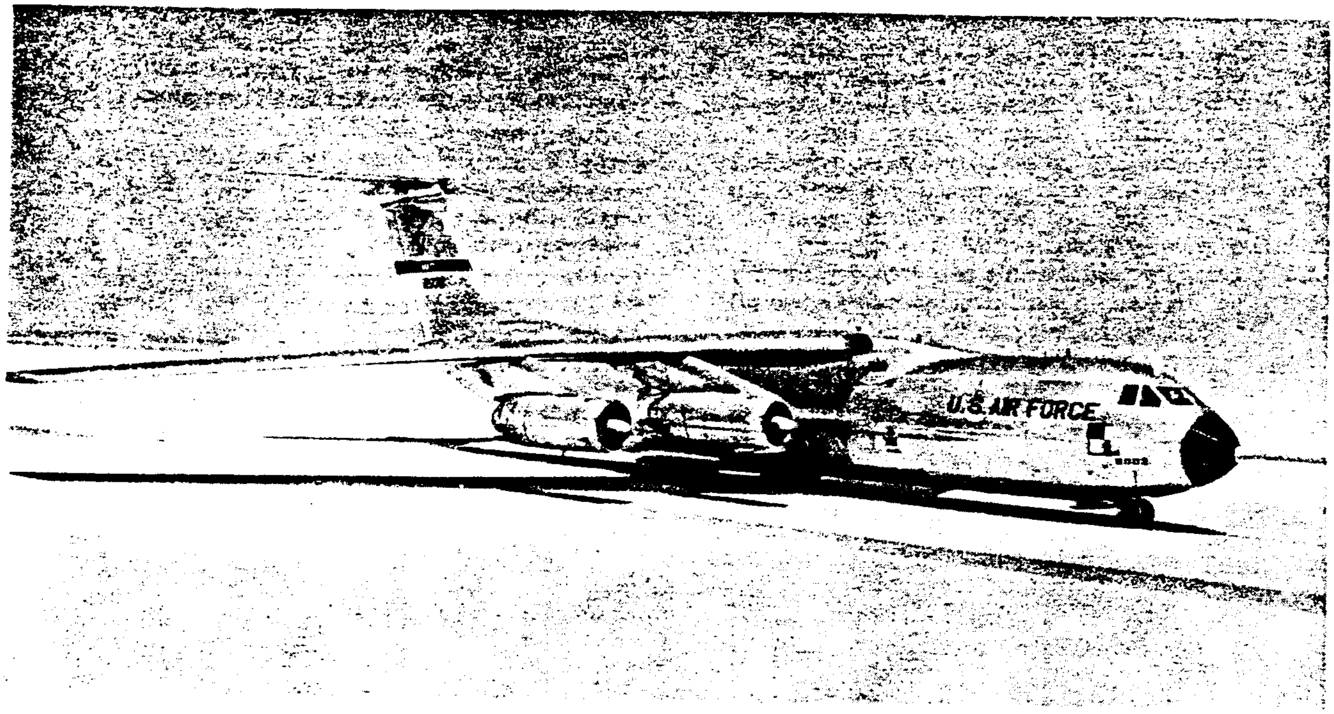
•1.3 TIMING

Proper utilization of time is always of paramount importance in conducting a test program. As a consequence, proper preflight preparation is absolutely essential. The complete test should be well in mind prior to takeoff. The pilot should always be thinking ahead and planning what he is to do next, keeping himself properly positioned in respect to the airfield. A flight that is well planned prior to flight has a very good possibility of working out well.

•1.4 PRIMARY OBJECTIVES

All stability flight tests will be flown with the prime objective of giving the prospective user the most information possible about the particular aircraft. This will be done by noting the aircraft's degree of compliance with the latest specification of flying qualities along with any other information that the test pilot feels essential for safe, effective use of the machine under all conditions.

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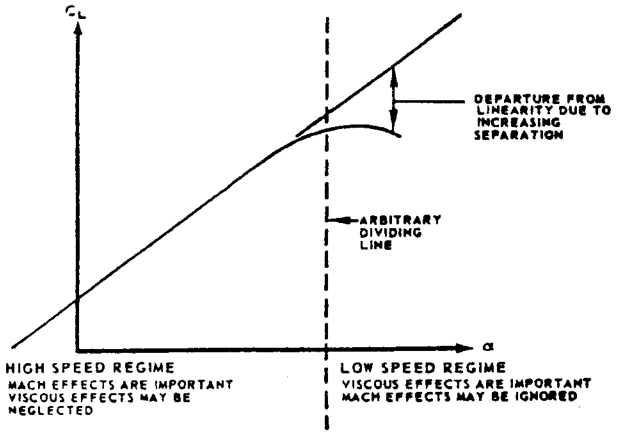
2.1 INTRODUCTION

Stall speed is the minimum steady speed attainable, or usable, in flight. A sudden loss of lift occurring at a speed just below that for maximum lift is considered the "conventional" stall, although it has become increasingly common for the minimum speed to be defined by some other characteristic, such as a high sink rate, an undesirable attitude, loss of control about any axis, or a deterioration of handling qualities.

For rather obvious safety and operational reasons, determination of stall characteristics is a first-order-of-business item in flight testing a new aircraft. Stall speeds are also required early in the test program for the determination of various test speeds.

2.2 SEPARATION

Figure 2.1



Separation, a condition wherein the streamlines fail to follow the body contours, produces a large disturbed wake behind the body and results in a pressure distribution greatly different from that of attached flow. On an aircraft, these changes in turn may produce:

a. A loss of lift
b. An increase in drag
c. Control problems due to:
   1. Control surfaces operating in the disturbed wake
   2. Changes in the aerodynamic pitching moment due to a shift in the center of pressure and an altered downwash angle.
d. A degradation of engine performance.

Separation occurs at a point where the boundary layer kinetic energy has been reduced to zero, therefore the position and amount of separation is a function of the transport of energy into and out of the boundary layer and of dissipation of energy within the boundary layer.

Some factors which contribute to energy transport are:

a. Turbulent (non-laminar) flow: Higher energy air from upper stream tubes is mixed into lower stream tubes. This type flow, characterized by a full velocity profile, occurs at high values of Reynolds number (R_e) and involves microscopic turbulence.

b. Vortex generators: These devices produce macroscopic turbulence to circulate high energy air down to lower levels.

c. Slats and slots: These devices inject high energy air from the underside of the leading edge into the upper surface boundary layer.

d. Boundary Layer Control: The blowing type of boundary layer control (BLC) injects high energy air into the boundary layer; while the suction type removes low energy air.

Two examples of energy dissipation functions are:

a. Viscous friction: Energy loss varies with surface roughness and distance traveled.

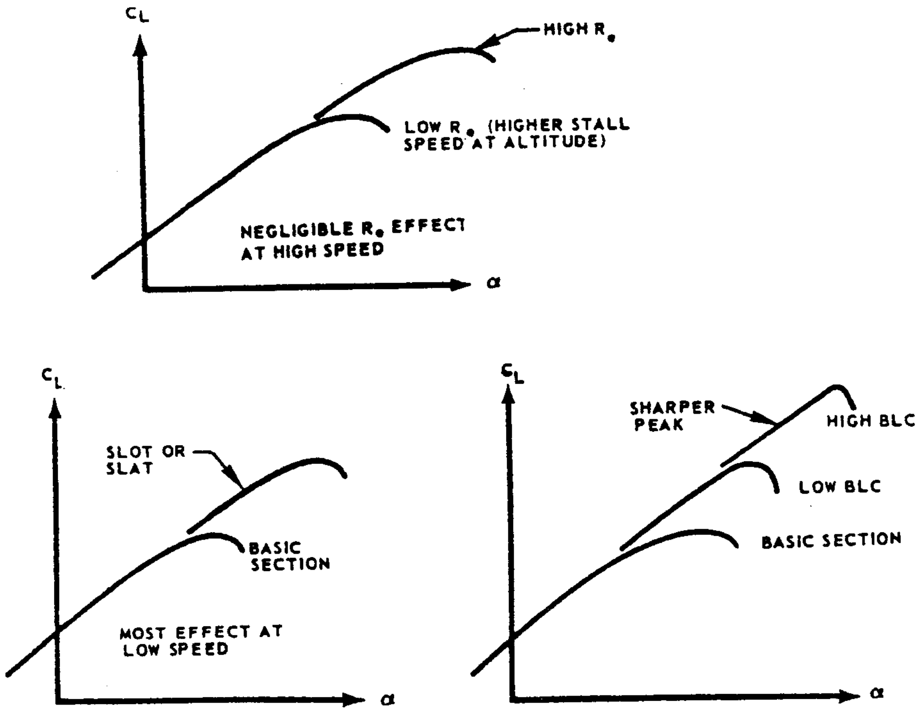
b. Adverse pressure gradient: Boundary layer energy is dissipated as the air moves against the adverse pressure gradient above a cambered airfoil section. The rate of energy loss is a function of:

1. Body contours - such as camber, thickness distribution, and sharp leading edges.

2. Angle of attack - Increased angle of attack steepens the adverse pressure gradient.

Some typical coefficient of lift versus angle of attack (C_L versus α) curves illustrating these effects are shown in figure 2.2.

Figure 2.2



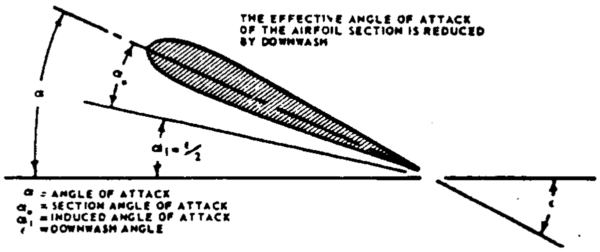
• 2.3 THREE-DIMENSIONAL EFFECTS

2.3 A three-dimensional wing exhibits aerodynamic properties considerably different from those of the two-dimensional airfoil sections of which it is formed. These differences are related to the planform and the aspect ratio of the wing.

<u>Planform</u>:

Downwash, a natural consequence of lift production by a real wing of less than infinite span, reduces the angle of attack at which the individual wing sections are operating.

Figure 2.3



An elliptical wing has a constant value of downwash angle along its entire span. Other planforms, however, have downwash angles that vary with position along the span. As a result, the lift coefficient for a particular wing section may be more or less than that of nearby sections, or that of the overall wing. Airfoil sections in areas of light downwash will be operating at high angles of attack, and will reach stall first. Stall patterns therefore depend on the downwash distribution, and vary predictably with planform as shown in figure 2.4.

Sweptback and delta planforms suffer from an inherent spanwise flow. This is caused by the outboard sections being located to the rear, placing low pressure areas adjacent to relatively high pressure areas.

Figure 2.4

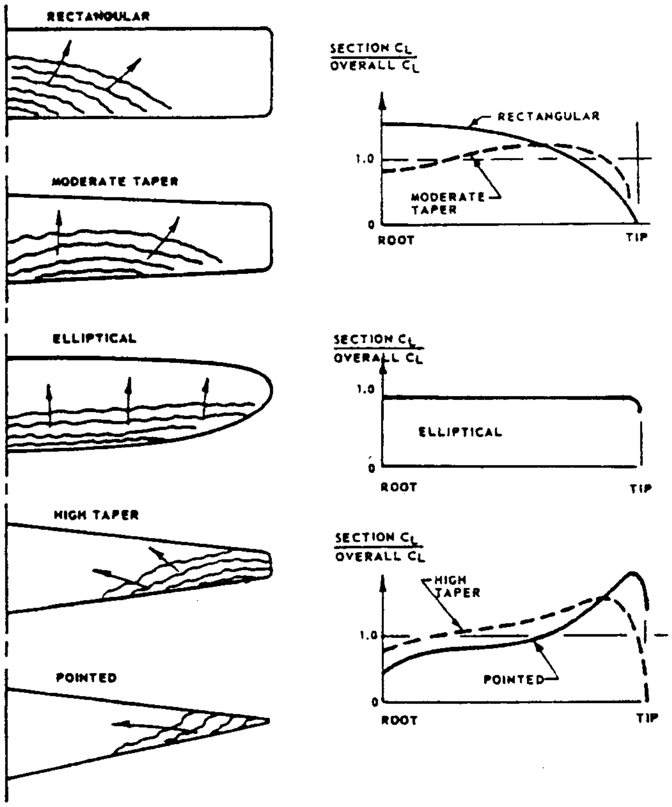
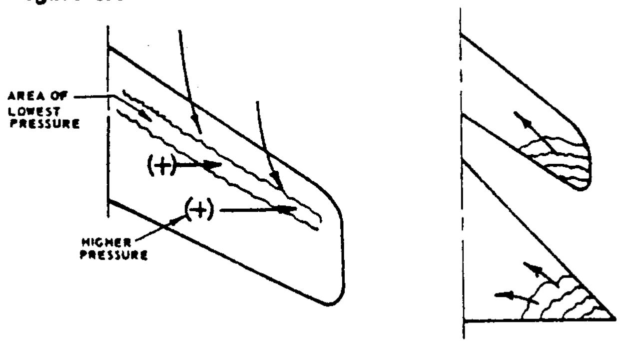


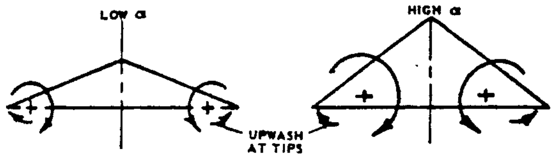
Figure 2.5



This spanwise flow transports low energy air from the wake of the forward sections outboard toward the tips, inviting early separation. Both the sweptback and delta planforms display tip-first stall patterns.

Pointed or low chord wing tips are unable to hold the tip vortex, which moves further inboard with increasing angle of attack.

Figure 2.8



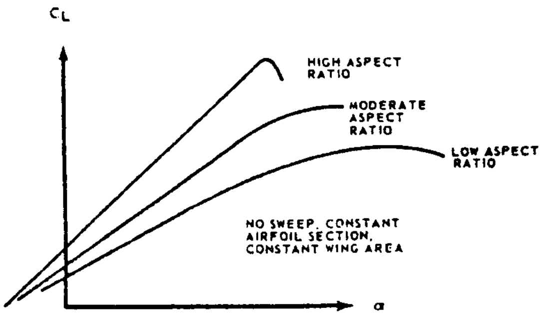
The extreme tips operate in upwash and in the absence of aerodynamic fixes such as twist or droop are completely stalled at most angles of attack.

Aspect Ratio:

Aspect ratio may be considered an inverse measure of how much of the wing is operating near the tips. Wings of low aspect ratio (much of the wing near the tip) require higher angles of attack to produce a given lift.

The curves of figure 2.7 illustrate several generalities important to stall characteristics. High aspect ratio wings have relatively steep lift curve slopes with well defined peaks at $C_{L_{max}}$. These wings have a relatively low angle of attack (and hence pitch angle) at the stall, and are usually characterized by a rather sudden stall br^ak.

Figure 2.7



Low aspect ratio wings display the reverse characteristics; high angle of attack (high pitch angles) at slow speeds and poorly defined stalls. They can frequently be flown in a high sink rate condition to the right of $C_{L_{max}}$ where drag increases rapidly.

Aerodynamic Pitching Moment:

On almost all planforms the center of pressure moves forward as the stall pattern develops, producing a noseup pitching moment about the aircraft center of gravity (cg).

This moment is not great on most straight wing planforms and the characteristic root stall of these wings adds a compensating nosedown moment such that a natural pitchdown tendency exists at high angles of attack. This occurs because the stalled center section produces much less downwash in the vicinity of the horizontal tail, decreasing its download. If the tail actually enters the turbulent wake the nosedown moment may be further intensified due to a decrease in elevator effectiveness. This latter case usually provides a natural stall warning in the form of airframe and control buffet.

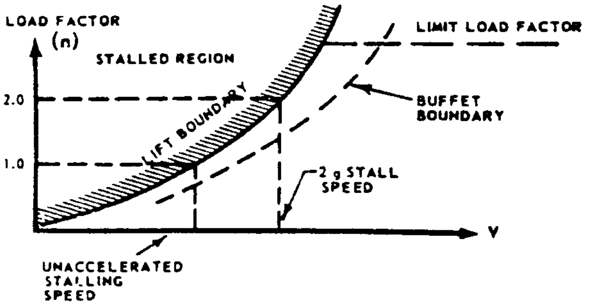
On swept-wing and delta planforms the moment produced by the center of pressure (c.p.) shift is usually more pronounced and the moment contributed by the change in downwash at the tail in this case is noseup. This occurs because the wing root section remains unstalled, producing greater lift and greater downwash as the angle of attack increases. The inboard movement of the tip vortex system also increases the downwash behind the center of the wing. Horizontal tails even in the vicinity of this increased downwash will produce more download. If the tail is mounted such that it actually enters the downwash area at high angles of attack, such as on the F-101, an uncontrollable pitchup may occur.

Many fixes and gimmicks have been used to alter lift distribution and stall patterns. Tip leading edge extensions, tip slots and slats, tip washout and droop, fences and root spoilers are but a few. Horizontal tail position is also subject to much adjustment such as has been necessary on the F-4C.

2.4 LOAD FACTOR CONSIDERATIONS

The relationship between load factor (n) and velocity may be seen on a V-n diagram.

Figure 2.8



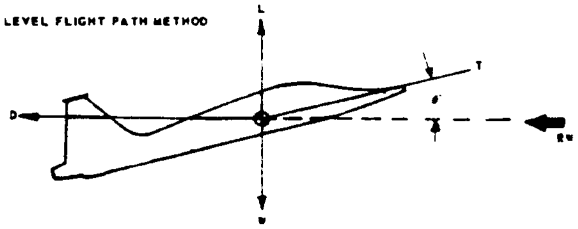
Every point along the lift boundary curve, the position of which is a function of gross weight, altitude, and aircraft configuration, represents a condition of $C_{L_{max}}$ (neglecting cases of insufficient elevator power). It is important to note that for each configuration, $C_{L_{max}}$ occurs at a particular α_{max}, independent of load factor, i.e., an aircraft stalls at the same angle of attack and C_L in accelerated flight, with n = 2.0, as it does in unaccelerated flight, with n = 1.0. The total lift (L) at stall for a given gross weight (W) does however vary with load factor since L = nW. The increased lift at the accelerated stall must be obtained by a higher dynamic pressure (q).

$$q_{stall} = 1/2\rho V^2_{stall} = \frac{nW}{C_{L_{max}}S}$$

Thus stall speed is proportional to n, making accurate control of normal acceleration of primary importance during stall tests.

Two flight test methods are described below. The first, involving a level flight path, is an older method that is valid only for unaccelerated stalls. It has several disadvantages that limit its application, but in certain cases such as VSTOL testing or initial envelope extension it might prove useful. It has been largely replaced by the second method that involves a curved flight path and is valid for both accelerated and unaccelerated stalls.

Figure 2.9



L + T sin θ = W and the flight path is straight. In order to slow the aircraft to stall speed, however, an acceleration (a_D) in the drag direction must be obtained by adjustment of thrust or drag such that D is greater than T cos θ. This represents a disadvantage of the method, since a particular trim power or drag configuration cannot be maintained to the stall.

Examination of figure 2.10 shows that the load factor will be at the desired value of 1.0 only if a_D is large enough. The size of a_D will be indicated by the rate of change of airspeed, termed the bleed rate. Experience has shown that undesirable dynamic effects are encountered if bleed rates much in excess of 1 or 2 knots per second are used. In practice, this usually restricts a_D to a value insufficient to close the acceleration diagram to the desired n = 1.0, another disadvantage of this method.

Figure 2.10

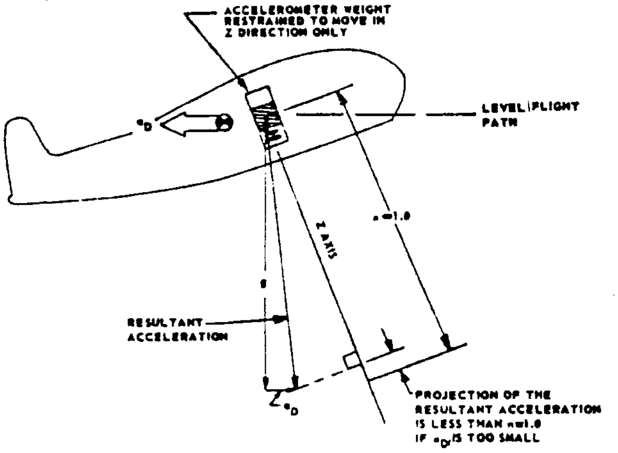
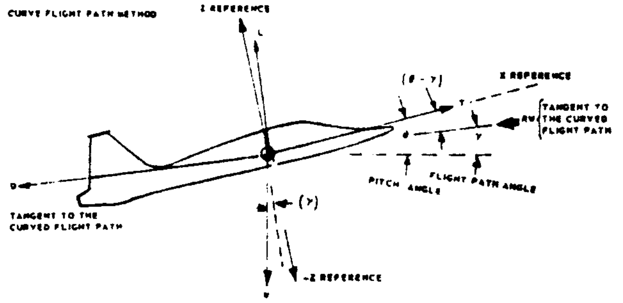


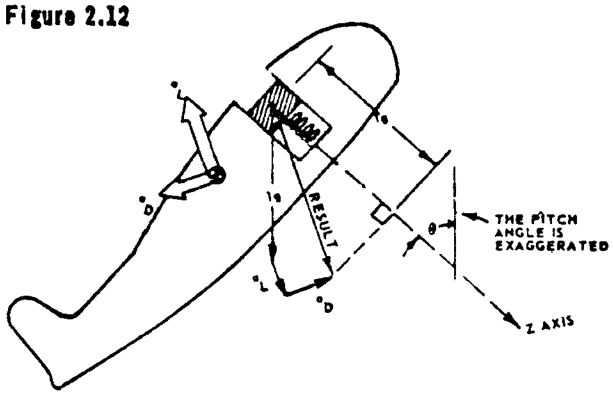
Figure 2.11



Here L + T sin (θ - γ) is greater than W cos(γ) and the aircraft is accelerating (a_L) toward the center of curvature. If trim power was set in level flight, D + W sin γ will be greater than T cos (θ - γ) and there will be an additional acceleration in the drag direction (a_D).

Note that a_L is closely aligned with the vertical reference and that adjustment of the radius of curvature may be used to close the acceleration diagram to the desired value of n. Thus load factor may be easily controlled with the elevator while maintaining trim power and configuration.

Figure 2.12



It is important to realize that the proper value of load factor may be maintained even with large bleed rates (a_D) simply by changing the radius of curvature (a_L), which will of course require a different pitch rate. The steady state diagrams above do not illustrate the need for a small bleed rate. They in fact indicate that the desired load factor may be obtained within a wide range of bleed rates. No rigorous limit for bleed rate can be calculated. In order to remove the possibility of dynamic effects, however, maximums of 1 knot/second for unaccelerated stalls, and 2 knots/second for accelerated stalls have been arbitrarily set on the basis of experience.

2.5 STALL FLIGHT TESTING

General:

Stalls, a familiar maneuver mastered by every pilot when he first learned to fly, must not be taken for granted in a test program. There is a rather large collection of examples from flight test history to document the need for caution. Designs that combine an inherent pitchup tendency with miserable spin characteristics have contributed much to these examples. Stalls are usually first demonstrated by a contractor pilot, but it is possible for a military test pilot to find himself doing the first stalls in a particular configuration, especially on test bed research programs where frequent modifications and changes are made after the vehicle has been delivered by the contractor.

The cautious approach starts with good preplanning. Discuss with the appropriate engineering talent the predicted stall characteristics. Develop with them the most promising recovery technique for each stage of the stall, to include possible post-stall gyrations. In marginal cases, a suggestion for further wind tunnel testing or other alternative investigations might be warranted. Determine the most favorable loading and configuration to be used in the initial stages. Stall and spin practice in trainer aircraft will enhance pilot performance during any out-of-control situations that might develop.

If pitchup or other control problems seem remotely possible, the first runs should terminate early in the approach to the stall and the data carefully examined (on the ground) for trends such as lightening or reversal of control, excessive attitudes or sink rates. Advance this data systematically on subsequent flights - avoid the mistake of suddenly deciding in flight, because things are going well, to take a bigger step than planned.

A stall test point will in general involve three phases; the approach, the stall, and the recovery.

Approach to the Stall:

As will be described later, the aircraft must be flown through this phase in a manner to insure that the stall occurs at the desired altitude and load factor.

Stall warning, if any, will occur during this phase. This requires a subjective judgement by the pilot - only he can tell when he has been warned. This judgement should be extrapolated to the conditions under which the aircraft will be used in service, when distractions such as combat maneuvering may be present. A warning barely discernable during the test program would be of little use under these conditions. Excessive warning is also not desirable; MIL-F-8785 specifies definite upper and lower airspeed limits within which warning should occur. Control shake or airframe buffet is desired although artificial warning devices such as stick and rudder shakers are becoming increasingly common.

The Stall:

Stall has been defined as the minimum steady speed attainable, or usable, in flight. This minimum may be set by a variety of factors, for example:

a. Reaching $C_{L_{max}}$ - the conventional stall.

b. Insufficient longitudinal control to further decrease speed - lack of elevator power.

c. Onset of control problems. (Loss of control about any axis.)

   1. Pitchup

   2. Insufficient lateral-directional control to maintain attitude

   3. Poor dynamic characteristics

d. Back-side problems.

   1. High sink rate

   2. Insufficient wave-off capability

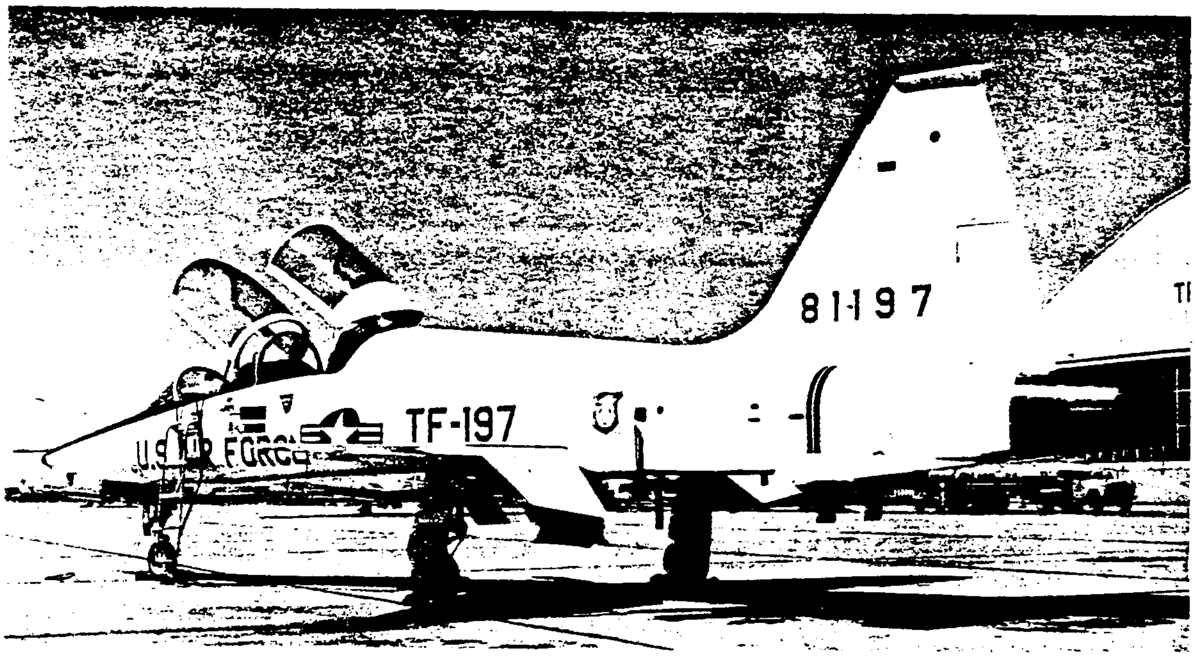
   3. Excessive pitch attitude

Aircraft with lift curves having sharp peaks may be prone to wing drop near the stall if local gusts or control motion cause a high angle of attack to occur on one wing before the other. MIL-F-8785 prescribes definite limits on pitch and bank angle at the stall. The test pilot should describe any other undesirable characteristics that may be evident.

The Recovery:

The recovery is started when the stall or minimum steady speed has been attained. For a conventional stall this is indicated by the inability to maintain the desired load factor - usually a sudden break is apparent on the cockpit accelerometer.

The goal of the recovery must be specified. For example, it might be to keep the altitude lost to a minimum or to obtain the fastest acceleration to a maneuver speed. In a test program all promising recovery procedures consistent with the objectives should be tried. It is important to have the recovery specified in detail before each stall - do not wait until the stall breaks to decide what procedure is to be used. There are no iron-clad rules for recovery - a "standard procedure" such as full military power could be disastrous in certain vehicles. Keep the instrumentation running throughout the recovery until the goal has been attained. In the case of minimum altitude loss this would be when rate of descent is zero and the aircraft is under control (the altimeter is the first indication of R/C = 0).



Stall characteristics must be evaluated in relation to their influence on mission accomplishment. Thus, both normal and accelerated stalls must be performed under entry conditions which could result from various mission tasks. However, prior to evaluating stalls entered from these conditions, a more controlled testing approach should be employed. This approach allows lower deceleration rates into the stall and lower pitch attitudes at the stall, thereby reducing chances for "deep-stall" penetration without adequate buildup. After the controlled stall investigation, if stall characteristics permit, simulated inadvertent stalls should be investigated under conditions representative of operational procedures.

The Controlled Stall Test Technique

The easiest and safest approach to controlled stall testing is to divide the investigation into three distinct parts:

1. Approach to the stall
2. Fully developed stall
3. Stall recovery.

Approach to the Stall

During this phase of the investigation, adequacy of stall warning and retention of reasonable airplane controllability are the primary items of interest. Assessment of stall warning required subjective judgement by the pilot. Only the pilot can decide when he has been adequately warned. Warning must occur sufficiently in advance of the stall to allow prevention of the stall by normal control applications after a reasonable pilot reaction time. However, stall warning should not occur too far in advance of the stall. For example, it is essential that stall warning for approach configuration occur below normal approach speed. Stall warning which occurs too early is not only annoying to the pilot but is meaningless as an indication of proximity to the stall.

The type of stall warning is very important. Primary stall warning is generally in the form of airframe buffet, control shaking, or small amplitude airplane oscillations in roll, yaw, or pitch. Other secondary cues to the approach of the stall may be high pitch attitude, large longitudinal control pull forces (of course, this cue can be destroyed by "trimming into the stall"), large control deflections or sluggish control response. In any case, stall warning, whether natural or artificial, should be unmistakable, even under conditions of high pilot workload and stress and under conditions of atmospheric turbulence. If an artificial stall warning device is installed, approach to the stall should be evaluated with the device operative and inoperative to determine if the device is really required for normal operations.

During this phase of the evaluation, the test pilot must evaluate stall warning with the intended use and operational environment in mind. He must remember that he is specifically looking for the stall warning under controlled conditions. The operational pilot probably will not be. This question must be answered: Will the operational pilot, preoccupied by other tasks and not concentrating on stalls, recognize approach of the stall and be able to prevent the stall?

The general flying qualities of the airplane should be investigated during the approach to the stall as well as stall warning characteristics. Longitudinal, lateral, and directional control effectiveness for maintaining a desired attitude may deteriorate significantly during the approach to the stall. Loss of control about any axis such as uncontrollable pitch-up or pitch-down, "wing drop", or directional "slicing" may define the actual stall. During the approach to the stall, the test pilot should be particularly aware of the amount of longitudinal nose-down control available because of the obvious influence of this characteristic on the ability to "break" the stalled condition and make a successful recovery.

This phase of stall investigation usually begins with onset of stall warning and ends at the stall, therefore the test pilot will certainly be concerned with the manner in which the airplane stalls and the ease of recovery. However, primary emphasis is placed on obtaining an accurate assessment of stall warning and general flying qualities during the approach to the stall. During initial investigations, it may be prudent to terminate the approach short of the actual stall, penetrating deeper and deeper with each succeeding approach until limiting conditions or the actual stall are reached. In addition, the rate of approach should be low initially, approximately one knot per second for normal stalls and two knots per second for accelerated stalls. As experience is gained, deeper penetrations at faster deceleration rates must be performed unless safety considerations dictate otherwise.

The test pilot should record at least the following cockpit data during the approach to the stall:

1. Airspeed and angle of attack at stall warning
2. Type and adequacy of stall warning
3. Longitudinal control force at stall warning (either measured or estimated)
4. Qualitative comments regarding controllability and control effectiveness.

Fully Developed Stall

During this phase of the investigation, the primary objective is to accurately define the stall and the associated airplane behavior. The stall should be well-marked by some characteristic, such as pitch-up or pitch-down or lateral or directional divergence. In general, any pitch-up or directional divergence at the stall is undesirable because pitch-up may precipitate a deep stall penetration and directional divergence may lead to a spin. Pitch-down at the stall and lateral divergence may be acceptable, however severe rolling, pitching, or yawing or any combination of the three are obviously poor characteristics.

Control effectiveness as evidenced by the pilot's ability to control or induce roll, pitch, or yaw should be evaluated in the stall, if airplane behavior permits this to be done safely. Obviously, control effectiveness should be evaluated with a suitable build-up program. Initially, control inputs only large enough to effect an immediate coordinated recovery should be used. As experience is gained, the airplane should be maintained in the stalled condition for longer and longer periods of time, and the effectiveness of all controls evaluated with larger and larger control deflections.

The test pilot should record at least the following cockpit data regarding the stall:

1. Airspeed and angle of attack at stall
2. Load factor (accelerated stalls only)
3. Characteristic which defines the stall
4. Longitudinal control force at the stall (either measured or estimated). The ratio of longitudinal control forces at stall and stall warning is a rough indication of longitudinal stability in the high angle of attack region and an indication of the ease of inadvertent stalling.
5. Qualitative descriptive comments.

Stall Recovery

During this phase of the investigation, primary items of interest are the ease of recovery (the pilot's task), general flying qualities during the recovery, altitude required for recovery and the determination of an optimum recovery technique. The definition of stall recovery may vary with the configuration under investigation. For example, the goal of recovery for configurations commensurate with combat maneuvering may be to regain sufficient control effectiveness about all three axes to perform offensive or defensive maneuvering tasks; the attainment of level flight may not be critical in these configurations. The goal of recovery for takeoff and approach configurations should be the attainment of level flight with a minimum loss of altitude and the regaining of sufficient control effectiveness to safely maintain stall-free conditions. In each case, the test pilot must clearly define "stall recovery."

During initial investigation, the stall recovery procedures specified in pertinent publications should be utilized and the ease of effecting recovery evaluated. If no procedure has been developed, initial recovery must be accomplished with a "preliminary" technique formulated from all available technical information. As experience is gained, various modifications to the recovery procedure should be made until an optimum procedure is determined. In arriving at an optimum procedure for use by the operational pilot, the test pilot must not only consider the effectiveness of the technique (in terms of altitude lost or maneuverability regained), but must also consider the simplicity of the technique.

The test pilot should record at least the following data regarding stall recovery:

1. Qualitative comments on ease of recovery
2. Optimum recovery technique
3. Altitude lost in recovery
4. Qualitative comments on control effectiveness.

2.7 DATA

Data to be Recorded:

A continuous photo panel recording should be taken from before stall warning until after recovery on all stalls on which quantitative data are collected. The photo panel will be used to record the trim points. Alpha is one parameter of primary interest from both the trim points and the accelerated stalls. It will be used to develop a C_L vs α curve so that α_{0L} can be determined. α_S and α_{0L} will be used in the determination of MIL-F-8785 compliance for accelerated stall warning.

The following data should be hand recorded for each stall:

a. Indicated airspeed at stall warning (also actuate the event marker).

b. Indicated airspeed at stall (also actuate the event marker).

c. Altitude lost in the recovery.

d. Qualitative comments on:

   1. Type and adequacy of stall warning.
   2. Stall characteristics such as pitch and roll.
   3. Control characteristics during all the three phases of the stall.
   4. Type and effectiveness of recovery techniques.

e. Fuel counter readings.

f. Correlation number.

The format of the flight data cards is not specified. However, the stall mission is a very busy one and it will tax the pilot's concentration and agility. Extra time should be spent to devise data cards that will aid in keeping track of the details. A space for every type of comment desired should be provided beforehand; then the pertinent information may be rapidly entered during the clean-up phase. Do not crowd the cards.

Data Presentation:

A table and a time history similar to figures 2.13 and 2.14 may be presented. The time history should be of a particularly well flown stall, and/or of one during which some unusual characteristic was observed.

The results should include a discussion of the qualitative findings and an evaluation of the aircraft in comparison to the requirements of MIL-F-8785. (The School flight profile was chosen to demonstrate as much as possible in a reasonable time - it does not necessarily fulfill all the requirements of the Military Specification.)

Figure 2.14

TYPICAL STALL TIME HISTORY

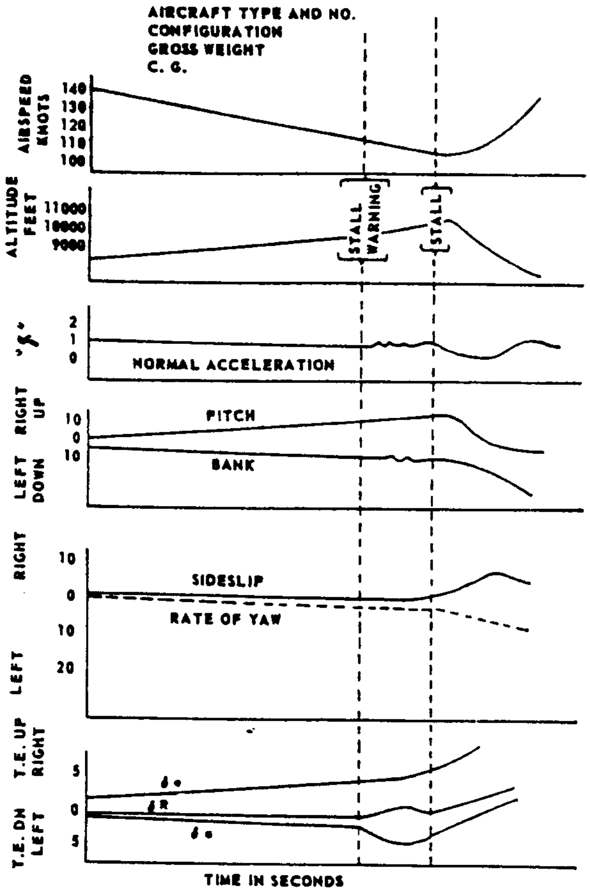


Figure 2.13

Conf	n	Weight	CG	TRIM M	TRIM IAS	ALTITUDE	IAS Warn	IAS Stall	$C_{L_{Max}}$	V_W/V_S	Remarks
CR											
etc.											

2.8 DETERMINATION OF THE LIFT BOUNDARY

The purpose of this test is to determine the limiting normal acceleration or g's that can be pulled at various speeds and Mach numbers. This may be determined by buffet and/or pitchup. From this data it is possible to determine the best maneuvering Mach number.

Data Recording

The data for this test will be hand recorded. Normally this test would be flown at several center of gravity positions.

Test Techniques

Place the aircraft in a steady level turn and increase power in an attempt to hold constant altitude, trim velocity and Mach number. Continually increase the load factor until initial buffet is reached and note the load factor at this time. Continue to increase the bank and load factor until moderate buffet is reached and again note this load factor.

The same technique will be used at varying airspeeds. At the higher airspeeds, when it may be impossible to hold constant altitude at full power, plan the entry from a higher altitude so that the aircraft will reach buffet at the required speed and altitude. A tolerance of plus or minus 500 feet will be allowed.

Care should be taken not to increase load factor more than one half g per second in order to minimize dynamic effects.

If any intolerable condition of flight is experienced prior to initial or heavy buffet the run will be discontinued and appropriate mention made of this fact.

Data Reduction Outline

The data reduction is as follows:

	Parameter	Source	How Obtained
(1)	V_i	Instrument Panel	Data Card
(2)	ΔV_{ic}	Calibration	
(3)	V_{ic}		(1) + (2)
(4)	ΔV_{pc}	Position Error	
(5)	V_c		(3) + (4)
(6)	H_i	Instrument Panel	
(7)	ΔH_{ic}	Calibration	
(8)	H_{ic}		(6) + (7)

	Parameter	Source	How Obtained
(9)	ΔH_{pc}	Position Error	
(10)	H_c	Calibration Altitude	(8) + (9)
(11)	M	From V_c and H_c	
(12)	n (Initial buffet)	Instrument Panel	
(13)	n (Moderate buffet)	Instrument Panel	
(14)	W_t	Gross Weight	
(15)	δ	Pressure Ratio	Appropriate charts at (10)
(16)	W/δ		(14) ÷ (15)
(17)	nW/δ (Initial)		(12) x (16)
(18)	nW/δ (Moderate)		(13) x (16)

Plot (17) and (18) versus (11) showing lines of initial buffet and moderate buffet.

CHAPTER 3 LONGITUDINAL STATIC STABILITY

REVISED FEBRUARY 1977

3.1 INTRODUCTION

The purpose of this flight test is to determine the longitudinal static stability characteristics of an aircraft. These characteristics include gust stability, speed stability, flight-path stability, and the associated terms static margin, neutral point, and friction/breakout.

An aircraft is said to be statically stable longitudinally (positive gust stability) if the moments created when the aircraft is disturbed from trimmed flight tend to return the aircraft to the condition from which it was disturbed. Longitudinal stability theory shows the flight test relationships for stick-fixed and stick-free gust stability, dC_m/dC_L, to be

stick-fixed

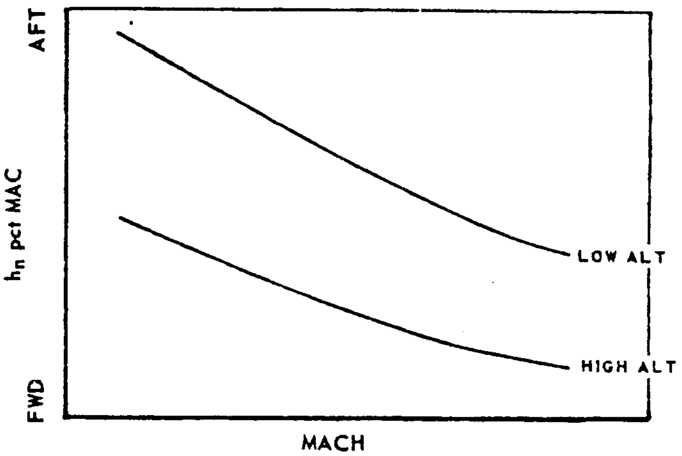
$$\frac{d\delta_e}{dC_L} = -\frac{dC_m/dC_L}{C_{m_{\delta_e}}} \text{ Fixed} \quad (3.0)$$

$$\frac{d(F_s/q)}{dC_L} = -A\frac{C_{h_{\delta_e}}}{C_{m_{\delta_e}}} \frac{dC_m}{dC_{L_{Free}}} \quad (3.1)$$

Stick force (F_s), elevator deflection (δ_e), equivalent velocity (V_e) and gross weight (W) are the parameters measured to solve the above equations. When $\frac{d\delta_e}{dC_L}$ is zero, an aircraft is defined as having neutral stick-fixed longitudinal static stability. As $\frac{d\delta_e}{dC_L}$ increases the stability of the aircraft increases. The same statements about stick-free longitudinal static stability can be made with respect to $\frac{d(F_s/q)}{dC_L}$. The neutral point is the cg location which gives neutral stability, stick-fixed or stick-free. These neutral points are determined by flight testing at two or more cg locations, extrapolating the curves of $\frac{d\delta_e}{dC_L}$ and $\frac{d(F_s/q)}{dC_L}$ versus cg to zero.

The neutral point so determined is valid for the trim altitude and airspeed at which the data were taken and may vary considerably at other trim conditions. A typical variation of neutral point with subsonic Mach number and altitude is shown below.

Figure 3.1 STICK-FIXED NEUTRAL POINT versus MACH AND ALTITUDE



The use of the neutral point theory to define gust stability is therefore time consuming and of limited practical value except for initially predicting aft cg limits. This is especially true for aircraft that have a large airpseed envelope and aeroelastic effects.

Speed stability is the variation in control stick forces with airspeed changes. Positive stability requires that increased aft stick force be required with decreasing airspeed and vice versa. It is related to gust stability, but may be considerably different on aircraft that employ a certain combination artificial feel and stability augmentation. Speed stability is the longitudinal static stability characteristic most apparent to the pilot and it therefore receives the greatest emphasis.

Flight-path stability is defined as the variation in flight-path angle when the airspeed is changed by use of the elevator alone. Flight-path stability generally applies only to the power approach flight phase and is basically determined by aircraft performance characteristics. Positive flight-path stability ensures that the aircraft will not develop large changes in rate of descent when corrections are made to the flight path with the throttle fixed. The exact limits are prescribed in MIL-F-8785B(ASG), paragraph 3.2.1.3. An aircraft likely to encounter difficulty in meeting these limits would be one whose power approach airspeed was far up on the "backside" of the power required curve. A corrective action might be to increase the power approach airspeed, thereby placing it on a flatter portion of the curve or installing an automatic throttle to improve handling qualities.

3.2 MILITARY SPECIFICATION REQUIRMENTS

The 1954 version of MIL-F-8785 established longitudinal stability requirements in terms of the neutral point. While the neutral point criteria is still valid for testing certain types of aircraft, this criteria was not optimum for aircraft operating in flight regimes where other factors were more important in determining longitudinal stability. The 1969 version of MIL-F-8785B does not even mention neutral points, instead, section 3.2.1 of MIL-F-8785B specifies longitudinal stability with respect to speed and flight-path. The requirements of this section are relaxed in the transonic speed range except for those aircraft which are designed for prolonged transonic operation.

3.3 TEST METHODS

There are two general test methods (stabilized and acceleration/deceleration) used to determine either speed stability or neutral points. There is an additional test method used for determining flight-path stability which is discussed later.

3.3.1 Stabilized Methods

This is used for aircraft with a small airspeed range in the cruise flight phase and virtually all aircraft in the power approach, landing or takeoff flight phases. Propeller type aircraft are normally tested by this method because of the effects on the elevator control power caused by thrust changes. It involves data taken at stabilized airspeed at the trim throttle setting with the airspeed maintained constant by a rate of descent or climb. As long as the altitude doesn't vary excessively (typically ±1,000 feet) this method gives good results, but it is time consuming.

The aircraft is trimmed carefully at the desired altitude and airspeed and a trim shot is recorded. Without moving the throttle or trim setting, the pilot changes aircraft pitch attitude to achieve a lower or higher airspeeds (typically in increments of ± 10 knots) and maintains that airspeed.

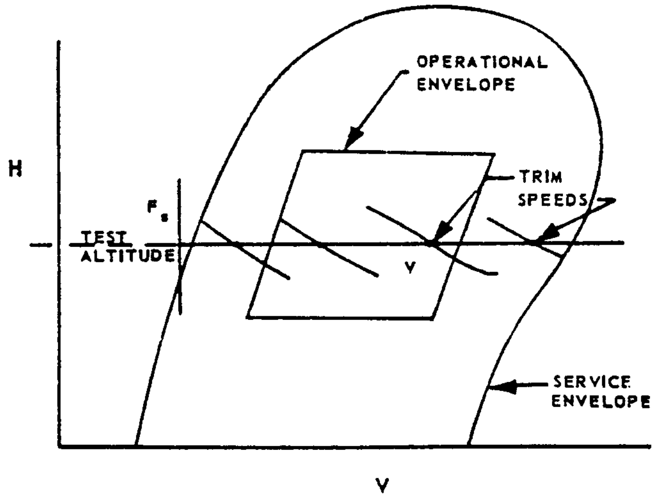
Since the pilot has usually moved the control stick fore and aft through the friction band, he must determine which side of the friction band he is on before recording the test point data. The elevator position for this airspeed will not vary, but stick force varies relative to the instantaneous position within the friction band at the time the data is taken. Therefore, the pilot should (assuming an initial reduction in airspeed from the trim condition) relax the force until the first indication that the nose is beginning to drop and then increase force carefully until the nose starts to rise. This defines the magnitude of the friction band. Since it is generally advisable to record data below trim airspeed points on the backside of the friction band, the pilot should relax the stick force and then increase stick force to a point estimated to be close to, but not on, the backside of the friction band. If the backside of the friction band is reached, there is a good possibility that the elevator will move and the point will no longer be stable. Once this exercise has been completed, the stick is frozen and the data recorded. The same technique should be used for all other airspeed points below trim, although the examination of the friction band may not be required to ensure that the stick force is on the backside. For airspeed points above the trim airspeed, the same technique is employed, although now the front side of the friction band is preferred.

3.3.2 Acceleration/Deceleration Method

This is commonly used for aircraft that have a large airspeed envelope. It is always used for transonic testing. It is less time consuming than the stabilized method but introduces thrust effects. The U.S. Navy uses the acceleration/deceleration method but maintains the throttle setting constant and varies altitude to change airspeed. The Navy method minimizes thrust effects but introduces another consideration because of the change in altitude.

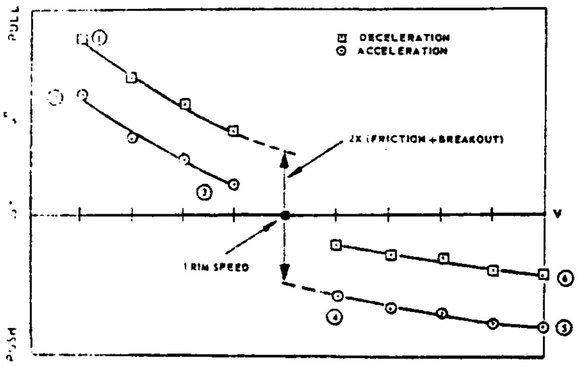
The same trim shot is taken as in the stabilized method to establish the trim conditions. MIL-F-8785B(ASG) requires that the aircraft exhibit positive speed stability only within ±50 knots or ±15 percent of the trim airspeed, whichever is less. This requires very little power change to traverse this band and maintain level flight unless the trim airspeed is near the backside of the thrust required curve. Before the 1968 revision to MIL-F-8785, the flight test technique commonly used to get acceleration/deceleration data was full military power or idle, covering the entire airspeed envelope. Unfortunately this technique cannot be used to conclusively determine the requirements under the current specification with the non-linearities that usually exist in the control system. Therefore a series of trim points must be selected to cover the envelope with a typical plot (friction and breakout excluded) shown in figure 3.2.

Figure 3.2 SPEED STABILITY DATA



The most practical method of taking data is to note the power setting required for trim and then either decrease or increase power to overshoot the data band limits slightly. Then turn on the instrumentation and reset trim power and a slow acceleration or deceleration will occur back towards the trim point. A few percent change in the trim power setting may be required to obtain a reasonable acceleration or deceleration without introducing gross power effects. The points near the trim airspeed point will be difficult to obtain but they are not of great importance since they will probably be obscured by the control system breakout and friction (figure 3.3).

Figure 3.3 ACCELERATION DECELERATION DATA ONE TRIM SPEED, cg, ALTITUDE



Throughout the acceleration or deceleration, the primary parameter to control is stick force. It is important that the friction band not be reversed during the test run. A slight change in altitude is preferable to a reversal of stick force. It is therefore advisable to let the aircraft climb slightly throughout an acceleration to avoid the tendency to reverse the stick force by over-rotating the nose. The opposite is advisable during the deceleration.

There is a relaxation in the requirement for speed stability in the transonic area unless the aircraft is designed for continued transonic operation. The best way to define where the transonic range occurs is to determine the point where the F_s versus V goes unstable. In this area, MIL-F-8785B(ASG) allows a specified maximum of instability in the stick force and a rate of change of instability. The purpose of the transonic longitudinal static stability flight test in the transonic area is to determine the degree of instability.

The transonic area flight test begins with a trim shot at some high subsonic airspeed. The power is increased to maximum thrust and an acceleration is begun. (Note that this applies also to aircraft that are normally termed subsonic, such as the T-33 and B-57.) It is important that a stable gradient be established before entering the transonic area. Once the first sensation of instability is felt by the pilot, his primary control parameter changes from stick force to attitude. From this point until the aircraft is supersonic, the true altitude should be held as closely as possible. This is because the unstable stick force being measured will be in error if a climb or descent occurs. A radar altimeter output on an over-water flight is the most precise way to hold constant altitude, but if this is not feasible the pilot will have to use his pitot-static instruments and outside references to maintain level flight.

Once the aircraft goes supersonic, the test pilot should again concern himself with not reversing the friction band and with establishing a stable gradient. The acceleration should be continued to the limit of the Service Envelope to test for supersonic speed stability. The supersonic data will also have to be shown at ±15 percent of the trim airspeed, so several trim shots may be required.

A deceleration from V_{max} to subsonic speed should be made with a careful reduction in power to decelerate supersonically and transonically. The criteria for decelerating through the transonic region are the same as for the acceleration. Power reductions during this deceleration will have to be figured carefully to minimize thrust effects and still decelerate past the Mach drag rise point to a stable subsonic gradient.

3.4 DATA REDUCTION

Longitudinal stability stability flight tests serve the same two purposes as all other flight tests: to verify compliance with military specifications and gather data to determine the aircraft's flying qualities. These two purposes are equally important, but unfortunately require slightly different approaches and quite different data reduction techniques.

3.4.1 Speed Stability

3.4.1.1 subsonic and supersonic

MIL-F-8785B(ASG) requirements for speed stability are relatively easy to examine. The stick force versus equivalent airspeed data from either test method is plotted. Since all that is required is for the gradient to be stable, it may save time to plot indicated velocity versus stick force; if this is obviously stable, then conversion to equivalent velocity is not necessary. It is essential, however, to identify the breakout and friction forces to separate them from the gradient of stick force versus velocity. Example data are shown in figure 3.3 for an acceleration and deceleration on each side of the trim airspeed. The acceleration from the trim airspeed to the high side of the band and the deceleration from the trim airspeed to the low side would both introduce power effects and would not normally be required to show compliance. They are shown on this example to illustrate breakout and friction. Power effects are assumed to be negligible.

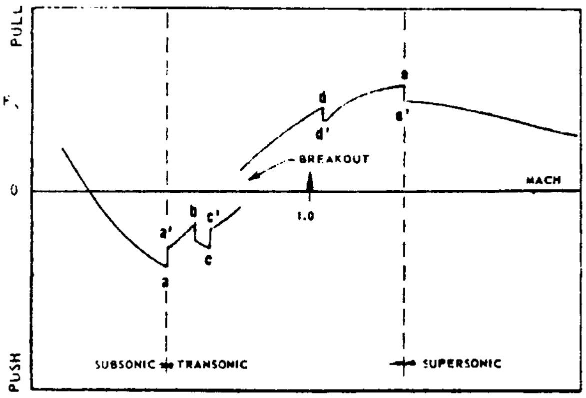
As the deceleration from trim airspeed is begun, the data points recorded are on the backside of the friction band. Upon reaching point 1 power is added to begin an acceleration. At the instant the airspeed starts to increase, the pilot senses the need to lower the nose. As he releases back stick pressure the stick traverses the friction band to point 2. It is not until this point is reached that the elevator starts to move to lower the nose of the aircraft. As the aircraft accelerates the pilot continues to release back stick pressure staying on the front side of the friction band. Somewhere between points 3 and 4 releasing back stick will not lower the nose as the point of hands-off trim is reached. The pilot now pushes forward on the stick to overcome the breakout force again and stay on the forward side of the friction band. As power is reduced to decelerate, the friction band is traversed again from points 5 to 6. If the outside curves are extrapolated to the trim point, the vertical distance between the two represents twice the breakout plus friction force. This is the best way to determine friction plus breakout force since there is no way to determine the location in the friction band where the control stick is at the time the trim shot is taken. This value can be used later in maneuvering flight data. It must be recognized that the friction force is not the same at all airspeeds because the mechanical parts of the longitudinal control system (that cause friction) are in different orientations as stick positions are changed.

For presentation purposes the friction and breakout values may be removed from the data. Data for a series of tests at one altitude are shown above in figure 3.2. The trim speeds were selected by MIL-F-8785B(ASG) standards with some overlap.

3.4.1.2 transonic

Speed stability in the transonic regime is generally unstable. The task is to see if the degree of instability exceeds the allowable limits. This involves investigation of the effects of friction that is not required in subsonic and supersonic speed stability if the tests are flown properly. Obviously, friction must be taken out of the measurements of unstable gradients since the gradient can theoreticaly have an infinite slope within the friction band. The value of the friction force must be known to determine whether the gradient of stick force has changed sign or the pilot has merely moved within the friction band. For an acceleration, the total instability should be made from the forward side of the friction band to avoid accounting for friction twice. An example measurement is shown below.

Figure 3.4 TRANSONIC STABILITY



Examination of the example raw data of F_S versus Mach in figure 3.4 shows the effects of friction in the transonic data. Speed instability occurs at point (a) which defines the beginning of the transonic regime. The pilot starts to release forward pressure until reaching the backside of the friction band at point (a'). He continues to release pressure until reaching point (b). Here speed stability is again present until reaching point (c) when the unstable stick forces occur again. The excursions from points (d) to (d') are less than the value of the friction band and are therefore not considered as a change in stability. When the aircraft passes out of the transonic regime - in this example supersonic speed stability occurs at the same point (e) - the pilot begins to release back pressure until the front side of the friction band is reached at point (e'). The unstable gradient may be measured between points (a') and (b) or points (c') and (e) with the excursions at point (d) excluded. The total instability is measured from points (a) to (e').

3.4.2 Neutral Point Determination

Data from acceleration/deceleration or stabilized methods are used to compute the stick-fixed and stick-free neutral points. The minimum requirements are two different cg positions flown at the same test point and trim airspeed.

3.4.2.1 stick fixed

The following plots are made:

FIGURE 3.5

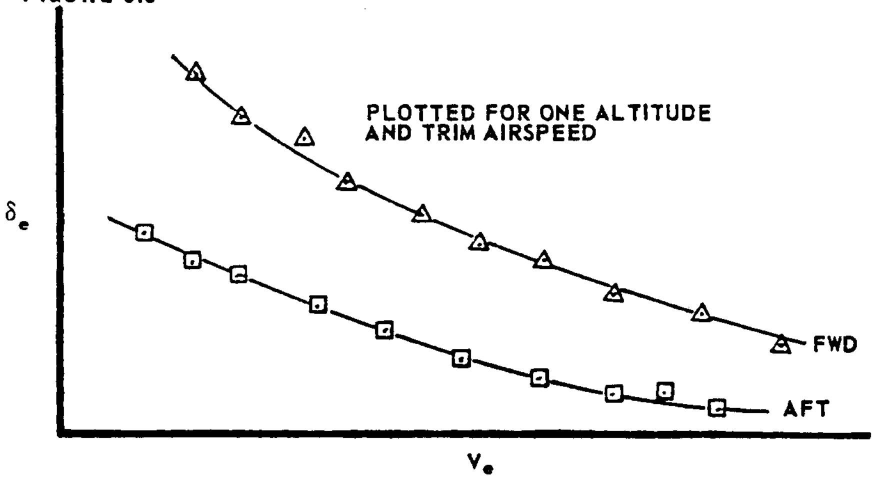
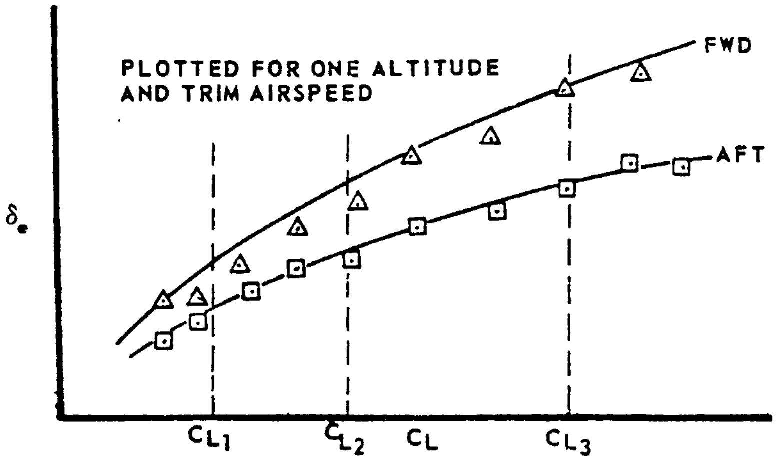


FIGURE 3.6

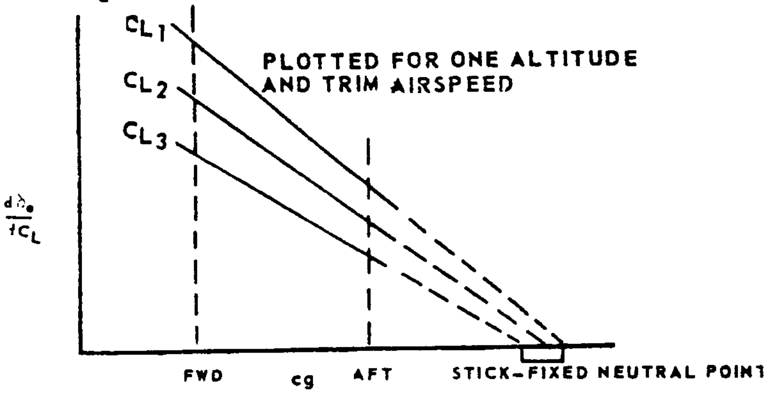


If elevator position plots linearly with lift coefficient, only one stick-fixed neutral point exists. Otherwise, the neutral point varies with angle of attack (or lift coefficient). The derivative $d\delta_e/dV_e$ (at one V_e and cg) does not change with weight unless the neutral point varies with angle of attack. The derivative $d\delta_e/dC_L$ serves better than $d\delta_e/dV_e$ as a plotting variable in locating neutral points because nonlinear weight effects are included.

The rate of change of elevator deflection with respect to lift force coefficient is measured from the plot of δ_e versus C_L. The slope is taken at three or more C_L's over the airspeed range for all cg loadings.

FIGURE 3.7

$d\delta_e/dC_L$ vs cg (WORKING PLOT)

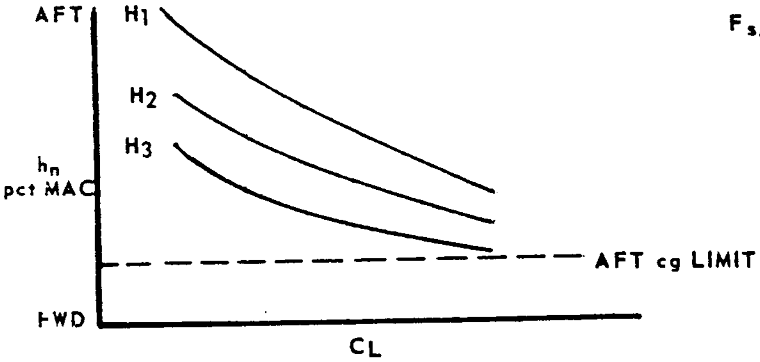


The point where $d\delta_e/dC_L = 0$, is the stick-fixed neutral point for that particular C_L. These neutral points for this one altitude are plotted versus C_L as the curve H_1 in figure 3.8 Additional altitude data would plot as curves H_2 and H_3.

Figure 3.8 indicates the change in stick-fixed stability as the aircraft traverses the speed range at three representative altitudes.

FIGURE 3.8

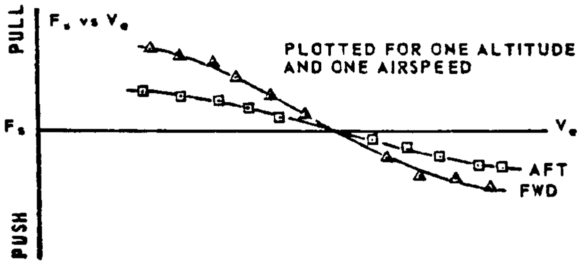
STICK-FIXED NEUTRAL POINT VERSUS LIFT COEFICIENT



3.4.2.2 stick free

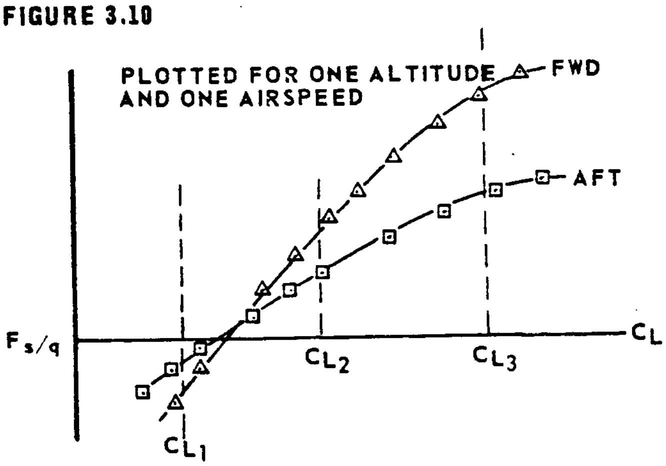
Stick-free neutral points are determined from data in the following manner with friction and breakout removed.

FIGURE 3.9



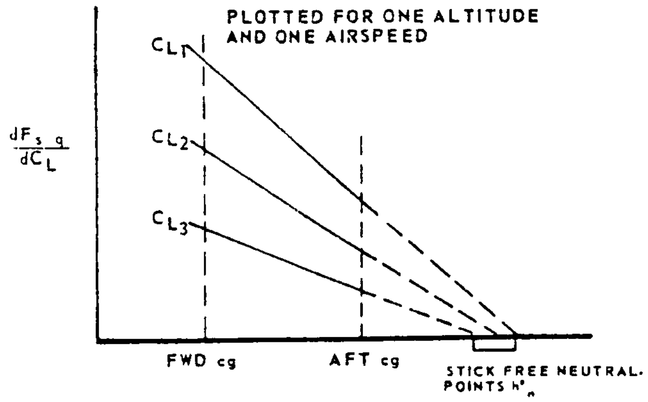
The derivative dF_s/dV_e is a function of aircraft trim as well as stability. This reduces the value of an extracted neutral point. When the stick force is divided by dynamic pressure, the derivative of this quantity, $\frac{dF_s/q}{dC_L}$, is a function of stability only and produces a more valid stick-free neutral point. (See Vol II, chapter 3, page 3.25.)

FIGURE 3.10



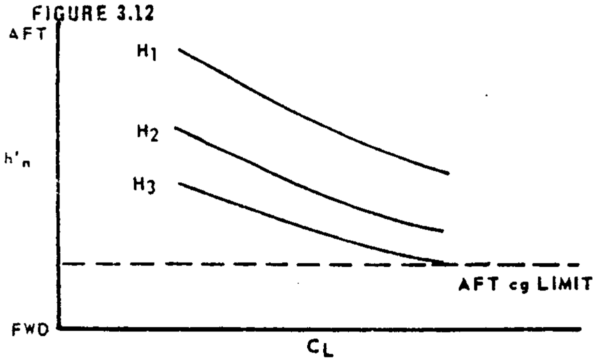
From the plot of F_s/q versus C_L, the rate of change of stick force with respect to lift coefficient is measured and the slope determined at three or more C_L's over the airspeed range for both cg loadings.

FIGURE 3.11



The point where $\frac{dF_s/q}{dC_L} = 0$ is the stick-free neutral point, h'_n, at that particular C_L. The neutral point movement with C_L for one altitude is curve H_1 in figure 3.12. Additional altitudes would plot as H_2 and H_3.

FIGURE 3.12



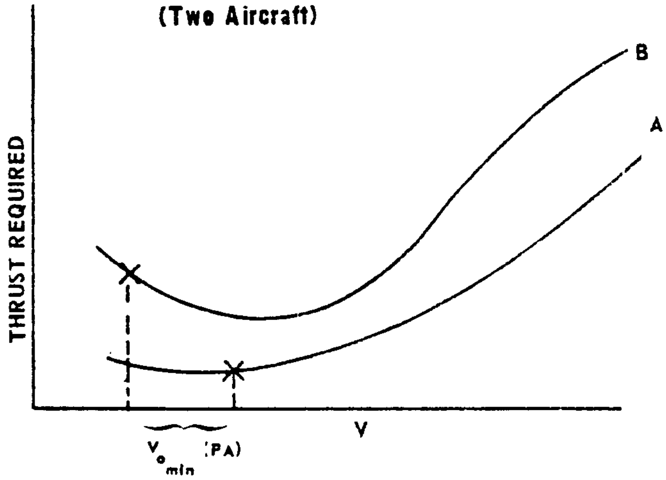
Neutral points vary with configuration, angle of attack, altitude, Mach number, and static elastic airframe distortion for a constant weight. No extension attempts should be made to locate transonic stick-fixed or stick-free neutral points because the cg would never be shifted to correct for transonic speed instability in any case. Also the dynamic stability is highly nonlinear with Mach number in this region and the neutral point concept has little utility other than to qualitatively specify an instability.

3.5 FLIGHT-PATH STABILITY

3.5.1 Definition

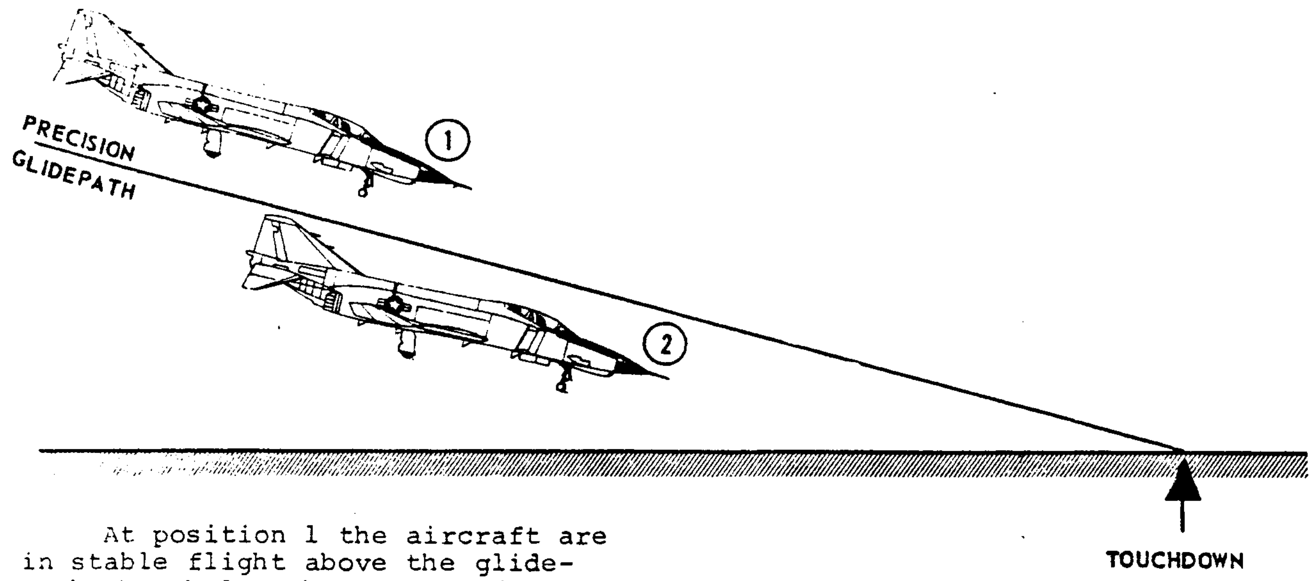
As stated earlier in this chapter, flight-path stability is a criterion applied to power approach handling qualities. It is primarily determined by the performance characteristics of the aircraft and related to stability and control only because it places another requirement on handling qualities. The following is one way to look at flight-path stability. Thrust required curves are shown for two aircraft with the recommended final approach speed marked.

Figure 3.13 THRUST REQUIRED vs VELOCITY (Two Aircraft)



If both aircraft A and B are located on the glidepath shown below their relative flight-path stability can be shown.

Figure 3.14 AIRCRAFT ON PRECISION APPROACH



At position 1 the aircraft are in stable flight above the glidepath, but below the recommended final approach speed. If aircraft A is in this position the pilot can nose the aircraft over and descend to glidepath while the airspeed increases. Because the thrust required curve is flat at this point, the rate of descent at this higher airspeed is about the same as before the correction, so he does not need to change throttle setting to maintain the glidepath. Aircraft B, under the same conditions, will have to be flown differently. If the pilot noses the aircraft over, the airspeed will increase to the recommended airspeed as the glidepath is reached. The rate of descent at this power setting is less than it was before so the pilot will go above glidepath if he maintains this airspeed.

At position 2 the aircraft are in stable flight below the glidepath but above the recommended airspeed. Aircraft A can be pulled up to the glidepath and maintained on the glidepath with little or no throttle change. Aircraft B will develop a greater rate of descent once the airspeed decreases while coming up to glidepath and will fall below the glidepath again. If the aircraft are in position 1 with the airspeed higher than recommended instead of lower, the same situation will develop when correcting back to flight-path, but the required pilot compensation is increased. In all cases aircraft A has better flight-path stability than aircraft B. As mentioned earlier in this chapter, aircraft which have unsatisfactory flight-path stability can be improved by increasing the recommended final approach airspeed or by adding an automatic throttle.

Another way of looking at flight-path stability is by investigating the difficulty that a pilot has in maintaining glidepath even when using the throttles. This problem is seen in large aircraft for which the time lag in pitching the aircraft to a new pitch attitude is quite long. In these instances, incorporation of direct lift allows the pilot to correct the glidepath without pitching the aircraft. Direct lift control will also affect the influence of performance on flight-path stability.

3.5.2 Test Method

Paragraph 3.2.1.3 of MIL-F-8785B(ASG) specifies the slope limits at two points in the flight-path angle (γ) versus true airspeed (V_T) plot. This plot is for the power approach flight phase at the normal glidepath with the throttle set at $V_{O_{min}}$ (PA). The steeper the glidepath, the more severe the tests, so 3 degrees is chosen as the steepest approach reasonable for a conventional aircraft precision approach. A steeper glidepath is appropriate for aircraft designed to approach in the STOL mode. The preflight planning involves selecting a test altitude so that $V_{O_{min}}$ (PA) indicated can be converted to V_T and the rate of descent (R/D) can be calculated to give the 3 degree glidepath. An altitude of approximately 10,000 feet MSL is usually selected as the mean altitude for the test. The pilot or engineer must compute the possible $V_{O_{min}}$ (PA) airspeeds for the aircraft gross weights that he will test. He converts these to true airspeeds and calculates the approximate R/D required to get a 3 degree glidepath in each case.

The pilot begins the test by configuring the aircraft for the power approach configuration at $V_{O_{min}}$ (PA) at about 12,000 feet MSL. Using a modified back side trim technique, the pilot reduces the power, maintaining airspeed, until the R/D stabilizes at the aim R/D $\pm$100 feet per minute. He hand records R/D, altitude, and airspeed for this point.

The pilot then raises the nose of the aircraft to slow it down about 5 knots and lowers the nose to an attitude that will hold this new airspeed. It is imperative that this airspeed be held within 1/2 knot so that the R/D will settle quickly. Once the R/D stabilizes, the pilot records the R/D, altitude, and airspeed again and repeats the procedure for another decrease of 5 knots. He then noses over to get a point at $V_{O_{min}}$ (PA) +5 knots and finally $V_{O_{min}}$ (PA) again.

At least four points are required to define the slope at $V_{O_{min}}$ (PA) and $V_{O_{min}}$ (PA) -5 knots. These points should be obtained quickly to avoid excessive altitude loss because changes in altitude affect the R/D for a given airspeed with the throttle fixed. Repeating the $V_{O_{min}}$ (PA) point at the bottom of the band allows a correction factor to be applied. In any case the data band should not exceed 2,000 - 3,000 feet.

If an aircraft appears to have marginal or unsatisfactory flight-path stability, the test method listed above, using hand recordings of standard aircraft instruments, will not be sufficiently accurate. More sophisticated types of instrumentation must be employed. Ground-based measurements are not practical because winds would affect the data and it would be difficult to account for this effect. Although untried as of this writing, the two most promising methods would be to fly the test over water and use the differentiated output or a radar altimeter to compute flight-path angle or to fly the test anywhere and integrate the output of a vertically mounted accelerometer.

3.5.3 Data Reduction

The first step in data reduction is to convert the airspeeds from indicated to true airspeed. The temperature at the flight level can be obtained from meterological data. Before using the R/D and V_T to compute the flightpath angle, a correction must be applied to account for the changes in R/D with altitude. This correction assumes a linear variation of R/D with altitude. Apply the following relationships for this correction:

R/D_1 = R/D at $V_{o_{min}}$ initial

R/D_2 = R/D at $V_{o_{min}}$ final

h_1 = altitude at $V_{o_{min}}$ initial

h_2 = altitude at $V_{o_{min}}$ final

$\Delta h = h_1 - h_x$ where x is the altitude at a test point

$$\Delta R/D_x = \frac{\Delta h}{h_1 - h_2} (R/D_1 - R/D_2)$$

$$R/D_{x_{corr}} = R/D_x + \Delta R/D_x$$

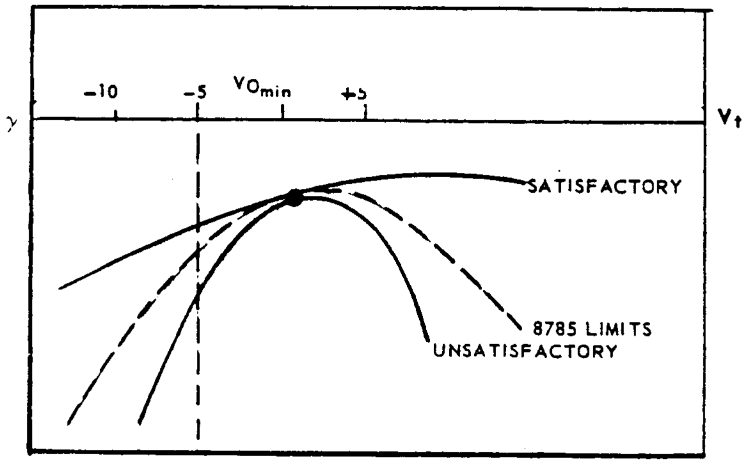
This was used in the following relationship:

$$\gamma = \text{flight path angle} = \sin^{-1} \frac{R/D_{corr}}{V_{True}}$$

Example Data Reduction

Temp (deg C)	CAS (kts)	TAS (kts)	R/D (ft/min)	ALT (ft)	ΔR/D (ft/min)	R/D_c (ft/min)	TAS (ft/min)	γ (deg)
-5	170	202	1,000	11,000	0	1,000	20,000	2.80
-4	164	192	800	10,300	+70	870	19,500	2.56
-3	160	186	700	9,900	+110	810	18,900	2.48
+0	170	199	1,000	8,800	+220	1,220	20,200	3.08
+1	170	194	750	8,500	-	-	-	-

Figure 3.15 SAMPLE DATA PLOT FLIGHT PATH STABILITY



CHAPTER 4 POST-STALL/SPIN FLIGHT TEST TECHNIQUES

REVISED FEBRUARY 1977

4.1 INTRODUCTION

This chapter summarizes the preparation for post-stall/spin investigations which test pilots assigned to such projects must undergo. The discussion is purposely general in nature· it will not specifically address the tests flown in the curriculum at the USAF Test Pilot School. These flights are described in detail by reference 1.

At this writing there are drastic philosophical and procedural changes being made in stall, post-stall, and spin testing within the USAF. The thrust of these changes is to provide simpler, more effective techniques for avoiding out-of-control conditions for all classes of aircraft and to force design of better high angle of attack capability into new weapons systems. References 2 and 3 provided some of the impetus for these changes and reference 4 has recently been revised to reflect this emphasis on spin prevention as opposed to spin recovery. Reference 5 has been replaced by reference 6. Clearly, such a shift in emphasis dictates careful attention to planning and execution of flight tests in an obviously hazardous area of flight testing.

4.2 SPIN PROJECT PILOT'S BACKGROUND REQUIREMENTS

4.2.1 PRELIMINARY DATA STUDY.

In previous spin test programs, the contractor was required to demonstrate satisfactory spin recoveries prior to the tests conducted by military project pilots. Under the current stall/post stall/spin demonstration specification (reference 6, paragraph 3.3) military project pilots will participate in high angle of attack investigations concurrently with the contractor's pilots. With this change in philosophy, it is imperative that the military test pilot assigned to a high angle of attack investigation thoroughly study all available background information.

Literature research should begin with the best and most current wind tunnel data available. Take careful note of any configuration or mass changes which were made since the available wind tunnel data were obtained. Look questioningly at the angle of attack and angle of sideslip ranges tested in the tunnel. Go over this data very carefully with the flight test engineers and try to ascertain the probable spin modes and optimum recovery techniques for each of them, as well as the optimum recovery procedure for post-stall gyrations if one is known. Start looking, even at this stage, for the simplest recovery technique possible. If possible, obtain analytical data to confirm or deny the possibility of using a common recovery procedure for both post-stall gyrations and spins (reference 6, paragraph 3.4.3). Spin test reports of similar aircraft should be reviewed thoroughly, but care must be exercised in extrapolating results. The spin characteristics of aircraft which are quite similar in appearance can vary drastically. Attempt to predict the effect that various loadings and configurations will have on post-stall/spin characteristics so that initial tests can be planned conservatively. As examples, the A-1 has loadings from which recovery is not acceptable (reference 7, page 6); and highly asymmetric loadings in the A-7D may prolong recovery to an unacceptable degree. Flight tests of the A-7D were not performed with loadings of greater than 13,000 foot-pounds of asymmetry (reference 8, page 11).

4.2.2 PILOT PROFICIENCY:

It is imperative that the test pilot engaged in a post-stall/spin test program have recent experience in stalls and in spinning aircraft as similar as possible to the test aircraft. Obviously, such aircraft should be those cleared for intentional spins. Coupled departures in a mildly spinning aircraft may be helpful in simulating the post-stall gyrations of an aircraft not cleared for intentional departures. Lack of spin practice for as little as three months will reduce the powers of observation of even the most skilled test pilot. Therefore, he should practice until he is at ease in the post-stall/spin environment immediately prior to commencing the data program. Centrifuge rides, with simulated instrumentation procedures and required data observations, can also be useful.

4.2.3 CHASE PILOT/AIRCRAFT REQUIREMENTS:

A highly qualified chase pilot in an aircraft compatible with the test aircraft increases the safety factor and adds another observer. The chase pilot should participate fully in the preparation phase. In fact it is preferable that more than one pilot be assigned to a given project. Not only does such an arrangement permit more than one qualitative opinion, but by alternating between post-stall/spin and chase assignments, each pilot gets at least two viewpoints. He can evaluate the post-stall/spin characteristics both as an in-the-cockpit observer and from the somewhat more detached chase position. Of course, from a flying safety viewpoint the benefits a competent chase pilot offer are obvious and immeasurable. In order to be useful, the chase pilot should be in an airplane with performance compatible with that of the test aircraft. His responsibilities include: staying close enough to observe and photograph departures, post-stall gyrations, and any spins; staying out of the way of an uncontrollable test aircraft; and being immediately in position to check any unusual circumstances, such as lost panels, malfunctioning drag/spin chutes, or control surface positions. And, of course, if necessary, he can call out canopy jettison/ejection altitudes. All these responsibilities point up the importance of a well-prepared, observant chase pilot in a similar aircraft.

4.3 DATA REQUIREMENTS

It would be presumptuous to suggest that a comprehensive set of required data can (or even should) realistically be set down within the space limitations in a manual intended as an instructional text for experimental test pilots. However, it is essential that a test pilot have at least a general idea of what parameters must be recorded. Hence, the following two paragraphs are intended to provide only general guidance. Naturally, the test plan for the specific project must be consulted for more detailed and specific requirements.

.3.1 DATA TO BE COLLECTED:

The flight test engineer will be highly concerned primarily with the required quantitative data. Rates of pitch, roll, and yaw, angular accelerations about each axis, control surface positions, angle of attack, indicated airspeed, and altitude are but a few of the typical time histories plotted meticulously by engineers. For the pilot, these data are not the main concern; they are available through automatic recording devices. His most important data gathering lies in a more qualitative area. Can all the necessary controls and switches be reached easily? What are the cockpit indications on production instruments of loss of control warning, departure, post-stall gyration, and spins? Can these indications be readily interpreted, or is the pilot so disoriented that he could not determine what action to take? What visual cues are available at critical stages of the recovery? Reference 8 gives an appropriate example of such a critical stage in the A-7D recovery sequence:

> On several occasions during recovery from fully developed spins, yaw rotation slowed, AOA decreased below 22 units, and roll rotation increased prior to release of anti-spin controls. Pilots found it easy to confuse roll rate for yaw rate leading to the "Auger" maneuver defined as rolling at unstalled AOA with anti-spin controls.

This sort of qualitative finding can be and usually is the most important kind of result from a spin test program. Hence, it is poor practice to ask the pilot to neglect cockpit observations to gather quantitative data which should be recorded by telemetry or on-board recording devices. Project pilots must guard against this pilot overload by looking carefully at the available instrumentation, both airborne and ground-based.

4.3.2 FLIGHT TEST INSTRUMENTATION:

The scope of the post-stall/spin test program will determine the extent of the instrumentation carried on board the aircraft. A qualitative program with a limited objective may require virtually no special instrumentation (reference 9, page 1); while extensive instrumentation may be mandatory for a full-blown stall/post-stall/spin investigation. Table I shows typical on-board instrumentation for a complete evaluation. Of course, this instrumentation is not appropriate for every investigation; each program is a special case.

The prospective test pilot should particularly note the kinds of parameters to be displayed in the cockpit. In this area he must protect his own interests by assuring that the indicators and controls available to him are complete, but that they do not overload his capacity to observe and to safely recover the aircraft. Simulations, preferable under stress of some kind (in a centrifuge, for example), may help the pilot decide whether or not the cockpit displays and controls are adequate.

Finally, reference 6, paragraph 6.4.2.3, directs preparation of a technical briefing film and suggests that an aircrew training film may be produced at the option of the procuring activity. Usually, it is advisable to have one or more movie cameras mounted on or in the test

Table I

TYPICAL FLIGHT TEST INSTRUMENTATION

Parameter	Time History[1]	Photopanel[2]	Pilot's Panel
Angle of attack	X		X
Production angle of attack	X	X	X
Angle of sideslip	X		
Swivel boom airspeed	X	X	X
Swivel boom altitude	X	X	X (coarse altimeter)
Production airspeed	X	X	
Production altitude	X	X	
Bank angle	X		
Pitch angle	X		
Pitch rate	X		
Roll rate	X		
Yaw rate	X		
Normal acceleration	X		X (sensitive indicator)
Accelerations at all crew stations	X		
All control surface positions	X		
Stick and rudder positions	X		
Stick and rudder forces	X		
All trim tab positions	X		
SAS input signals	X		
Engine(s) oil pressures	X	X	X
Hydraulic pressures	X	X	X
Fuel used (each tank)	X	X	X
Film, oscillograph, or tape-correlation and amount remaining	X	X	X
Event marker	X	X	X
Spin turn counter	X	X	X
Elapsed time	X	X	
Critical structural loads	X		X
Pilot warning signal(s)[3]			X
Emergency recovery device indicators	X	X	X

[1] Oscillographs, magnetic tape, telemetry.

[2] May not be necessary if time histories are complete and reliable.

[3] Pilot warning signals may include maximum yaw rate indicators, spin direction indicators, minimum altitude indicators, and other such devices to help lower the pilot's workload. They may take the form of flashing lights, horns, oversized indicators, etc.

aircraft to provide portions of this photographic coverage. Motion pictures taken over the pilot's shoulder may provide visualization of the departure motion, readability of production instruments, information about the adequacy of the restraint system, and other similar data. A movie camera taking pictures of the control surface positions can produce dramatic evidence of the effectiveness or lack of effectiveness of recovery controls. These cameras and recording devices should be made as "crash-proof" or at least as "crash-recoverable" as possible.

Further information on flight test instrumentation, cockpit displays, and cameras may be found in paragraphs 3.2.2, 3.2.3, and 3.2.4 of reference 6.

4.4 SAFETY PRECAUTIONS

Stall/post-stall/spin test programs are usually regarded with suspicion by program managers and flying supervisors. This suspicion is not altogether unreasonable since many such investigations have resulted in the loss of expensive, highly instrumented test aircraft and crew fatalities. This is a fact that awakens the prospective post-stall/spin test pilot's sensibilities, and the fact that such testing is hazardous is not questioned. However, careful attention to detail in several areas will at least minimize the dangers involved.

4.4.1 CONSERVATIVE APPROACH:

One of the most important ways to minimize hazards in such a program is to incrementally expand the areas of investigation, choosing safe increments until the aircraft's uncontrolled motions are better understood. Such a conservative approach is suggested in paragraph 3.4. of reference 6, but how can the test pilot help plan to assure that such an approach is actually followed?

First, the entire program is usually broken down into phases. Even the terms now in use - stall/post-stall/spin - suggest the basic phases of such an investigation, although in practice the phases are generally broken down in more detail. Table II (from reference 6, page 4) lists the recommended phases for such investigations.

Within these phases there are several smaller steps to be taken with successive departures, post-stall gyrations, or spins. For example, aircraft loadings are normally changed gradually from clean to symmetric store loadings to asymmetric store loadings. The effects of these loading changes must be evaluated both for the aerodynamic effects and the changes in mass distribution. Unfortunately, it is not often obvious which effect is most damaging until after the tests are completed. One would also be ill-advised to use full-pro-spin controls on the very first departure in phases B, C, or D. Delayed recoveries should be approached by sustaining the desired misapplication of controls in increments in each successive departure up to the maximum of 15 seconds as indicated in phase D. Such conservatism in flying these tests is essential and must be adhered to scrupulously. However, it is also necessary to consider aircraft systems in order to plan a safe post-stall/spin program.

Table II

TEST PHASES

Phase	Control Application
A - Stalls	Pitch control applied to achieve the specified AOA rate, lateral-directional controls neutral or small lateral-directional control inputs as normally required for the maneuver task.
	Recovery initiated after the pilot has a positive indications of: (a) a definite g-break or (b) a rapid angular divergence, or (c) the aft stick stop has been reached and AOA is not increasing.
B - Stalls with aggravated control inputs	Pitch control applied to achieve the specified AOA rate, lateral-directional controls as required for the maneuver task. When condition (a), (b), or (c) from above has been attained, controls briefly misapplied, intentionally or in response to unscheduled aircraft motions before recovery attempt is initiated.
C - Stalls with aggravated and sustained control inputs	Pitch control applied to achieve the specified AOA rate, lateral-directional controls as required for the maneuver task. When condition (a), (b), or (c) has been attained, controls are misapplied, intentionally or in response to unscheduled aircraft motions, and held for 3 seconds before recovery attempt is initiated.
D - Spin attempts (this phase required only for training aircraft which may be intentionally spun and for Class I and IV aircraft in which sufficient departures or spins did not result in Test Phase A, B, or C to define characteristics.)	Pitch control applied abruptly, lateral-directional controls as required for the maneuver task, when condition (a), (b), or (c) has been attained, controls applied in the most critical positions to attain the expected spin modes of the aircraft and held up to 15 seconds before recovery attempt is initiated, unless the pilot definitely recognizes a spin mode.

4.4.2 DEGRADED AIRCRAFT SYSTEMS:

All systems are under an often unknown amount of strain during high angle of attack maneuvering. If the aircraft goes out of control in this flight regime, system design limits may well be exceeded. The propulsion/inlet system is often not designed to allow reliable operation of the engines during extreme angles of attack and sideslip. Of course, if the engine(s) flame out, this failure may result in loss of control in modern aircraft with hydraulic flight control systems. Obviously, the test vehicle is a post-stall/spin program must have an alternate source of hydraulic power for the flight controls if there is even a hint that engine flameouts are likely to occur. However, do not overlook the behavior of the production hydraulic system: loss of production hydraulic pressure may be all that is necessary to prohibit intentional spins. In propeller-driven aircraft the hydraulic power used to govern the propeller pitch can also be a limiting factor, particularly during inverted spins. The electrical system may also be affected by engine flameout, and such a failure can render instrumentation inoperative at a critical time. In fact, even a momentary distruption of electrical power can destroy invaluable data. Hence, a reliable back-up electrical power source may be necessary. Other systems, such as the ejection system, pilot restraint system, or communications/nevigation system, may cause special problems during the post-stall/spin test program. The test pilot and test engineer must think through these special problems and where necessary add back-up systems to the test aircraft to assure safe completion of the program. Furthermore, any back-up systems that are required must not limit the range and scope of the tests; otherwise, they defeat their purpose.

4.4.3 EMERGENCY RECOVERY DEVICE:

The ultimate back-up system, some sort of emergency recovery device, is so important that it deserves a paragraph all its own. Failure of this "last-ditch" system has in the past contributed to the discomfort of test pilot, engineer, and SPO director all to often. Reference 3 suggests that more attention must be given to the design of this system, perhaps to the extreme of making emergency recovery system components government-furnished equipment (GFE). While the feasibility of this rather drastic suggestion is questionable, it is imperative that more reliable systems be designed. Some of the things that must be scrutinized by the test pilot are:

1. Has the deployment/actuation mechanism demonstrated reliability through the expected envelope of dynamic pressures?

2. Are the moments generated large enough for all predicted spin rates?

3. Has the jettison mechanism demonstrated reliability throughout the expected envelope?

4. Are maintenance inspection procedures adequate for this sytem? (This system should be checked just prior to takeoff.)

5. Does the emergency recovery system grossly alter the aerodynamic and/or inertia characteristics of the test aircraft?

Obviously, no such list is complete, but the test pilot must carefully evaluate every component of the emergency recovery system: spin chute, spin rockets, or any other device.

4.5 SPECIAL POST-STALL/SPIN TEST FLYING TECHNIQUES

Stall flight test techniques have been thoroughly discussed in Chapter II and in reference 6, consequently, the remainder of this chapter is devoted exclusively to flying techniques peculiar to post-stall phenomonea. In general, the test pilot must have indelibly fixed in mind what control actions he will take when the first departure occurs. An inadvertent departure can give just as meaningful (perhaps more meaningful) data as an intentional one - if the test pilot overcomes his surprise quickly enough to make preplanned and precise control inputs. The keys to avoiding confusion in the cockpit have already been mentioned, but they bear repeating. The test pilot must be recently proficient in post-stall gyrations and in spinning, and he must be so familiar with the desired recovery controls that they are second nature. Apart from overcoming the surprise factor through adequate preparation, the test pilot may need some other tricks in this highly specialized trade. For instance, entry to a desired out-of-control maneuver can be a very hit-and-miss proposition.

4.5.1 ENTRY TECHNIQUES:

4.5.1.1 Upright Entries.

For aircraft susceptible or extremely susceptible to spins, an upright spin may be easy to attain. In this case the test pilot's main concern may be how to produce repeatable characteristics: that is, he may seek to achieve the same entry g-loading, attitude, airspeed, and altitude in successive spins so that correlation between spins is easier. Of course, if the aircraft is resistant to spins, it may still be susceptible to departure and entry into a post-stall gyration. In this case, correlation of the data may be even more difficult since the random motions of a PSG seldom are repeatable. Again, the attempt usually is to achieve repeatable entry conditions so that over a large statistical sample the characteristics of the PSG become clear. Achieving several departures with repeatable entry conditions is one of the more demanding piloting tasks. Considerable proficiency is required to achieve the AOA bleed rates or airspeed bleed rates specified in reference 6, page 5. Once the baseline characteristics for a given configuration are relatively well known, the test pilot is called on to simulate entries appropriate to the operational use of the aircraft.

4.5.1.2 Tactical Entries.

These entry maneuvers must be carefully thought out in light of the expected role of the aircraft. It is often wise to consult directly with the using command, particularly if the aircraft has already entered operational service. Of course, reference 6 does suggest the types of tactical entries listed in table III, but past experience is no substitute for foresight in planning such tests. By carefully examining the tactics envisioned by operational planners, the test pilot should be able to recognize other possible tactical entries which may cause difficulty in the high angle of attack flight regime.

Table III

TACTICAL ENTRIES

1.	Normal inverted stalls (see paragraph 4.5.1.3).
2.	Aborted maneuvers in the vertical plane (vertical reversals, loops, or Immelmanns).
3.	High pitch attitudes (above 45 degrees).
4.	Hard turns and breaks as used in air combat maneuvering.
5.	Overshot roll-ins as for ground attack maneuvering.
6.	High-g supersonic turns and/or transonic accelerations/decelerations.
7.	Sudden idle power and/or speed brake decelerations.
8.	Sudden asymmetric thrust transients prior to stall.

4.5.1.3 Inverted Entries.

Obtaining entries into inverted post-stall gyrations or spins can be very difficult simply because aircraft often lack the longitudinal control authority to achieve a stall at negative angles of attack. The most straightforward way to depart the aircraft in an inverted attitude is to roll inverted and push forward on the stick until stall occurs at the desired g-loading. But, many aircraft have marginal elevator authority and it is necessary to misapply the controls to obtain an inverted departure. Pulsing the rudder or applying other pro-spin controls as the nose drops can help precipitate departure. In the OV-10, for example, the direction of applied aileron determines the direction of the inverted spin - provided full aileron deflection is used. However, if aerodynamic controls lack authority, the test pilot can also use inertial moments to precipitate inverted departures.

How the inertial terms can aid entry into a spin can best be seen by examining the $\dot{q}$ equation.

$$\dot{q} = \frac{M_{aero}}{I_y} + pr\,\frac{(I_z - I_x)}{I_y}$$

If the negative pitching acceleration generated by $\frac{M_{aero}}{I_y}$ was too small to produce a stalled negative angle of attack, and additional negative pitching acceleration can be produced from: $pr\,\frac{I_z - I_x}{I_y}$. All that is necessary is for p and r to have opposite signs. Typically, the roll momentum is built up by rolling for at least 180 degrees opposite to the desired direction of the inverted spin and then applying full pro-spin controls at the inverted position. Obviously, these control manipulations must be made at an angle of attack near the stall. Sometimes it is even advisable to apply a slight amount of rudder opposite to the roll during the roll momentum buildup period. A typical procedure designed to produce a left inverted spin is given below:

1. Establish a nose high pitch attitude.

2. Apply full right aileron and a slight amount of left rudder.

3. After a minimum of 180 degrees of roll (360 degrees or more may be advantageous in some aircraft), apply full left rudder, maintain full right aileron, and full forward stick (on some aircraft full aft stick may be used).

4. Recover using predicted or recommended recovery procedures.

Of course, this procedure must be modified to fit the characteristics of a particular aircraft, but it does illustrate the kind of control manipulation sometimes required in post-stall/spin investigations. Some aircraft will not enter an inverted spin using this sort of exaggerated technique, but using the inertial moments to augment aerodynamic controls has uncovered spin modes not obtained by other means. Reference 10 provides further information on the subject of inverted spinning.

4.5.2 RECOVERY TECHNIQUES:

4.5.2.1 Out-of-control Recoveries.

The underlying principle of all recovery techniques is simplicity (refer to paragraph 3.4.2 of reference 6). The procedure to be used must not require the pilot to determine the nature or direction of the post-stall gyration. In fact paragraph 3.4.2.2.2 of reference 4 requires recovery from both post-stall gyrations and incipient spins using only the elevator control. Engine deceleration effects must be tested. Any part of the flight control system (the SAS, for example) which hinders desired control surface placement must be identified and carefully evaluated. Care must be taken to ensure that the recovery controls recommended to recover from a post-stall gyration will not precipitate a spin. The test pilot is primarily responsible for identifying reliable visual and cockpit cues to distinguish between post-recovery angular motions (steep spirals, rolling dives) and the post-stall gyration. Taken together, these requirements demand that the test pilot be a careful observer of the motion. In fact, he is likely to become so adept at making these observations that he must guard against complacency. His familiarity with the motions may cause him to over-estimate the operational pilot's ability to cope with the out-of-control motions. Paragraph 3.4.2.2.2 of reference 4 specifies that the start of the recovery shall be apparent to the pilot within 3 seconds after initiation of recovery. This requirement is very stringent and will require very fine judgement on the part of the test pilot.

4.5.2.2 Spin Recoveries.

The criteria for recovery from a spin are specified in paragraph 3.4.2.2.2 of reference 4 and outlined in table IV. These criteria are applicable to any spin modes resulting from any control misapplication specified in reference 6. Timing of control movements should not be critical to avoiding spin reversals or an adverse mode change.

Table V outlines the so-called NASA Standard and NASA Modified recovery procedures. These recoveries are by no means optimum for all aircraft and they must not be construed to be. In contrast, the F-4E recovery technique now includes forward stick, which reflects the philosophy of simplifying out-of-control recovery procedures. Generally, forward stick is desirable for recovery immediately following a departure. The reason for retaining the forward stick is to keep the out-of-control recovery procedure like the spin recovery procedure. However, individual aircraft characteristics may dictate that out-of-control recovery procedures differ from spin recovery procedures. Such characteristics violate the specifications of reference 4 and 6, but the test pilot must evaluate the need for two recovery procedures. He cannot assume that any "canned" recovery procedure will work nor that the design meets the specifications. In summary, the test pilot's job is to assure that the operational pilot has a simple, reliable recovery procedure which will consistently regain controlled flight.

Table IV

RECOVERY CRITERIA

Class	Flight Phase	Turns for Recovery
I	Category A, B	1-1/2
I	PA	1
IV	Category A, B	2-1/2

[1]Not including dive pullout.

Table V

RECOVERY TECHNIQUES

NASA Standard	NASA Modified
(If ailerons were held during spin, neutralize)	Same
A. Full opposite rudder	A. Full opposite rudder and at the same time ease stick forward to neutral.
B. Stick full aft	B. Neutralize rudder when rotation stops.
C. When rotation stops - neutralize rudder (immediately)	
D. EASE stick forward to approximately neutral position.	

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4. MIL-F-8785B(ASG), Flying Qualities of Piloted Airplanes, 7 August 1969 (including Interim Amendment 1 (USAF), dated 31 March 1971).

5. MIL-S-25015(USAF), Spinning Requirements for Airplanes, 18 August 1954 (including Amendement 2, dated 22 May 1957).

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CHAPTER 5 MANEUVERABILITY

5.1 INTRODUCTION

The purpose of maneuvering flight is to determine the stick force versus load factor gradients and the forward and aft center of gravity limits for an aircraft in accelerated flight conditions.

To maneuver an aircraft longitudinally from its equilibrium condition, the pilot must apply a force, F_s, on the stick to deflect the elevator an increment, $\Delta\delta_e$. The requirements that must be met during longitudinal maneuvering are covered in MIL-F-8785B(ASG), hereafter referred to as MIL-F-8785, Section 3.2.2.

5.2 MIL-F-8785

MIL-F-8785 specifies the allowable stick/wheel force per "g" gradient during maneuvering flight. It also specifies that the stick/wheel force gradients be approximately linear with pull forces on the stick/wheel required to maintain or increase normal acceleration. The pilot must also have sufficient aircraft response without excessive cockpit control movement. These requirements and associated requirements of lesser importance provide the legitimate background for good aircraft handling qualities in maneuvering flight.

The backbone of any discussion of maneuvering flight is stick/wheel force per "g". The amount of stick/wheel force that the pilot must apply to maneuver his aircraft is an important parameter. If the force per "g" is very light, a pilot could overstress or overcontrol his aircraft with very little resistance from the aircraft. The T-38, for instance, has a 5 lb/g gradient at 25,000 feet, Mach 0.9, and 20 percent MAC cg position. With this condition, a ham-fisted pilot could pull 10 g's with only 50 lbs of force and bend or destroy the aircraft. The designer could prevent this possibility by making the pilot exert 100 lb/g to maneuver. This would be highly unsatisfactory for a fighter type aircraft, but perhaps about right for a cargo type aircraft. The mission and type of aircraft must therefore be considered in deciding upon acceptable stick/wheel force per "g". Furthermore, the gradient of stick/wheel force per "g" at any normal load factor must be within 50% of the average gradient over the limit load factor. If it took 10 lb to achieve a 4 g turn, it would be unacceptable for the pilot to reach the limit load factor of 7.33 g's with only a little additional force.

The position of the aircraft's cg is a critical factor in stick/wheel force per "g" consideration. The fore and aft limits of cg position may therefore be established by maneuvering requirements.

5.3 EXAMPLE TEST METHODS

Generally speaking, there are four flight test methods for determining maneuvering flight characteristics such as stick force gradients, maneuver points, and permissible cg locations. The names given to these different methods may vary among test organizations. Therefore, care should be exercised when discussing a particular test method to make certain that everyone involved is speaking the same language.

Stabilized g Method:

This method requires holding a constant airspeed and varying the load factor. The aircraft is trimmed at the test altitude for hands-off flight, and a trim shot is taken. The power setting is then noted, and the aircraft is climbed to the upper limit of the altitude band ($\pm$2,000 feet). The power is then reset to trim power and the aircraft is slowly rolled into a 15-degree bank while the nose is lowered slowly. Data is recorded when the aircraft has been stabilized on an airspeed and bank angle with no stick movements. The attitude indicator should be used to establish the bank angle. The bank angle is then increased to 30 degrees and data is again recorded when the aircraft has been stabilized. Stabilized data points are also obtained at bank angles of 45 and 60 degrees.

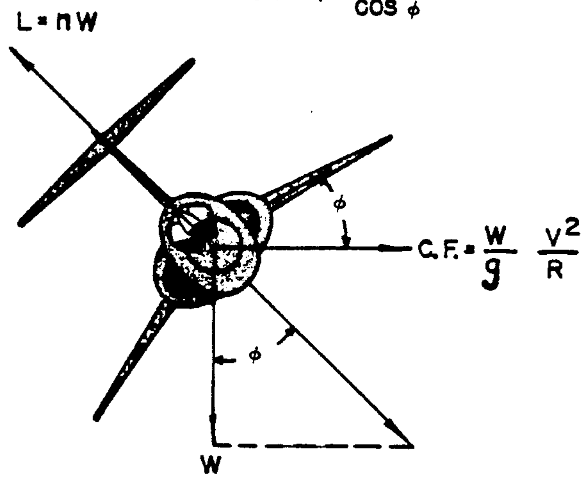
After taking data at the 60-degree bank angle, the bank angle should be increased so as to obtain 0.5 g-increments in load factor. At each 0.5 g-increment, the aircraft is stabilized and data recorded. The test should be terminated when heavy buffet or the limit load factor is reached. Above a load factor of 2.0, only slight increases in bank angle are needed to obtain 0.5-g increments. An idea of the approximate bank angle required can be reached by exploiting the relationship between load factor and bank angle for a constant altitude (figure 5.1).

Figure 5.1 LOAD FACTOR vs BANK ANGLE RELATIONSHIP

W = GROSS WEIGHT
n = LOAD FACTOR
ϕ = BANK ANGLE

$$\cos\phi = \frac{W}{nW} = \frac{1}{n}$$

OR $n = \dfrac{1}{\cos\phi}$



Little altitude is lost at the lower bank angles up to approximately 60 degrees, and thus more time may be spent stabilizing the aircraft. At 60 degrees of bank angle and beyond, altitude is being lost rapidly; therefore every effort should be made to be on speed and well stabilized as rapidly as possible in order to stay within the allowable altitude block (test altitude $\pm$2,000 feet). If the lower altitude band is approached before reaching limit load factor, the aircraft should be climbed to the upper limit and the test continued. No attempt should be made to obtain data at exact values of g since a good spread is all that is necessary.

The method of holding airspeed constant within a specified altitude band is recommended where Mach number is not of great importance. In regions where Mach number may be a primary consideration, every effort should be made to hold Mach number and airspeed constant.

If power has only a minor effect on the maneuvering stability and trim, altitude loss and the resulting Mach number change may be minimized by adding power as load factor is increased. At times, constant Mach number is held at the sacrifice of varying airspeed and altitude. For constant Mach number tests, a sensitive Mach meter is required or a programmed airspeed/ altitude schedule is flown. The stabilized g method is usually used for testing bomber and cargo aircraft and fighters in the power approach configuration.

Slowly Varying g Method:

The aircraft is trimmed as before at the desired altitude. The power is noted and the aircraft is climbed to the upper limit of the altitude band (±2,000 feet). Power is reset at the trimmed value. The data recording switch is activated and the aircraft is slowly banked into a turn. With the airspeed held constant, load factor is slowly increased by increasing bank angle and descending. The slow increase of bank angle and the resulting load factor increase continue until heavy buffet or limit load factor is reached. The rate of g onset should be approximately 0.1 g per second. Again airspeed is of primary importance and should be held to within ±1 knot of aim airspeed. Care should also be taken not to reverse stick forces during the maneuver.

If the airspeed varies excessively, or if the lower limit of the altitude band is approached, the data recorder should be turned off. The aircraft should then be restabilized at the upper altitude limit at a lower g loading. The maneuver should then be continued until heavy buffet onset or limit load factor is reached.

The greatest error made in this method is overbanking beyond 60 degrees of bank. Overbanking, causes the aircraft to traverse the g increments too quickly to be able to accurately hold airspeed. Good bank control is important to obtain the proper g rate of 0.1 g per second.

The slowly varying g method is more applicable to fighter aircraft. Often a combination of the two methods is used in which the stabilized g method is followed until a 60-degree bank angle is reached. The slowly varying g method is then followed from the 60-degree bank angle until heavy buffet or limit load factor is reached.

Constant g Method (Wind-Up Turn):

The aircraft is stabilized and roughly trimmed at the desired altitude and at maximum airspeed for the test. The aircraft is then placed in a constant g turn. Data recording is started and the aircraft is climbed or descended to obtain a 2 to 5 knot per second airspeed bleed rate at the desired constant load factor. Normally the aircraft is climbed to obtain a bleed rate at low load factors and descended to obtain a bleed rate at high load factors. For high thrust-to-weight ratio aircraft at low altitudes, the maneuver may have to be initiated at reduced power to avoid a too rapid traverse of the altitude band. Maintaining the aim load factor is the primary requirement while establishing the bleed rate is secondary. During the maneuver the aircraft should be kept within the altitude band of ±2,000 feet. The airspeed should be noted as the aircraft flies out of the altitude band. When the aircraft is returned to the altitude band, the maneuver is started at an airspeed above the just previously noted airspeed so that continuity of g and airspeed can be maintained for data purposes. Airspeed is again noted at buffet onset and the g break (when aim load factor can no longer be maintained). The buffet and stall flight envelope is determined or verified by this test method. The maneuver is then repeated at 0.5 g increments at high altitudes and 1-g increments at low altitudes.

<u>Symmetrical Pull-Up Method:</u>

The aircraft is trimmed at the desired test altitude and airspeed. The aircraft is then climbed to an altitude above the test altitude using power as required. Trim power is reset and the aircraft is pushed over into a dive. The dive angle is a function of the load factor to be applied (steeper angle for higher g values).

The aircraft is then maneuvered to reach a point, above the test altitude at a lead airspeed below the test airspeed, such that a "g pull" can be established that will place the aircraft at a given constant load factor while passing through the test altitude at the test airspeed. The lead airspeed is determined by the desired load factor – higher load factors and their resultant steeper dive angles require greater leads (lower lead airspeeds). The aircraft should pass through level flight (±15 degrees from horizontal) just as the airspeed reaches the trim airspeed with aim g loading and steady stick forces. Achieving the trim airspeed through level flight, ±15 degrees, and holding steady stick forces to give a steady pitch rate are of primary importance. The variation in altitude (±1,000 feet) at the pull-up is less important. The g loading need not be exact, but should be steady. Data is recorded as the aircraft passes through level flight ±15 degrees. The aircraft is then climbed to an altitude above the test altitude and the maneuver is repeated at another load factor at the same trim airspeed.

5.4 DATA REDUCTION METHODS

The maneuvering flight test is conducted at high, medium, and low altitudes at different airspeeds or Mach numbers throughout the flight envelope. The aircraft is flown with a forward and an aft cg. Oscillograph recordings give a readout of stick force, elevator deflection, angle of attack, and load factor obtained with the constant airspeed, varying g, flight test method. A sample data card is shown.

STUDENT FLIGHT RECORD — DATE:

NAME OF STUDENT: — NAME OF INSTRUCTOR:

AIRCRAFT NUMBER: T-38 596 — TEST: MAN FLT

CONFIGURATION: CRUISE – COMBAT AFT CG

PRESS ALTITUDE — RUNWAY TEMP — TAKEOFF ROLL — TAKEOFF V_1

GROUND BLOCK AT TAKEOFF	ALTIMETER (at 29.92)	TEMP	TOD	CAM NR	OSC NR

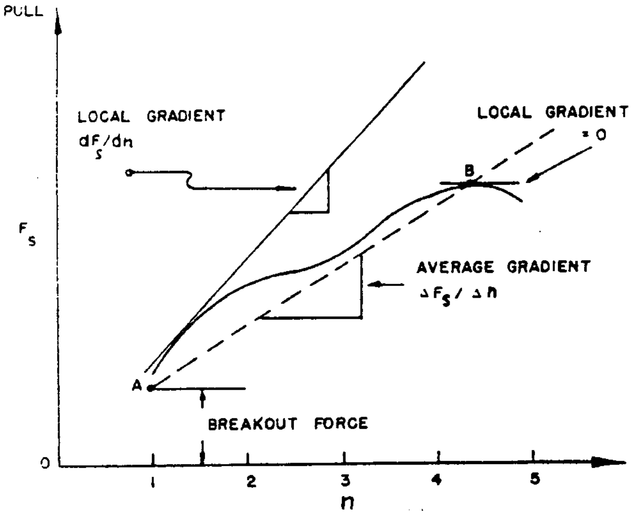
Ui	Hi	TRIM C/N	START C/N	END C/N	LEFT FUEL	RIGHT FUEL
.45	15M					
.60	"					
.75	"					
.66	"					
.75	25M					
.85	"					
.15	35M					
.85	35M					
1.1	"					

GROUND BLOCK AT LANDING	ALTIMETER (at 29.92)	TEMP	TOD	CAM NR	OSC NR

AFFTC FORM JAN 63 0-112 PREVIOUS EDITION OF THIS FORM WILL BE USED UNTIL STOCK IS EXHAUSTED.

The first step in data reduction is to plot stick force, F_s, against load factor for each test point. A sample plot is shown in figure 5.2. In this figure, the breakout force is determined and labeled Point A. Point B is located where the stick force curve becomes erratic. This point (approximately 85 percent of the limit load factor) may be defined by heavy buffet or change of sign of stick force gradient. The line connecting points A and B is the average gradient. The local gradient is the slope at any point along the curve.

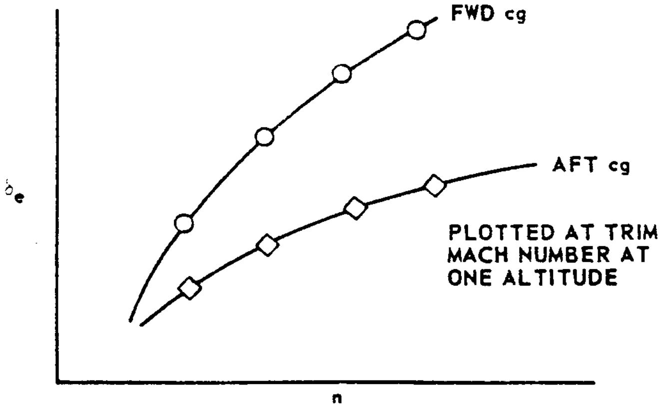
Figure 5.2 DETERMINATION OF AVERAGE GRADIENT FOR IRREGULAR CURVE OF F_s vs n



Stick-Fixed Maneuvering Flight:

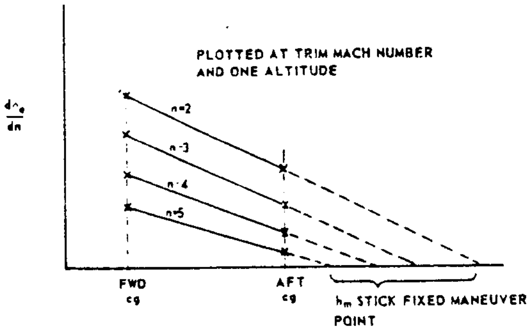
Elevator deflections, δ_e, are plotted versus load factor, n, for each trim airspeed at a particular altitude.

FIGURE 5.3 ELEVATOR DEFLECTION vs LOAD FACTOR



The slope, $d\delta_e/dn$, is determined for several load factors at the two cg positions and is plotted as shown in figure 5.4.

FIGURE 5.4 SLOPE OF ELEVATOR DEFLECTION PER LOAD FACTOR vs cg POSITION (WORKING PLOT)



The cg position where the slope $d\delta_e/dn$ is zero, is the maneuver point location for that load factor at the designated trim Mach and airspeed. Plots of δ_e vs n and $d\delta_e/dn$ vs cg yield the summary plots in Figure 5.5.

Stick-Free Maneuvering Flight:

Stick force, F_s, is plotted versus load factor, n, for each trim Mach number.

FIGURE 5.5
STICK-FIXED MANEUVER POINT
VARIATION WITH LOAD FACTOR

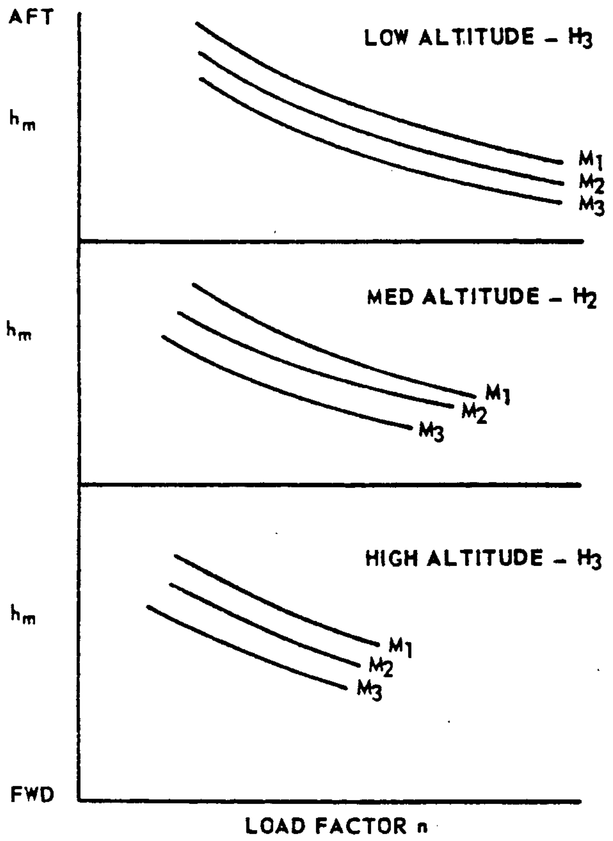
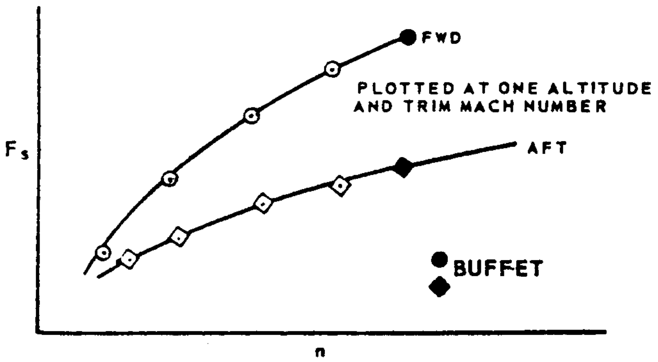


FIGURE 5.6
STICK-FORCE VERSUS LOAD FACTOR



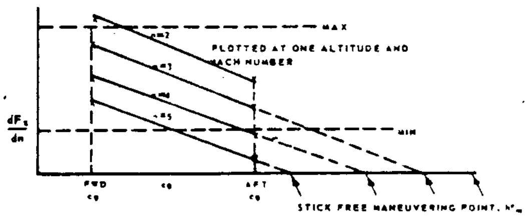
One plot for a single trim Mach, M_1, at a particular altitude, H_1, is shown in figure 5.6. The load factor at buffet onset should also be indicated.

It should be noted that the above curves do not necessarily go through zero stick force at one g. This is because there will usually be some breakout force required to allow movement of the longitudinal control.

The average stick force gradient should be found and the local gradient examined to determine whether or not the local gradient is within 50 percent of the average gradient.

In Figure 5.7 the plot of dF_s/dn versus cg position is derived from the stick force versus load factor curves to determine the stick-free maneuver points.

FIGURE 5.7
SLOPE OF STICK-FORCE PER LOAD FACTOR
VERSUS cg POSITION (WORKING PLOT)



The cg location where dF_s/dn equals zero is the stick-free maneuver point for that particular load factor at the given trim Mach and altitude.

Figure 5.8 is a plot of stick-free maneuver point variation with load factor. A cross plot of figure 5.8 can also be made to show how stick-free maneuver points vary with Mach number.

FIGURE 5.8 STICK FREE MANEUVER POINTS vs LOAD FACTOR

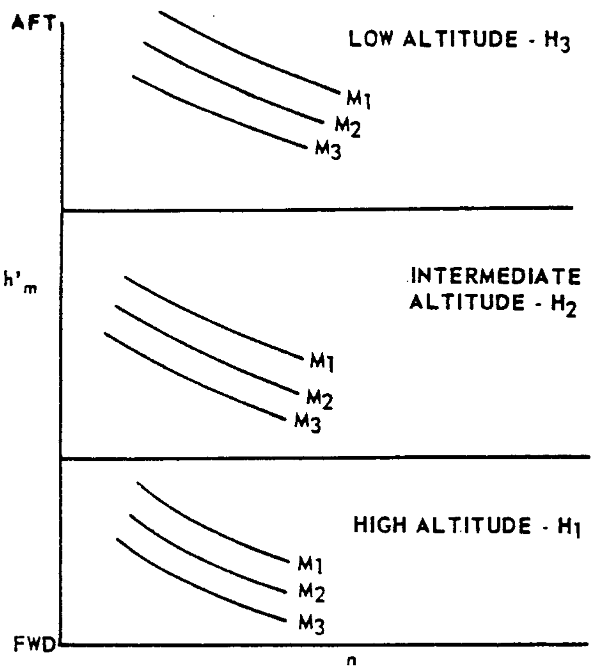


Figure 5.9 STICK FORCE PER "g" vs n_z/α

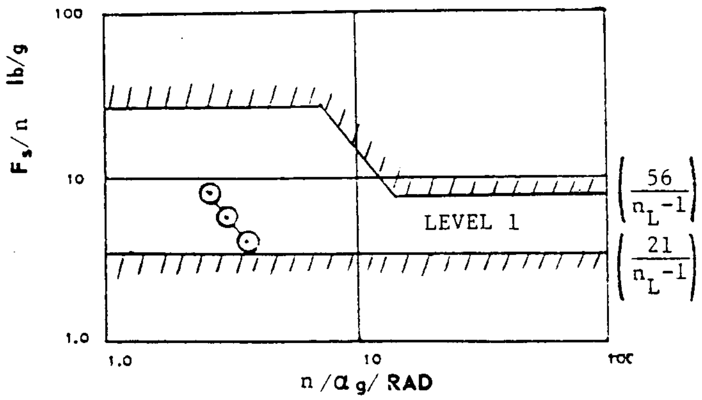


Figure 5.9 is extracted from Figure 11, page 75, MIL-F-8785B(ASG) Military Specification - Flying Qualities of Piloted Airplanes, Aug 69.

Stick Force Gradients:

It is not sufficient to say that local stick force per "g" gradients do not differ from the average gradient by more than 50%. MIL-F-8785 specifies minimum and maximum values of the local gradient which must be met. These gradients are specified in absolute numbers, $x/n_L - 1$, or in $x/n/\alpha$. Requirements for local gradients specified in terms of $x/n_L - 1$ or in absolute numbers can be determined from the data used to plot figure 5.2. To satisfy the requirements in terms of n/α, another type of plot is required. First form the ratio of n/α, in lb/radian.

From figure 5.2 determine the F_s/g gradient at the corresponding test point. Finally plot F_s/g vs n/α on a log-log plot as shown on Figure 5.9 to see if it lies within the MIL-F-8785 requirements.

CHAPTER 6

TRIM CHANGES

REVISED FEBRUARY 1977

6.1 PURPOSE

The purpose of this test is to ascertain the magnitude of control force changes associated with normal configuration changes, trim system failure, or transfer to alternate control systems in relation to specified limits. It must also be determined that no undesirable flight characteristics accompany these configuration changes.

6.2 TEST CONDITIONS

Pitching moments on aircraft are normally associated with changes in the condition of any of the following: landing gear, flaps, speed brakes, power, bomb bay doors, rocket and missile doors, or any jettisonable device. The magnitude of the change in control forces resulting from these pitching moments is limited by Military Specification F-8785, and it is the responsibility of the testing organization to determine if the actual forces are within the specified limits.

The pitching moment resulting from a given configuration change will normally vary with airspeed, altitude, cg loading, and initial configuration of the aircraft. The control forces resulting from the pitching moment will further depend on the aircraft parameter being held constant during the configuration change. These factors should be kept in mind when determining the conditions under which the given configuration change should be tested. Even though the specification lists the altitude, airspeed, initial conditions, and parameter to be held constant for most normal configuration changes, some variations may be necessary on a specific aircraft to provide information on the most adverse conditions encountered in operational use of the aircraft. The altitude and airspeed should be selected as indicated in the specifications or for the most adverse conditions. In general, the trim change will be greatest at the highest airspeed and the lowest altitude. The effect of cg location is not so readily apparent and usually has a different effect for each configuration change. A forward loading may cause the greatest trim change for one configuration change, and an aft loading may be most severe for another. For this reason, a mid cg loading is normally selected since rapid movement of the cg in flight will probably not be possible. If a specific trim change appears critical at this loading, it may be necessary to test it at other cg loadings to ascertain its acceptability.

Selection of the initial aircraft configurations will be dependent on the anticipated normal operational use of the aircraft. The conditions given in the specifications will normally be sufficient and can always be used as a guide, but again variations may be necessary for specific aircraft. The same holds true for selection of the aircraft parameter to hold constant during the change. The parameter that the pilot would normally want to hold constant in operational use of the aircraft is the one that should be selected. Therefore, if the requirements of MIL-F-8785 do not appear logical or complete, then a more appropriate test should be added or substituted.

In addition to the conditions outlined above, it may be necessary to test for some configuration changes that could logically be accomplished simultaneously. The force changes might be additive and could conceivably be objectionably large. For example, on a go-around, power may be applied and the landing gear retracted at the same time. If the trim changes associated with each configuration change are appreciable and in the same direction, the combined changes should definitely be investigated.

The specifications require that no objectional buffet or undesirable flight characteristics be associated with normal trim changes. Some buffet is normal with some configuration changes, e.g., gear extension; however, it would be considered objectionable in this case if this buffet tended to mask the buffet associated with stall warning. The judgment of the pilot is the best measure of what actually constitutes "objectionable" but anything that would interfere with normal use of the aircraft would certainly be considered objectionable. The same is true for "undesirable flight characteristics." An example would be a strong nose-down pitching moment associated with gear or flap retraction after takeoff.

The specification also sets limits on the trim changes resulting from transfer to an alternate control system. The limits vary with the type of alternate system and the configuration and speed at the time of transfer, but in no case may a dangerous flight condition result. It will probably be necessary for the pilot to study the operation of the control systems and methods of effecting transfer in order to determine the conditions most likely to cause an unacceptable trim change upon transferring from one system to the other. As in all flight testing, a thorough knowledge of the aircraft and the objectives of the test will improve the quality and increase the value of the test results.

6.3 EXAMPLE TEST METHODS

The pre-flight preparation for the trim change test should start with a study of the applicable paragraphs of Military Specification F-8785. By comparing the specification requirements with the expected operational use of the aircraft, it will be possible to determine all the configuration changes and the conditions under which they should be tested.

When all the required changes have been determined, some time should be spent in laying out the sequence in which to test the various items in order to conserve flight time. Most of the configuration changes can be planned so that at the end of one test, the aircraft will be ready for the condition desired on the next test. This will result in a minimum delay. Table XIV in the specification can be used as an excellent guide in establishing the most advantageous sequence, but this sequence may vary depending on the specific aircraft. In an actual test program, much of the trim change information would probably be obtained during other tests as the aircraft was placed in the required test conditions. For example, the trim change with gear extension might be tested in the landing pattern after completion of another test.

After the required configuration changes and the sequence has been determined, a data card or cards should be prepared. Many forms are possible but the card should include all the information required by the pilot in flight and should be clearly legible. This test does not lend itself to the standard data card format, therefore a card will have to be made up from a blank one. A suggested form follows:

TRIM CHANGES

STUDENT ____________ INSTRUCTOR ____________

DATE ________ A/C No ________ CAMERA No ________

GRND SHOT ____________ TEST ____________

TOTAL FUEL ______ CONFIGURATION ________ CG _____

RUN NO	ALT	INITIAL TRIM CONDITIONS				CONFIG CHANGE	HOLD CONSTANT	C.N.	REMARKS
		SPEED	GEARS	FLAPS	POWER				
1	5M	140	UP	UP	PLF	GEAR ON	ALT.		
2	5M	140	ON	UP	PLF	FLAPS ON	ALT.		

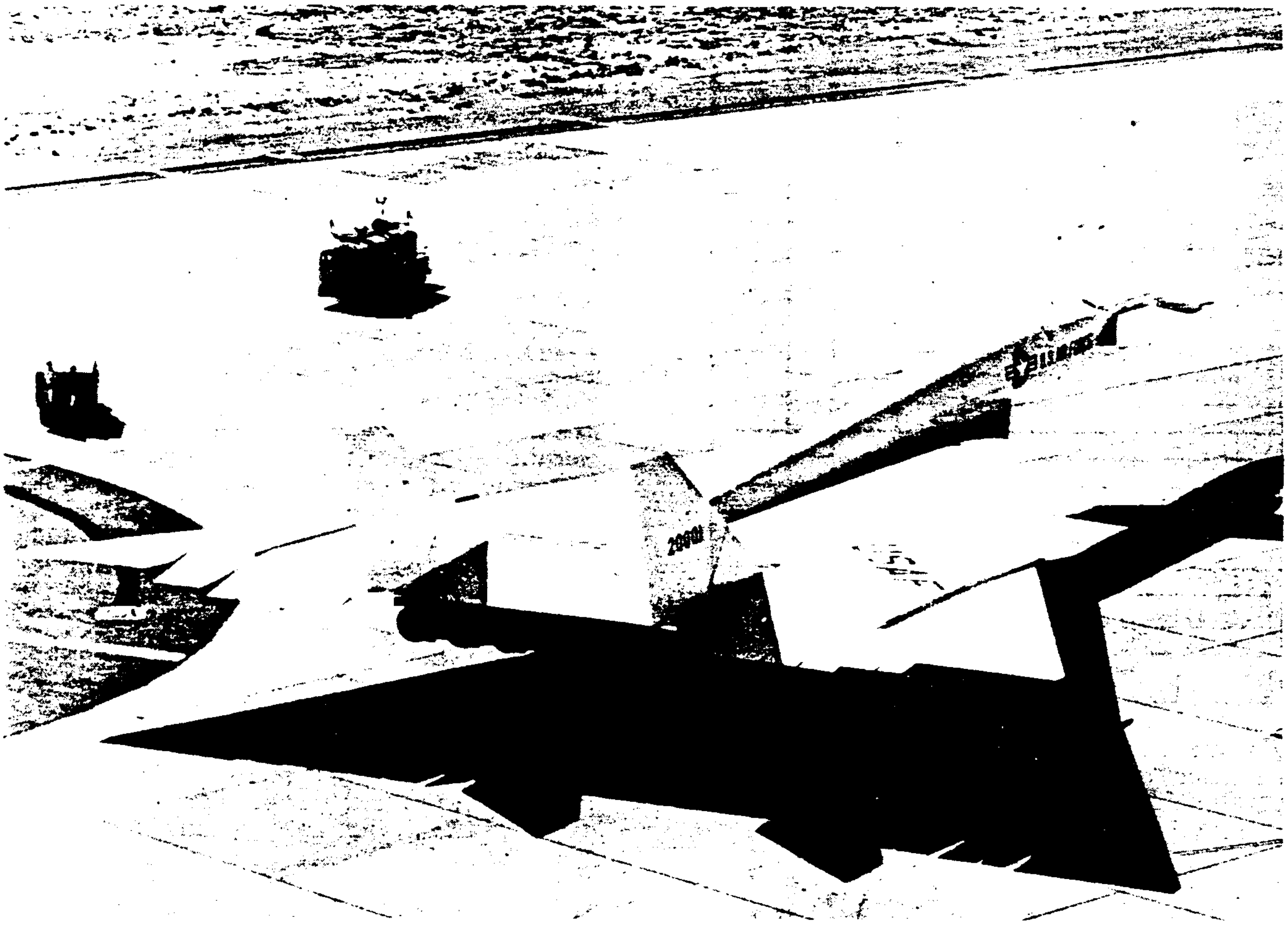
Prior to leaving the line a normal ground block will be taken to insure that all force and position measuring equipment is functioning properly. Following the flight data card, the aircraft should be placed in the desired condition and trimmed for pilot comfort. Excessive time should not be wasted on being exactly trimmed since a change in control forces is the objective rather than the magnitude of the total force. When trimmed and stabilized the oscillograph should be started, and then the desired configuration change is made. Actuating the event marker at the same time the configuration change is initiated will facilitate interpretation of the recorded data. The desired parameter (attitude, altitude, rate-of-climb, etc.) should be held constant with force alone for approximately 5 seconds and then normal trimming action employed to relieve the forces. It is important to pick out the most logical parameter to hold constant, and then smoothly hold that parameter constant. The control force changes which occur are often strongly affected by overcontrolling or allowing the supposedly constant parameter to vary. Both extremes must be avoided. While trimming, a qualitative evaluation of the adequacy of the aircraft trim should be made. If it is too fast or too slow or there is insufficient trim available to relieve the forces, a notation should be made on the data card. It should also be possible to trim the control forces to zero throughout the operational envelope of the aircraft. While there is no specific test for this requirement, the pilot should be alerted to note non compliance during other tests. When approximately trimmed, the oscillograph should be turned "OFF" and the aircraft readied for the next test condition.

This same general procedure will be repeated until all the points listed on the data card have been accomplished. The test for transfer to alternate control modes will have to be done twice. The first one done with the controls free and the second test done with the pilot maintaining attitude and zero sideslip throughout. As always, fuel, time, and orientation should be kept in mind.

The oscillograph traces will be used to determine the changes in forces and control deflections associated with each configuration change. If all the force changes are within the specification limits, all that is necessary is a table of values similar to the example below. However, for one configuration change a time history will be included. This should be for a configuration change that exceeded the limits of the specification. If none exceeded the specification, time histories are optional. The time history will include, but not be limited to, airspeed, altitude, load factor, angle of pitch, angle of bank, elevator force, aileron position, rudder position, and rudder force plotted against time. The table will be as shown below.

TABULAR SUMMARY OF RESULTS

RUN No.	ALT	INITIAL TRIM CONDITIONS A/S	GEAR	FLAPS	POWER	CONFIG CHANGE	HELD CONSTANT	F_s Lbs	δ_e Deg
1	5M	195	UP	UP	PLF	GEAR DN	ALTITUDE	−7	−1.5
2	5B	165	DN	UP	PLF	FLAPS DN	ALTITUDE	+16	−1.0
ETC									



CHAPTER 7 LATERAL-DIRECTIONAL FLIGHT TESTS

REVISED FEBRUARY 1977

7.1 INTRODUCTION

7.1 With lateral-directional theory as a background, it is possible to look at the flight test techniques used to investigate the actual lateral-directional static stability of an aircraft.

Basically, the lateral-directional characteristics of an aircraft are determined by two different flight tests; the Sideslip Test and the Aileron Roll Test. The tests do not measure lateral and directional characteristics independently. Rather, each test yields information concerning both the lateral and the directional characteristics of the aircraft.

In this chapter, both the Sideslip Test and the Aileron Roll Test will be covered. As each test is discussed, the required results, as determined by MIL-F-8785 will also be covered. Finally, an example test mission in the T-33 aircraft will be discussed.

7.2 SIDESLIP FLIGHT TEST TECHNIQUE

7.2 The purpose of the sideslip test is to investigate the static lateral and directional stability characteristics of a particular aircraft in each of several configurations. Since the static lateral and directional stabilities are functions of Mach number, angle of attack, elasticity and configuration, it is important to check the aircraft in various configurations at several altitude-airspeed combinations. In so doing, the sense (±) of the lateral and directional stabilities and the characteristics of the side force can be determined throughout the flight envelope.

All equations relating to the static directional stability of an aircraft were developed under the assumption that the aircraft was in a "steady straight sideslip." This is the maneuver used in the Sideslip Test. To develop a ground for discussion, it is appropriate to discuss the basic mechanics the pilot must perform to establish a "steady straight sideslip." First, the aircraft is trimmed at the desired altitude-airspeed combination. The rudder is then depressed and an increment of sideslip is developed. In order to maintain "straight" or constant heading flight, it will then be necessary for the pilot to bank the aircraft in a direction opposite that of the applied rudder. The aircraft is then stabilized in this condition. Thus, the pilot establishes a "steady straight sideslip." To understand the forces and moments at play in this condition, consider figure 7.1. In this figure the aircraft is in a steady sideslip. Thus, the moment created by the rudder, n_{δ_r}, must equal the moment created by the aerodynamic forces acting on the aircraft, n_β. It can be seen, however, that in this condition the side forces are unbalanced and that this will cause an acceleration. The force, F_{y_β}, will always be greater than $F_{y_{\delta_r}}$. Thus, in the case depicted, the aircraft will accelerate, or turn, to the right. In order to stop this turn, it is necessary to bank the aircraft; in this case to the left (figure 7.2). The bank allows a component of aircraft weight, $W\sin\phi$, to act in the y direction and thus balance the previously

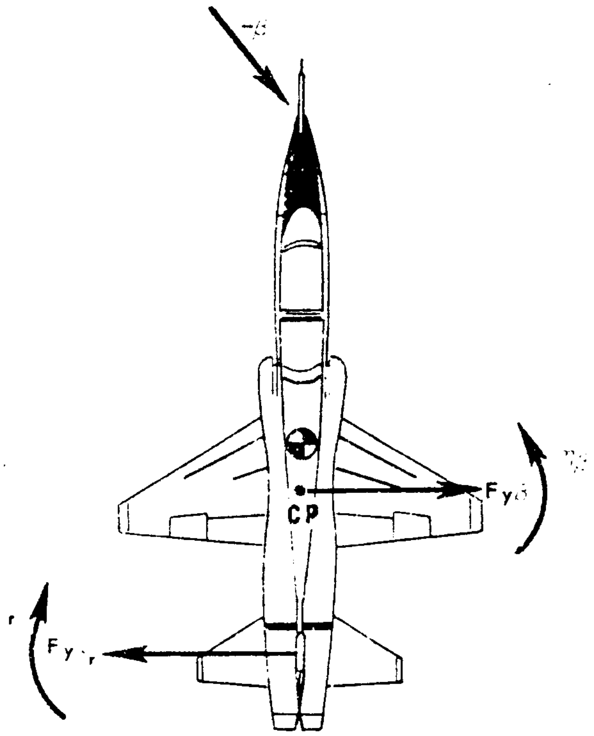


FIGURE 7.1 STEADY SIDESLIP

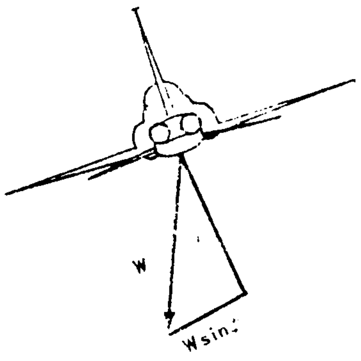


FIGURE 7.2 STEADY STRAIGHT SIDESLIP

unbalanced side forces. Thus, the pilot establishes a "straight sideslip." By holding this condition constant with respect to time, or varying it so slowly in a continuously stabilized condition that rate effects are negligible, he establishes a "steady straight sideslip" - the condition that was used to derive the flight test relationships in static directional stability theory.

By simply establishing a steady straight sideslip, the side force characteristics of the aircraft can be investigated. MIL-F-8785, paragraph 3.3.6.2 states that "an increase in right bank angle must accompany an increase in right sideslip," where right sideslip is defined by the incident airflow approaching from the right side of the plane of symmetry.

One property of basic importance in the sideslip test is the directional stiffness of an aircraft or its static directional stability. To review, the static directional stability of an aircraft is defined by the initial tendency of the aircraft to return to or depart from its equilibrium angle of sideslip when disturbed from the equilibrium condition. In order to determine if the aircraft possesses static directional stability, it is necessary to determine how the yawing moments change as the sideslip angle is changed. Thus, the slope of a line in a plot such as figure 7.3 is of interest. For positive directional stability a plot of C_{n_β} must have a positive slope.

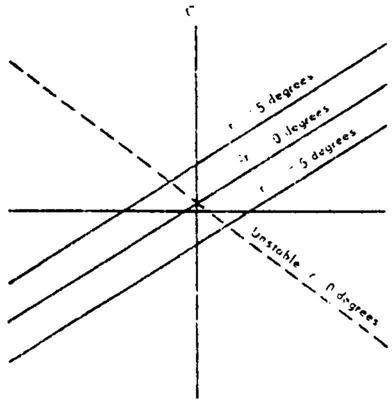


Figure 7.3 WIND TUNNEL RESULTS OF YAWING MOMENT COEFFICIENT vs SIDESLIP ANGLE

Plots like those presented in figure 7.3 would be obtained from wind tunnel data. The aircraft model would be placed at various angles of sideslip with various angles of rudder deflection, and the unbalanced moments would be measured. However, it is impossible to determine from flight tests the unbalanced moments at varying angles of sideslip. It was shown in static directional theory, however, that the amount of rudder deflection required to fly in a steady straight sideslip is considered to be an indication of the amount of yawing moment present tending to return the aircraft to or remove it from its original trimmed angle of sideslip. Thus, in order to determine the sign of the rudder fixed static directional stability, C_{n_β}, a plot is made of rudder deflection required versus sideslip angle.

The apparent stability parameter, $\partial\delta_r/\partial\beta$, for a directionally stable aircraft is shown in figure 7.4. For a stable aircraft, this plot has a negative slope.

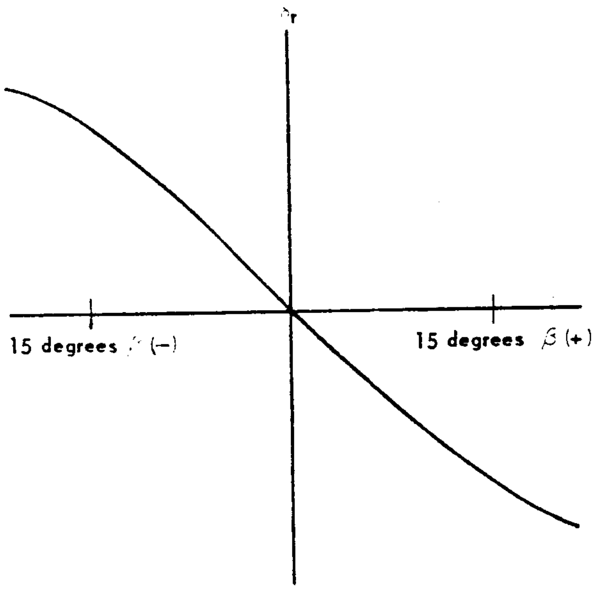


FIGURE 7.4 RUDDER DEFLECTION vs SIDESLIP

MIL-F-8785, paragraph 3.3.6.1 requires that right rudder pedal deflection ($+\delta_r$) accompany left sideslips ($-\beta$). Further, for angles of sideslip between $\pm$15 degrees, a plot of $\partial\delta_r/\partial\beta$ should be essentially linear. For larger sideslip angles, an increase in β must require an increase in δ_r. In other words, the slope of $\partial\delta_r/\partial\beta$ cannot go to zero.

It should be remembered that drastic changes occur in the transonic and supersonic speed regions. In the transonic region where the flight controls are most effective, a small δ_r may give a large β and thus $\partial\delta_r/\partial\beta$ may appear less stable. However, as speed increases, control surface effectiveness decreases, and $\partial\delta_r/\partial\beta$ will increase in slope. This apparent change in C_{n_β} is due solely to a change in control surface effectiveness and can give an entirely erroneous indication of the magnitude of the static directional stability if not taken into account.

It is now necessary to investigate the control free stability of an aircraft. As discussed in the theory of static directional stability, a plot of rudder force required versus sideslip, $\partial F_r/\partial\beta$, is an indication of the rudder-free static directional stability of an aircraft. It was shown that a plot of $\partial F_r/\partial\beta$ must have a negative slope for positive rudder-free static directional stability. MIL-F-8785 paragraph 3.3.6.1 requires that a plot of $\partial F_r/\partial\beta$ be essentially linear between $\pm$10 degrees of β from the trim condition. However, at greater angles of sideslip, the rudder forces may lighten but may never go to zero, or overbalance. These requirements are depicted in figure 7.5.

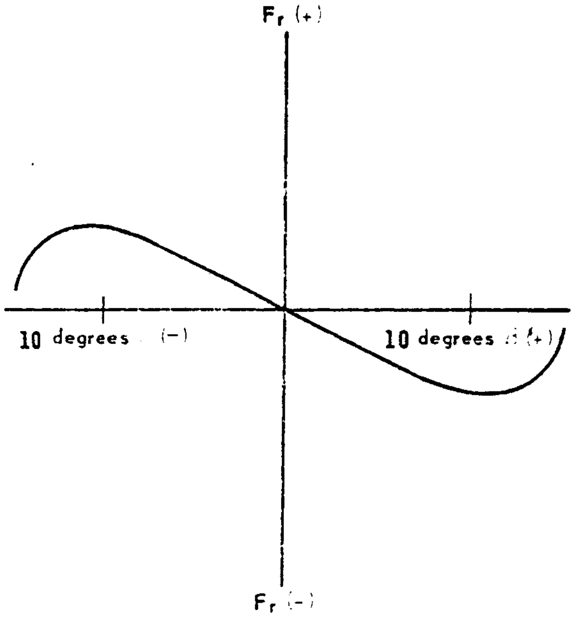


FIGURE 7.5 CONTROL FREE SIDESLIP DATA

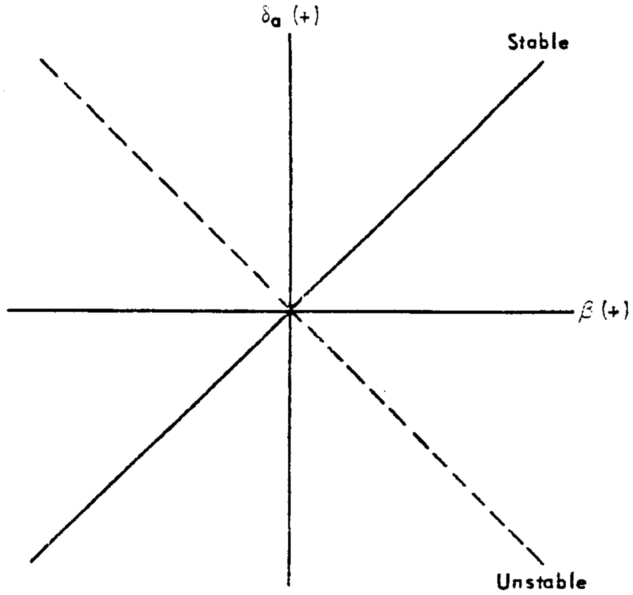


FIGURE 7.6 CONTROL FIXED SIDESLIP DATA

The control force information in figure 7.5 is acceptable in accordance with MIL-F-8785 as long as the algebraic sign of F_r/β is negative. It can be seen that at very large sideslip angles, the slope $\partial F_r/\partial\beta$ may be positive. This is acceptable as long as the rudder force required does not go to zero.

Static lateral characteristics are also investigated during the sideslip test. It was shown in the theory of static lateral stability that $\partial\delta_a/\partial\beta$ may be taken as an indication of the control-fixed dihedral effect of an aircraft, $C_{\ell_{\beta}(Fixed)}$. For stable dihedral effect, it was shown that a plot of $\partial\delta_a/\partial\beta$ must have a positive slope. MIL-F-8785 specifies that right aileron control deflection shall accompany right sideslips and left aileron control shall accompany left sideslips. A plot of $\partial\delta_a/\partial\beta$ for stable dihedral effect is presented in figure 7.6.

MIL-F-8785, paragraph 3.3.6.3.2 limits the amount of stable dihedral effect an aircraft will exhibit by specifying that no more than 75 percent of full aileron cockpit control deflection will be required in any of the mandatory sideslip tests required by paragraph 3.3.6. MIL-F-8785 paragraph 3.3.6.3.2 limits the amount of dihedral effect an aircraft may have. It states that no more than 10 pounds of aileron stick force or 20 pounds of aileron wheel force is allowed for sideslips which may be experienced in operational employment.

The theoretical discussion of control free dihedral effect revealed that $\partial F_a/\partial\beta$ will give an indication of $C_{\ell_{\beta}(Free)}$, and that for stable dihedral effect $\partial F_a/\partial\beta$ is positive. Refer to figure 7.7. MIL-F-8785 paragraph 3.3.6.3 states that the left aileron force should be required for left sideslips and that a plot of $\partial F_a/\partial\beta$ should be essentially linear for all of the mandatory sideslips tested.

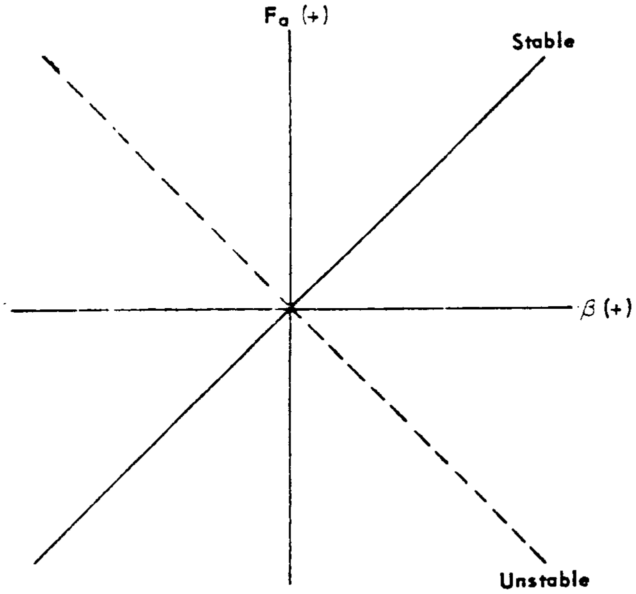


FIGURE 7.7 CONTROL FREE SIDESLIP DATA

MIL-F-8785, paragraph 3.3.6.3.1 does permit an aircraft to exhibit negative dihedral effect in wave-off conditions as long as no more than 50 percent of available roll control or 10 pounds of aileron control force is required in the negative dihedral direction.

A longitudinal trim change will most likely occur when the aircraft is sideslipped. MIL-F-8785 paragraph 3.2.3.7, places definite limits on the allowable magnitude of this trim change. It is prefered that an increasing pull force accompany an increase in sideslip angle and that the magnitude and direction of the trim change should be similar for both left and right sideslips. The specification also limits the magnitude of the control force accompanying the longitudinal trim change depending on the type of controller in the aircraft (stick or wheel). A plot of elevator force versus sideslip angle that complies with MIL-F-8785 is presented in figure 7.8.

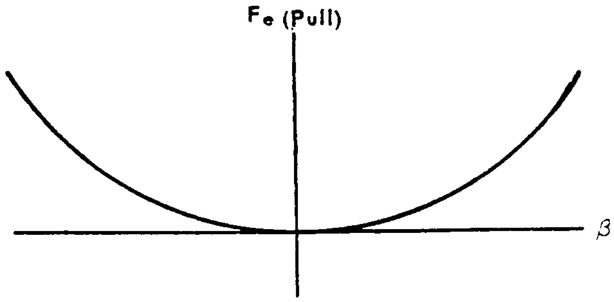


FIGURE 7.8 ELEVATOR FORCE VERSUS SIDESLIP ANGLE

Paragraph 3.3.6 of MIL-F-8785 outlines the sideslip tests that must be performed in an aircraft. The specification requires that sideslips be tested to full rudder pedal deflection, 250 pounds of rudder pedal force, or maximum aileron deflection, whichever occurs first. Often sideslips must be discontinued prior to reaching these limits due to controllability or structural problems.

The following is a complete list of MIL-F-8785 paragraphs that apply to sideslip tests:

3.2.3.7

3.3.6

3.3.6.1

3.3.6.2

3.3.6.3, 3.3.6.3.1, 3.3.6.3.2

7.3 LIMITATIONS

On student data missions at the US Air Force Test Pilot School it is not possible to conform to all aspects of MIL-F-8785 that concern sideslip tests. Therefore, certain general limitations must be applied to all student sideslip tests.

1. In the cruise configuration, sideslip tests will be conducted at three different altitudes. At each altitude, three different airspeeds will be investigated.

2. In the power approach configuration, sideslip tests will be conducted at 10,000 feet AGL, at the minimum speed for normal final approach.

3. Sideslip tests will be conducted within the limitations outlined in the appropriate aircraft Flight Manual.

4. Within these limitations, the student test program will be set up to investigate all of the requirements concerning sideslip tests that are outlined in the applicable paragraphs of MIL-F-8785.

5. To void excessive data reduction, all results will be plotted against V_c.

Sample data plots of sideslip test results are presented in figures 7.9 and 7.10.

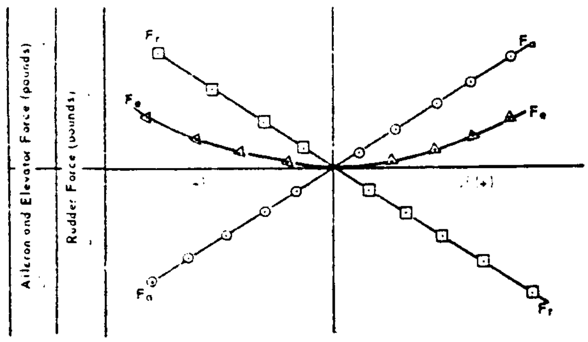


FIGURE 7.9 STEADY STRAIGHT SIDESLIP CHARACTERISTICS CONTROL FORCES VERSUS SIDESLIP

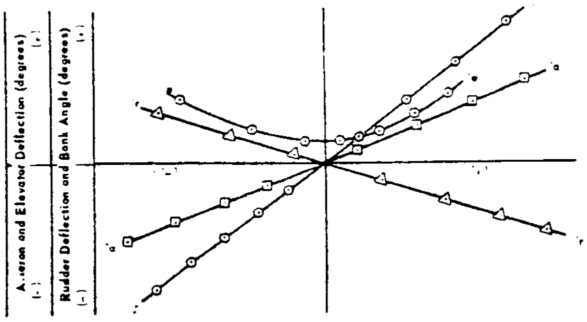


FIGURE 7.10 STEADY STRAIGHT SIDESLIP CHARACTERISTICS CONTROL DEFLECTIONS AND BANK ANGLE VERSUS SIDESLIP

7.4 AILERON ROLL FLIGHT TEST TECHNIQUE

Generally, the aileron roll flight test technique is used to determine the rolling performance of an aircraft and the yawing moments generated by rolling. Roll coupling is another important aircraft characteristic normally investigated by using the aileron roll flight test technique. The roll coupling aspect of the aileron roll test will not be investigated at the USAF Test Pilot School. However, the theoretical aspects of roll coupling will be covered in a later course.

It is necessary to understand the basic mechanics involved in the aileron roll flight test technique. The aircraft is first trimmed in the desired configuration at the desired altitude-airspeed combination. Then, the lateral control is abruptly placed to a particular control deflection (1/4, 1/2, 3/4, or full). Normally, the desired control deflection is obtained by using some mechanical restrictor such as a chain stop. With the lateral control at the desired deflection, the aircraft is rolled through a specified increment of bank. For control deflections less than maximum, the aircraft is normally rolled through 90 degrees of bank. Because of the higher roll rates obtained at full control deflection, it is usually desirable to roll the aircraft through 360 degrees of bank when using maximum lateral control deflection. To facilitate aircraft control when rolling through a bank angle change of 90 degrees, start the roll from a 45-degree bank angle. During the roll, a mechanical recorder, such as an oscillograph, is used to record the following information: aileron position, aileron force, bank angle, sideslip and roll rate. Aileron rolls are normally conducted in both directions to account for roll variations due to engine gyroscopic effects. Aileron rolls are performed with rudders free; with rudders fixed; and are coordinated with $\beta = 0$ throughout roll.

Caution should be exercised in testing a fighter type airplane in rolling maneuvers. The stability of the airplane in pitch and yaw is lower while rolling. The incremental angles of attack and sideslip that are attained in rolling can produce accelerations which are disturbing to the pilot and can also cause critical structural loading. The stability of an airplane in a rolling maneuver is a function of Mach number, roll rate, dynamic pressure, angle of attack, configuration, and control deflections during the maneuver.

The most important design requirement imposed upon ailerons or other lateral control devices is the ability to provide sufficient rolling moment at low speeds to counteract the effects of vertical asymmetric gusts tending to roll the airplane. This means, in effect, that the ailerons must provide a minimum specified roll rate, and a rolling acceleration such that the required rate of roll can be obtained within a specified time, even under loading conditions that result in the maximum rolling moment of inertia (e.g., full tip tanks). The steady roll rate and the minimum time required to reach a particular change in bank angle are the two parameters presently used to indicate rolling capability. Pilot opinion surveys reveal that time to roll a specified number of degrees provides the best overall measure of rolling performance.

The minimum rolling performance required of an aircraft is outlined in MIL-F-8785, table IX This rolling performance is expressed as a function of time to reach a specified bank angle. Table IX is supplemented further by roll performance required of Class IV airplanes in various flight phases. The specific requirements for Class IV airplanes are spelled out in MIL-F-8785, paragraphs 3.3.4.1.1, .2, .3, .4 . Paragraph 3.3.4.2 and table X of MIL-F-8785 specify the maximum and minimum aileron control forces allowed in meeting the roll requirements of table IX and the supplemental requirements concerning Class IV aircraft. Paragraph 3.3.2.5 specifies the maximum rudder force permitted for coordinating the required rolls.

In addition to examining time required to bank a specified number of degrees and aileron forces, F_a, it is necessary to examine the maximum roll rate, P_{max}, to get a complete picture of the aircraft's rolling performance. Therefore, in any investigation of aircraft rolling performance, the maximum roll rate obtained at maximum lateral control displacement is normally plotted versus airspeed.

Paragraph 3.3.4.3 of MIL-F-8785 states that there should be no objectionable nonlinearities in roll response to small aileron control deflections or forces. To investigate this area, it is necessary to observe the roll response to aileron deflections less than maximum - such as 1/4 and 1/2 aileron deflections.

MIL-F-8785, paragraph 3.3.4.5 states that it should be possible to raise a wing by using the rudder pedal alone, and that right rudder pedal force should be required for right rolls. Further, it states that with the aileron cockpit control free, it should be possible to produce a roll rate of 3 degrees per second by use of rudder pedal forces of 50 pounds or less. Turn coordination requirements are spelled out in MIL-F-8785, paragraph 3.3.2.6 for steady turning maneuvers.

The other area of prime interest in the aileron roll flight test is the amount of sideslip that is developed in a roll and the phasing of this sideslip with respect to the roll rate. Associated with this characteristic is the roll rate oscillation. These factors influence the pilot's ability to accomplish precise tracking tasks.

The following is a complete list of MIL-F-8785 paragraphs that apply to aileron roll tests:

3.3.2.3

3.3.2.4

3.3.4

3.3.4.1, 3.3.4.1.1, 3.3.4.1.2, 3.3.4.1.3, 3.3.4.1.4

3.3.4.2

3.3.4.3

3.3.4.4

3.3.4.5

7.5 SCHOOL TEST LIMITATIONS

The following limitations will apply to all student data missions at the Aerospace Research Pilot School:

1. In the cruise configuration, aileron roll tests will be conducted at three different altitudes. At each altitude, three different airspeeds will be investigated.

2. In the power approach configuration, aileron roll tests will be conducted at final approach speed at 10,000 feet AGL.

3. Aileron roll tests will be conducted within the limitations outlined in the appropriate aircraft Flight Manual.

4. Within these limitations, the student test program will be set up to investigate all of the requirements concerning aileron roll tests that are outlined in the applicable paragraphs of MIL-F-8785(ASG).

5. To avoid excessive data reduction, all results will be plotted against V_c. Sample plots of aileron roll data are presented in figure 7.11.

7.6 DEMONSTRATION MISSION

To unify all that has been said concerning the sideslip and aileron roll flight test techniques, a complete description of an example demonstration mission in the T-33 will be presented.

1. After engine start, and with aileron boost on, visually position the ailerons to approximately 1/4, 1/2 and 3/4 deflections. Mark each position on the instrument panel with masking tape. Prior to leaving the ramp, a ground shot will be taken to ascertain proper functioning of the force and control deflector indicator. The ground shot will consist of a con-

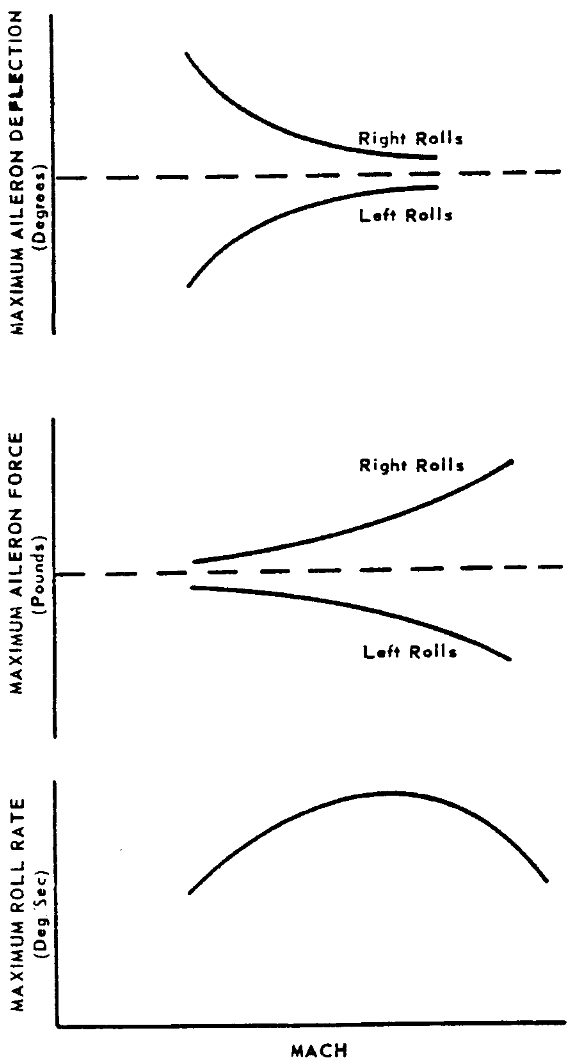


FIGURE 7.11 MAXIMUM AILERON ROLL TEST RESULTS ONE ALTITUDE

tinuous recording of all control deflections and forces from neutral to full positive, to full negative, and then back to neutral. This will help eliminate confusion in reading the film and the oscillograph paper. While taxiing, the pilot should practice making 1/4, 1/2, 3/4, and full aileron deflections.

2. Climb to 20,000 feet and trim the aircraft in the cruise configuration at 250 KIAS using the back side trim technique. Obtain a photo panel shot with the aircraft trimmed in this condition.

3. If a yaw string is available and it is not centered during the trim shot, align it with the longitudinal axis of the aircraft and obtain a photo panel shot. If a sideslip indicator is available for inflight use, this zero sideslip shot can be used to calibrate it. The photo panel shot with zero sideslip should be examined on the ground to ascertain that the zero sideslip indication, as obtained from the calibration book, is correct.

4. During the sideslip test, altitude will be lost if trim power and 250 KIAS are maintained. Therefore, this test will be conducted at 20,000 ±1,000 feet. After the trim shot at 20,000 feet has been obtained, note trim power and climb to 21,000 feet.

5. The instructor pilot will demonstrate the stabilized sideslip flight test technique. Starting from a zero sideslip condition, establish a steady straight sideslip of approximately two degrees while holding trim airspeed. This may best be done by applying a small amount of rudder and then coming in with just enough bank to hold a constant heading. A point on the outside horizon will provide the pilot with the most accurate means of holding a constant heading. The needle of the turn and slip indicator may also be an aid in holding constant heading, zero yawing velocity flight.

Constant trim velocity should be maintained as the sideslip is increased in approximately 2 degree increments until reaching the maximum sideslip obtainable (if no other restriction applies), or until a dangerous flight condition is anticipated. A photopanel shot will be taken at each stabilized sideslip condition. Sideslips in the T-33 will be discontinued at rudder buffet, or at plus or minus 14 degrees of sideslip in the cruise configuration to prevent inadvertent tumbling and possible structural damage. The T-33 is restricted from full rudder deflection sideslips. If no sideslip gauge is available in the cockpit, the following guide may be used: At approximately 12 degrees of sideslip, the standard airspeed indicator will jump approximately 5 knots. The pilot may continue the sideslip investigation approximately two degrees past this point if no other adverse indications are noted. If an airspeed calibration is available for the boom airspeed system at various angles of sideslip, this correction should be used while attempting to hold a constant airspeed. If such a calibration is not available, an indication of the magnitude of the position error due to sideslip can be obtained by placing the aircraft in a sideslip and rapidly coming back to zero sideslip while noting the magnitude of the airspeed change. This correction can be ignored if this change is only one or two knots. It should be immediately apparent whether back or forward stick is necessary to hold a constant airspeed as the sideslip is increased, and thus the correct control movement should be anticipated as the sideslip is increased. Throughout the test, lead all inputs with the rudder. This technique, coupled with slow, deliberate control inputs, will help keep Dutch roll at a minimum. If a Dutch roll should develop, stop it with the rudder; aileron inputs will only reinforce the motion. After the maximum sideslip point is reached, the aircraft should be smoothly returned to trim and then similar sideslip points should be made in the opposite direction. Smoothness in this test, as in all tests, is imperative in order to get good stabilized points quickly. Anticipation of correct control movement is a great aid in establishing good test points quickly. A record should be kept of beginning and final photo correlation numbers as well as a list of any unreliable points.

6. The pilot will practice the stabilized sideslip flight test technique. Maintain altitude at 20,000 $\pm$1,000 feet.

7. The instructor pilot will demonstrate the slowly varying sideslip method. This is an alternate method of obtaining sideslip information. Starting from zero sideslip, continuously increase the sideslip angle at not more than one degree per second while maintaining heading and velocity. A continuous photo record is taken out to the maximum sideslip angle. This method is considerably more difficult to fly properly than the stabilized sideslip method.

8. The pilot will practice the slowly varying sideslip method. Maintain 20,000 ±1,000 feet.

9. At the completion of the sideslip practice, the pilot will return the aircraft to 21,000 feet and 250 KIAS. The instructor pilot will demonstrate the aileron roll flight test technique. Using trim power and holding 250 KIAS, roll the aircraft into a 45 degree bank. Stabilize the aircraft in this condition, holding the rudder pedals fixed to hold trim sideslip. The recording trigger is depressed prior to starting a roll to permit some leader on the oscillograph paper. The stick is rapidly moved to 1/4 aileron deflection, and the aircraft is rolled to 45 degrees of bank in the opposite direction. This control input should be a "step input." The pilot may avoid stick "bobble" by using both hands on the stick. The recording trigger is held depressed throughout the roll. The airspeed should be held as close as possible to the aim V_i throughout the roll. When the roll is complete, rapidly re-establish a 45-degree bank at 250 KIAS. This will prevent needless altitude loss and airspeed excursion. This procedure is repeated in the opposite direction. When the aircraft has been rolled in both directions at 1/4 aileron deflection, repeat the procedure at 1/2 aileron deflection. The full deflection aileron roll is started from wings level flight at 250 KIAS. Apply full aileron deflection in the desired direction of roll. In order to determine exactly how much aileron force it required to hold full aileron deflection, it will be necessary to slowly relax the control force prior to completing 360° of roll. Once on the ground, the oscillograph trace will allow the pilot to match a control force against the point where the aileron deflection first starts to decrease. The full deflection aileron roll will be repeated in the opposite direction. For the sake of demonstration, the instructor pilot may roll in only one direction for each control deflection tested. The instructor will then demonstrate a step aileron input that will accomplish a 30°-30° roll in approximately 6 seconds.

10. The pilot will practice the aileron roll test with rudders fixed. When complete, he will repeat the full deflection roll with rudders free. He will then practice the step aileron input needed to get a 30°-30° roll in approximately 6 seconds.

11. The pilot will descend to 12,000 feet pressure altitude in an area that will allow at least 10,000 feet terrain clearance. The aircraft will be placed in the power approach configuration, i.e., gear down, full flaps, speed brakes up. Obtain a trim shot at 12,000 feet and 120 knots plus fuel using the backside trim technique. Obtain a photopanel shot with the aircraft trimmed in this condition. Also, obtain a photopanel shot in a zero sideslip condition.

12. This test will be conducted at 12,000 ±1,000 feet. After the 12,000 foot trim shot has been obtained, note trim

power and climb to 11,000 feet.

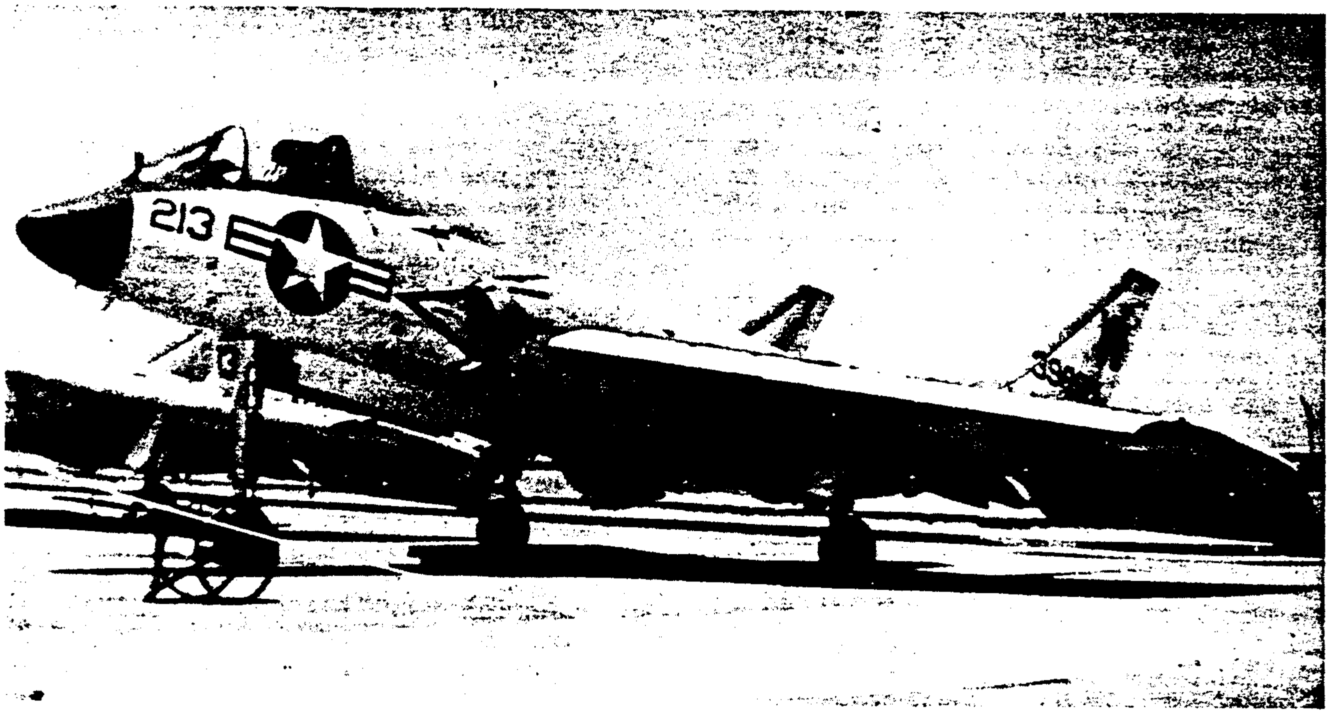
13. The instructor pilot will demonstrate the stabilized sideslip flight test technique in the power approach configuration. Because of the low "q", rudder forces will be very light and care should be exercised to avoid overcontrolling. Sideslips in the T-33 will be discontinued at rudder buffet or at plus or minus 10 degrees of sideslip in the power approach configuration to prevent inadvertent tumbling. Care should be exercised in returning from maximum sideslip to the trim condition. Climb when necessary in order to remain within the allowable altitude band (±1,000 feet).

14. The pilot will practice the stabilized sideslip flight test technique in the power approach configuration.

15. The instructor pilot will demonstrate the aileron roll flight test technique in the power approach configuration. During this low "q" condition, considerable sideslip will develop. Recover from the roll using aileron only. Overcontrolling or putting in incorrect rudder inputs can create a hazardous situation. Therefore, 360-degree rolls will not be accomplished in the power approach configuration.

16. The pilot will practice the stabilized sideslip flight test technique in the power approach configuration with the rudders fixed. When this is completed, he will repeat the 1/2 aileron deflection point with rudders free. Recovery will be made with rudders free.

17. Landing will be made from a simulated flameout pattern set up by the instructor pilot.



CHAPTER 8

REVISED FEBRUARY 1977

ENGINE-OUT OPERATION

8.1 INTRODUCTION

The problems associated with an engine failure in a multi-engine aircraft may be classified into two types; control problems and performance problems. The severity of one may greatly overshadow the other in certain aircraft; but in general, the pilot is confronted with a generous portion of both.

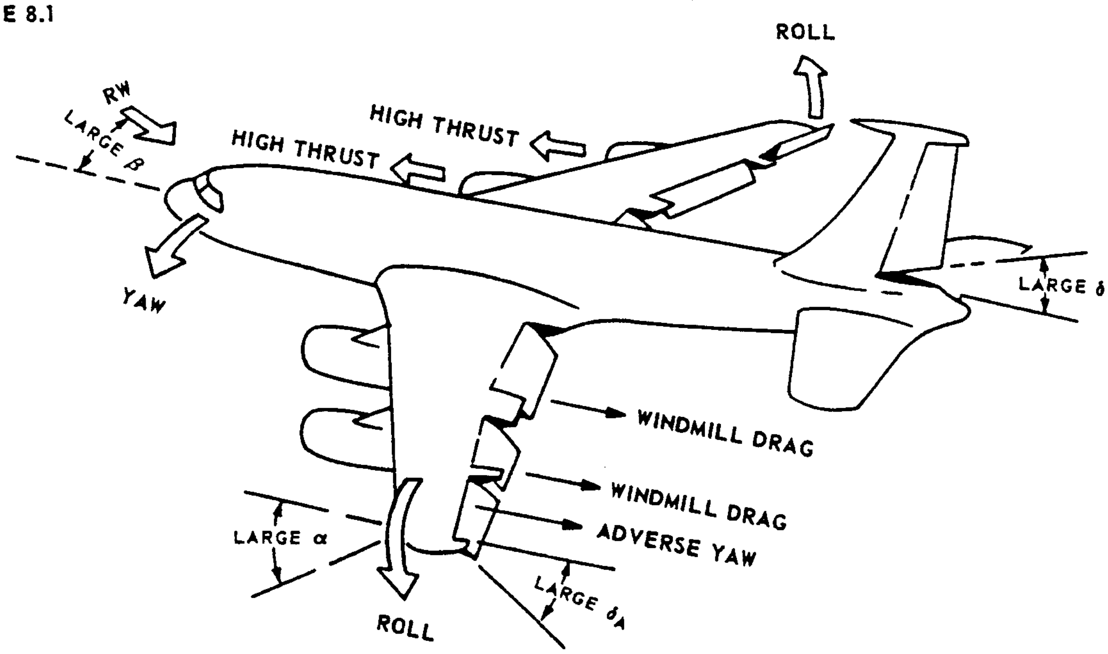
8.2 THE CONTROL PROBLEM

The control problem may be simply stated - the pilot must be able to achieve and maintain straight unaccelerated flight following the loss of an engine. Thus the engine-out control problem can be divided into cases; the non-steady state dynamic case and the steady state equilibrium case. The dynamic case begins when an engine fails and terminates when the equilibrium case has been achieved.

When a pilot intentionally shuts down an engine in an aircraft with adequate control authority to maintain equilibrium, the dynamic case is usually not severe and the transients encountered are mild. If, however, an engine fails suddenly on takeoff, or the pilot makes a sudden application of go-around power to asymmetric engines, a potentially divergent rolling and yawing motion can ensue.

These hazardous dynamic situations are caused by a rapid sequence of events, as illustrated

FIGURE 8.1



in the following hypothetical sequence:

1. The aircraft is in a critical flight phase such as takeoff or go-around when a large yawing moment due to asymmetric thrust appears very suddenly. The aircraft yaws rapidly through a large angle.

2. The pilot allows a large sideslip angle to develop because of the high yaw rate and the surprise factor. A rolling moment into the bad engines is generated by the dihedral effect. This rolling moment is augmented by wing blanking on swept-wing configurations and by asymmetric slipstream effects on propeller aircraft.

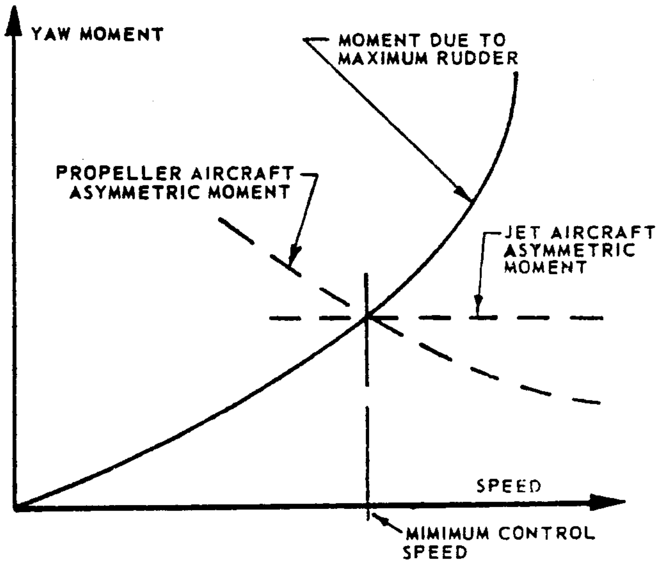
3. As the angular momentum builds up in roll and yaw, larger compensating moments, over and above the steady state requirements, are required to arrest the motion. Large control deflections are required because of the reduced control effectiveness at slow speed, and adverse yaw adds to the forcing moment. If full control is insufficient to achieve equilibrium, a power reduction on the good engines will be required.

4. But a power reduction aggravates an already critical performance problem. Speed is difficult to maintain because of decreased thrust and increased drag. If the down-going wing, which is at a high angle of attack because of the slow speed and the rolling velocity, is allowed to reach stall, the dynamic case may terminate without ever reaching equilibrium.

The severity of such responses is difficult to predict by theoretical analysis, and flight test of critical situations is required to establish safe flight boundaries. Slow speed restrictions due to decreased control effectiveness are most common, although others may exist in the supersonic range due to reduced stability. The dynamic case boundaries are usually (although not necessarily) more restrictive than those due to the equilibrium case.

For every given set of asymmetric conditions there is a speed below which aerodynamic control is insufficient to maintain the equilibrium case. This is called the minimum control speed.

FIGURE 8.2



Obviously, this minimum speed will vary with the prevailing conditions. Not so obvious, however, is the fact that for a given condition the equilibrium case can be maintained with different combinations of bank angle, sideslip angle, and rudder deflection and that the minimum speed will vary according to the combination used.

Engine-out definitions and terminology are not standard throughout the aviation industry and in any discussion it must be clearly understood what the conditions are, and that everyone is talking about the same thing. Several more or less standard definitions are discussed below.

Minimum Control Speed:

It is possible that there will be no minimum control speed for a multi-engine aircraft because it can be controlled up to aerodynamic stall. This is the desired situation. MIL-F-8785 (para 3.3.9.2) specifies that straight flight must be possible during takeoff at any speed above minimum takeoff speed and further specifies the control forces and deflections that may be used to accomplish this. This establishes the minimum control speed required for every possible gross weight condition. It might therefore seem that a minimum control speed is only of academic interest, but there may be instances where a multi-engine aircraft could meet the specifications at takeoff but be operated at a speed in some operational or approach flight phase which would be lower than minimum takeoff speed and hence the asymmetric thrust minimum control speed must still be determined by flight test.

Ground Minimum Control Speed:

Control of asymmetrically powered multi-engine aircraft on the ground also presents a problem that must be considered. If a pilot loses the most critical engine during takeoff roll, he must decide whether to continue the takeoff or abort. MIL-F-8785 (para 3.3.9.1) specifies that the pilot must be able to maintain a path on the runway that does not deviate more than 30 feet from the original path if he decides to continue the takeoff and is above the refusal speed (based on the shortest runway from which the airplane is designed to operate). If the pilot decides to abort, the directional control requirements are still specified but the pilot is allowed to use additional controls such as nosewheel steering and differential braking. Flight (ground) testing is required to show compliance with the specification so a ground minimum control speed can be determined. This is defined as the lowest speed at which directional control can be maintained on the ground when the most critical engine fails during takeoff roll.

8.3 THE PERFORMANCE PROBLEM

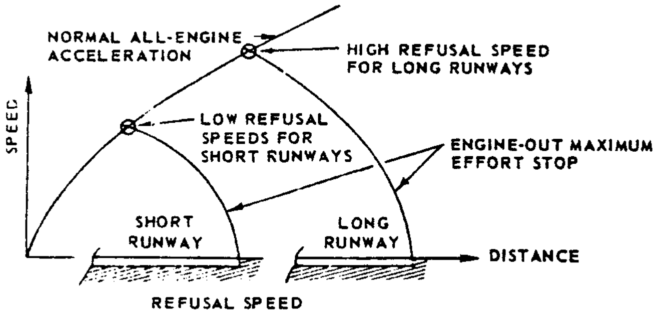
Reduced climb performance, service ceiling and range capability accompany an engine failure as a natural result of decreased thrust and increased drag. The effect of engine failure on takeoff performance, however, is a complex subject requiring additional definitions and operational techniques.

Takeoff Performance:

The basic requirement is simple; at every instant throughout the takeoff roll the pilot must have an acceptable course of action available to him in the event of engine failure. During the first part of the roll, this action will be to abort the takeoff and stop. Beyond a certain point the action will be to continue the takeoff with the engine failed. The dividing point between these courses of action is a function of aircraft performance.

Consider an aircraft at a particular configuration and gross weight starting its takeoff roll on a given day. For a given runway length there is a maximum speed to which it can accelerate on all engines, lose the critical engine and then just complete a maximum effort stop at the far end of the runway. This speed, called the refusal speed, is relatively high for long runways and relatively low for short ones.

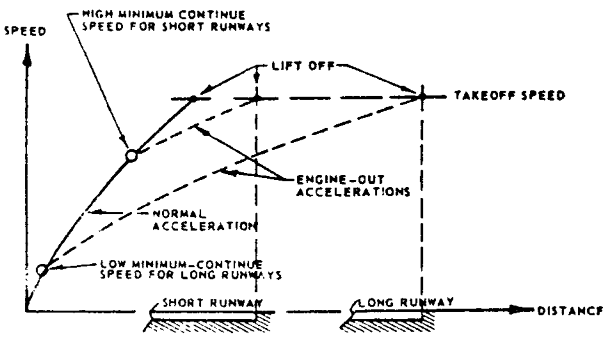
FIGURE 8.3



REFUSAL SPEED is thus defined as the maximum speed to which the aircraft can make a normal takeoff acceleration, lose the critical engine at that speed and then stop on the remaining runway. Stopping technique and devices to be used must be specified.

Now consider the same aircraft making the same takeoff under identical conditions. For a given runway length there is a minimum speed to which it can accelerate on all engines, lose the critical engine at that speed and then continue the takeoff with the engine failed, getting airborne just at the far end of the runway. This speed (a "minimum-continue" speed) varies with runway length in a manner opposite that of refusal speed, i.e., it is relatively low for long runways.

FIGURE 8.4



The gap between the minimum-continue speed and the Refusal Speed reflects the size of the safety margin provided by a given runway for the particular conditions.

FIGURE 8.5

SAFE TAKEOFF

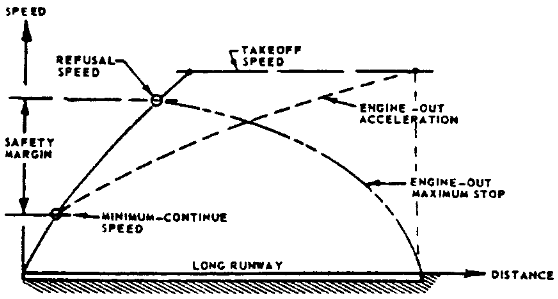


FIGURE 8.6

UNSAFE TAKEOFF

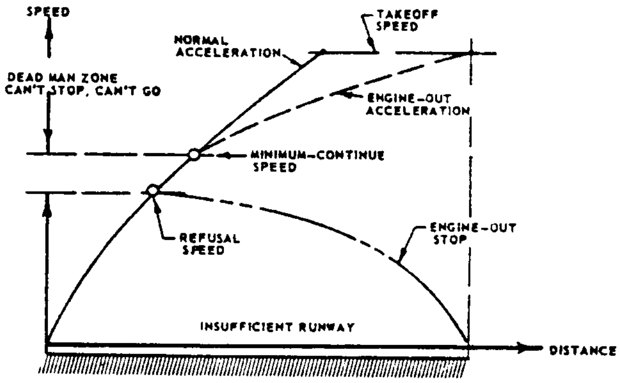
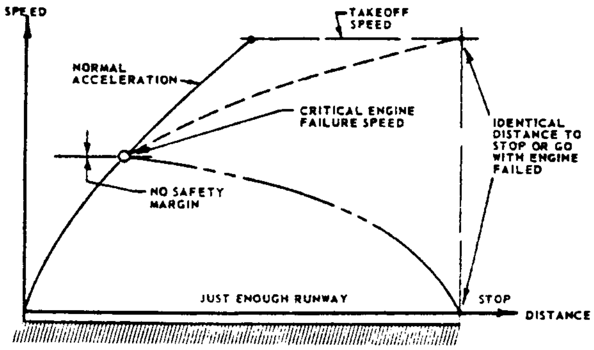


FIGURE 8.7

CRITICAL FIELD LENGTH



Critical Engine Failure Speed:

If the critical engine fails at this speed, the distance required to complete the takeoff is identical to the distance required to stop. The total runway required to accelerate to this speed, lose an engine, and then stop or go is the Critical Field Length.

Initial Climb Performance:

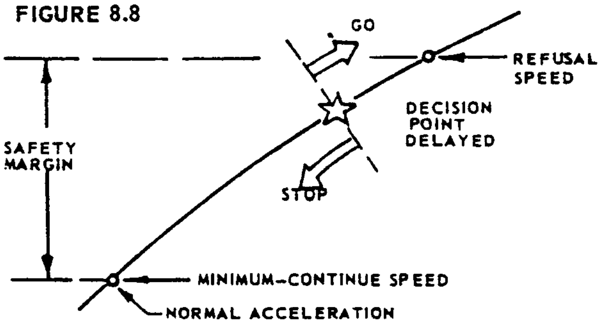
The period between lift-off and attainment of best engine-out climb speed can be very critical. Major air commands normally specify a minimum authorized rate of climb between 200 and 500 feet per minute, for engine out operations. Obviously, this level of performance allows little margin for mis-management of attitude or configuration. Flap retraction may have to be accomplished incrementally on a very tight speed schedule to keep sufficient lift for a positive climb gradient without excessive drag. Unexpected characteristics may be encountered in this phase. For example, the additional drag due to doors opening might make it desirable to delay gear retraction until late in the clean-up phase or in another instance, the time available to obtain the clean configuration might be limited by the supply of water injection fluid if dry thrust is insufficient to maintain the climb. Careful flight test exploration of this phase is an obvious requirement.

Decision Speed/Distance:

All the performance discussions above are concerned with what the aircraft will actually do. It still remains for the pilot or operational authority to decide at what particular speed or distance the course of action will change from stop to go in the event of engine failure. This defines the decision point.

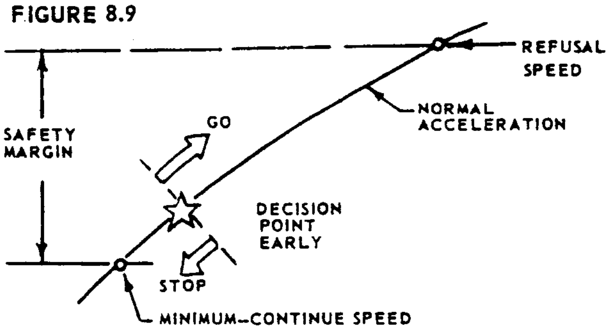
If the initial climb performance is going to be critical on the takeoff in question, the decision point may be near the higher speed end of the safety margin.

FIGURE 8.8



The B-47 illustrates the opposite case. This aircraft has a very poor record for successful aborts and is operated with the decision point relatively early (near the low speed end) of the safety margin.

FIGURE 8.9



Other cases may be decided by the nature of the overrun or the terrain beyond the runway, i.e., is it better to go off the far end almost stopped or almost flying?

8.4 ENGINE-OUT FLIGHT TESTING

Military aircraft are usually designed with relatively low safety margins in order to attain the desired performance - the marginal engine-out climb capability previously mentioned is an example. In fact, during war emergency operation the gross weight may be so high that engine-out operation is not possible at all. Flight tests of these critical phases, on or near the ground, require a high level of crew skill and proficiency; each point must be carefully planned and flown.

Such tests are a normal part of the developmental testing of a new aircraft. They also play a vital part in side-by-side evaluations of assault or VSTOL transports where the ability to carry a useful load in and out of a given landing area is frequently limited by engine-out performance. Individual evaluations to determine if an aircraft meets the contractor's guarantees may also hinge on this area of operation.

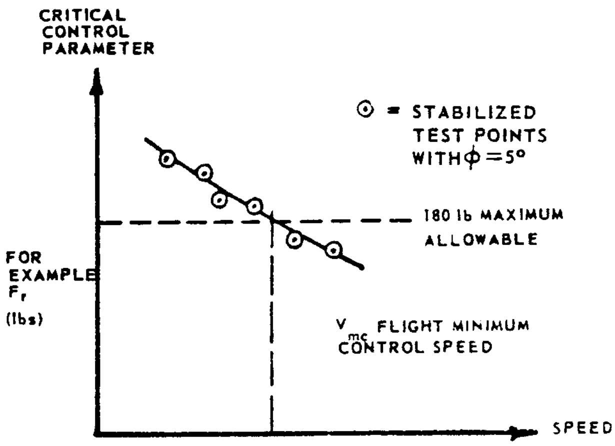
Flight Minimum Control Speed:

It will be shown later in paragraph 8.5 that an aircraft with an engine inoperative can be stabilized in straight (unaccelerated) flight in various combinations of bank angle, sideslip angle, and rudder deflection. For determination of minimum control speed, the maximum bank angle of 5 degrees allowed by MIL-F-8785 will be held constant, and the other two parameters adjusted as necessary to achieve straight flight.

The critical engine is always an outboard engine. For prop-driven aircraft with clockwise rotating propellers (looking forward), the left outboard is critical due to propeller fin effect. Assuming there is no angular motion of the aircraft to provide gyroscopic couples from rotating engines, left or right is usually not critical on a jet powered aircraft (the distinction may be important for dynamic points, however).

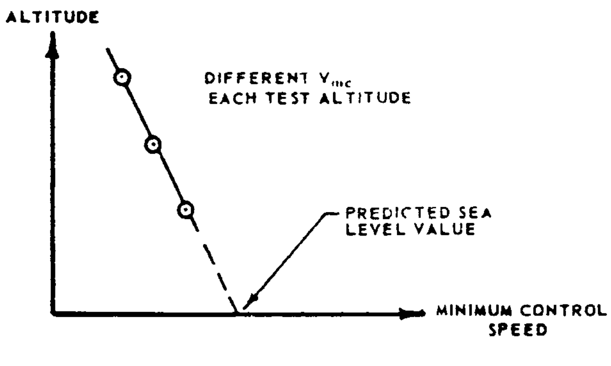
With the aircraft in the specified configuration, and with the critical engine failed, a series of stabilized points are recorded at decreasing speeds. A plot of the critical control parameter (this will most frequently be rudder force or deflection) versus airspeed is made to determine the minimum control speed.

FIGURE 8.10



The minimum control speed usually increases at lower altitude due to increased engine thrust; the test must be accomplished at more than one altitude, including one as low as is safely possible, to provide an extrapolation to sea level.

FIGURE 8.11



Care must be exercised to obtain points that are unaccelerated and well stabilized. The outside visual attitude is primary for maintaining airspeed, bank angle, and zero yaw rate, using the cockpit instruments for cross check. The ball, which in unaccelerated flight will always be at the bottom of the race, is the primary reference used to eliminate accelerations that result from unbalanced forces in the y direction. These lateral translations are difficult to discern visually.

Dynamic Engine Failure:

The military specifications (MIL-F-8785 para 3.3.9.3) require that a pilot be able to avoid dangerous conditions that might result from the sudden loss of an engine during flight. The method to test compliance with this specification is to stabilize with symmetrical power and suddenly fail the most critical engine. After observing a realistic time delay for pilot realization and diagnosis, the pilot arrests the aircraft motion and achieves the equilibrium engine-out condition. Since it obviously requires more control to arrest the motion than to maintain equilibrium, this dynamic situation must be considered in determining the minimum control speed.

Minimum control speed should not be set by any factor other than insufficient control. If the aircraft stalls before reaching the minimum control speed, a statement that "at this gross weight, the aircraft is controllable down to the stall" is preferable to calling the stall speed the "minimum control speed."

Ground Minimum Control Speed:

The ground minimum control speed will differ from the flight value because of:

1. The inability to use sideslip and the restriction on the use of bank angle.
2. Cross wind components.
3. The additional yaw moments produced by the landing gear, which in turn vary with: the landing gear configuration; the amount of steering used; the vertical loads on each gear; and runway condition.

There are two basic test methods, one involving acceleration and the other deceleration. If the aircraft will decelerate with the asymmetric power condition set up (symmetrical pairs of non-critical engines may also be retarded) the "back-in" method may be used. The test is started at a ground speed in excess of the expected minimum and the power condition is set. As the speed decreases, more aerodynamic control deflection is required; the speed where directional control cannot be maintained is the minimum control speed.

Some high performance aircraft accelerate in the test condition and the acceleration method is required. The asymmetric yawing moment is gradually increased (by throttle manipulation) as increasing speed provides more control. The speed where sufficient control is available to hold the full asymmetric power condition is the minimum control speed. This method requires considerable skill and coordination to obtain good results - the aircraft is essentially at minimum control speed throughout the acceleration.

Both of the methods above determine equilibrium control speeds. When sufficient experience has been obtained, sudden engine cuts are performed to determine how much more restrictive margins must be for dynamic failures.

In-Flight Performance:

Normal performance flight test methods may be used to determine the climb, range, and endurance at altitude with engines inoperative.

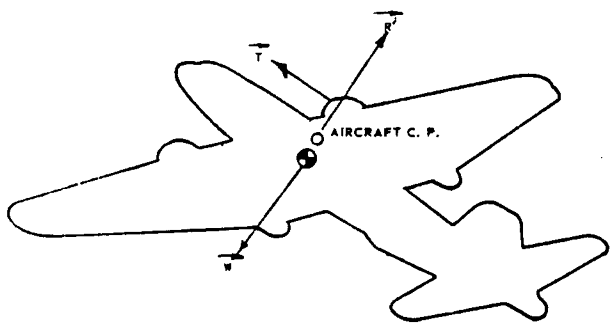
Landing Performance:

Restricted reversing capability and possible higher approach speeds required to maintain minimum safe speeds will affect landing performance. Normal flight test methods are valid, but caution must be exercised in go-around situations, especially at light gross weights.

● 8.5 EFFECT OF BANK ANGLE ON THE EQUILIBRIUM CASE

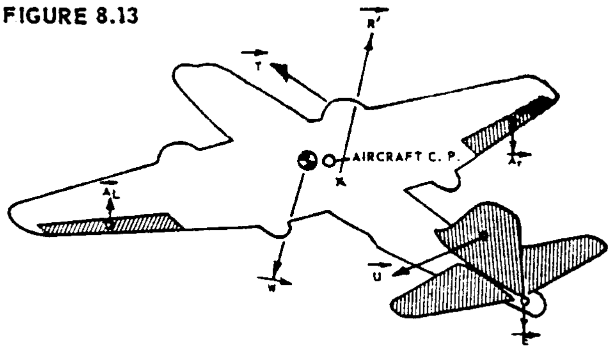
If torque and gyroscopic effects due to rotating engines or propellers are neglected, all the forces acting on an aircraft in flight with an engine inoperative are shown in figure 8.12.

FIGURE 8.12



The vector $\vec{R}$ is the total aerodynamic reaction acting at the aircraft center of pressure. This vector may be thought of as the sum of all the smaller reactions acting on the separate parts of aircraft. For the present discussion it is convenient to handle separately the smaller reactions that act on the ailerons ($\vec{A_r}$ and $\vec{A_L}$), the vertical fin and rudder ($\vec{U}$) and the horizontal stabilizer and elevator ($\vec{E}$), as shown in figure 8.13.

FIGURE 8.13

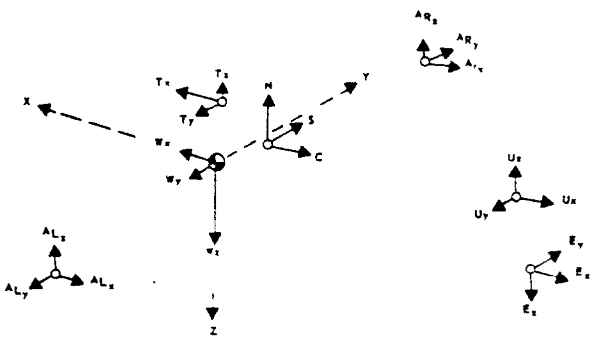


$\vec{R}$ is now the remaining portion of the total aerodynamic reaction, such that:

$$\vec{R} = \vec{R} + \vec{A_R} + \vec{A_L} + \vec{E} + \vec{U}$$

$\vec{R}$ has a point of action that is near, but not necessarily at, the aircraft c.p. The weight vector acts through the cg at the origin of the body axis system. Because of the possibility of sideslip, none of the aerodynamic force vectors necessarily pass through a body axis, i.e., they may all produce moments in three directions. When all forces are resolved into components parallel to the body axes, the representation in figure 8.14 is obtained.

FIGURE 8.14



If the restriction of equilibrium (unaccelerated) flight is now imposed, six equations result:

Longitudinal	Lateral-Directional
(1) $F_x = 0$	(4) $F_y = 0$
(2) $F_z = 0$	(5) $L = 0$
(3) $M = 0$	(6) $N = 0$

The longitudinal equations are not critical in the achievement of equilibrium; they are balanced by the usual technique of stabilized points, i.e., variation of pitch angle (θ), angle of attack (α), and elevator deflection (δ_e).

If the forces of figure 8.14 are all moved to the cg and the moments lost by the move are replaced, the six equations can be expanded as shown below.

$$(1)\quad W_x + \underbrace{E_x + U_x + A_{R_x} + A_{L_x}}_{\text{control drag}} + \underbrace{C + T_x}_{\text{other drag}} = 0$$

W_x → equation is balanced by θ

$$(2)\quad N + \underbrace{E_z + U_z + A_{R_z} + A_{L_z}}_{\text{very small}} + W_z + T_z = 0$$

N → equation is balanced by α

$$(3)\quad M/_E + M/_T + M/_N + M/_C + \underbrace{M/_U + M/_{A_{R,L}}}_{\text{very small}} = 0$$

$M/_E$ → equation is balanced by δ_e

The lateral-directional equations are of most interest in achieving equilibrium. The roll equation (5) is usually not critical, although in some cases lack of roll authority may be the limiting factor.

$$(5)\quad L/_{A_{R,L}} + L/_T + L/_U + \underbrace{L/_N + L/_S + L/_E}_{\text{very small}} = 0$$

$L/_{A_{R,L}}$ → equation is balanced by δ_a

It now remains to be shown that the side force equation (4) and the yaw equation (6) can be simultaneously balanced using an infinite number of combinations of bank angle (ϕ), sideslip (β), and rudder deflection (δ_r).

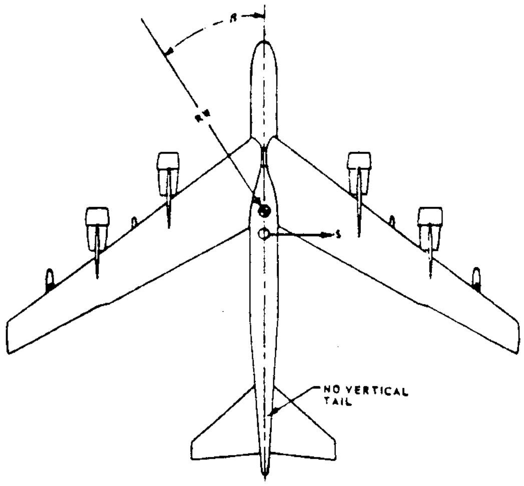
$$(4)\quad \underbrace{W_y}_{W \sin\phi} + U_y + S + \underbrace{E_y + A_{R_y} + A_{L_y}}_{\text{small values combined into S}} = 0$$

Side Force Equation: $W \sin\phi + U_y + S = 0$

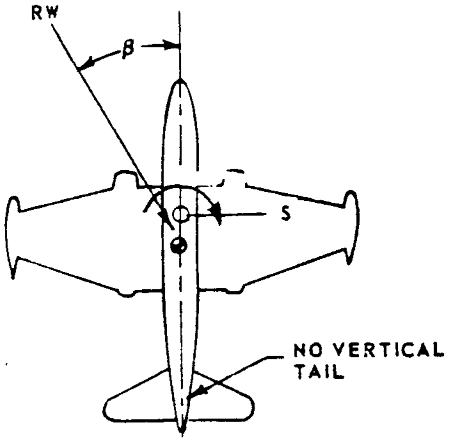
$$(6)\quad \underbrace{N/_T + \underbrace{N/_C + N/_E}_{\text{small}} + \underbrace{N/_{A_{R,L}}}_{\text{adverse yaw}}}_{\text{lumped together as } N_{Forcing}} + N/_U + N/_S = 0$$

Yaw Equation: $N_{Forcing} + N/_U + N/_S = 0$

The point of action of S is related to the directional stability with the vertical tail removed. Certain aircraft, such as B-52 with its long, slab-sided aft fuselage and swept wings, might have some directional stability without the vertical tail installed, in which case S would operate aft of the cg.



Aircraft more generally will be directionally unstable in this condition, and S will operate ahead of the cg.



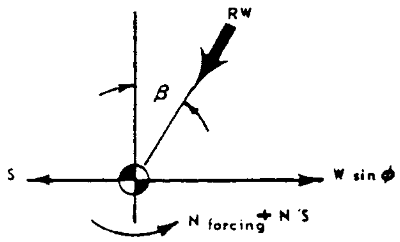
In either case S will have a short arm compared to that of U_Y and the sign of $N/_S$ (which in the discussion below will be considered unstable) or the effect of the other simplifying assumptions will not alter the diagrams below.

Equilibrium with $\delta_r = 0$

β from good engine side

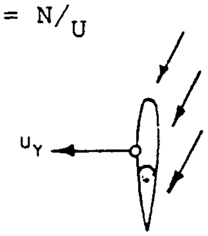
φ large

$\delta_r = 0$



$W \sin \phi = U_Y + S$

$N_{Forcing} + N/_S = N/_U$

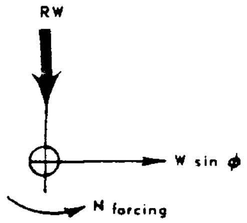


Equilibrium with β = 0

β = 0

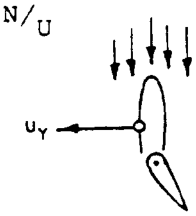
φ = reduced

δ_r = moderate



$W \sin \phi = U_Y$

$N_{Forcing} = N/_U$

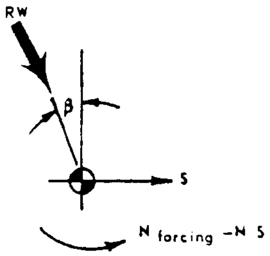


Equilibrium with φ = 0

β from bad engine side

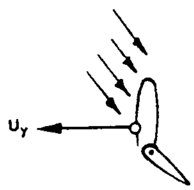
φ = 0

δ_r = large



$U_Y = S$

$N_{Forcing} - N/_S = N/_U$

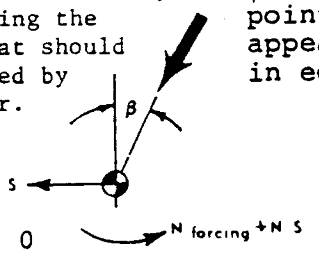


It may be shown by trial and error that the arrangements above are the only ones possible for the conditions specified provided the aircraft is truly following an unaccelerated flight path. For example, if the $\beta = 0$ condition is attempted with β from the good engine side, equilibrium cannot be obtained.

False $\phi = 0$ Point

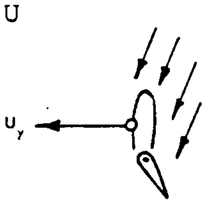
Aircraft is accelerating laterally because of insufficient rudder, i.e., weathercock stability is providing the moment that should be provided by the rudder.

Because $\psi = 0$ the point may appear to be in equilibrium



$U_y + S \neq 0$

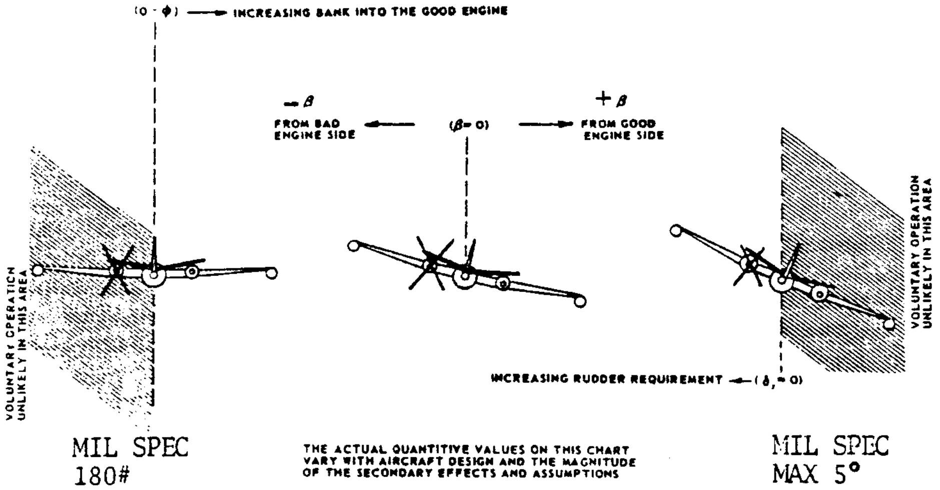
$N_{Forcing} + N/_S = N/_U$



Compared to the true $\phi = 0$ condition, more U_y is required to balance the yaw equation (since $N/_S$ is now added to $N_{Forcing}$) but it is obtained with less δ_r because of the favorable β at the tail. But the side force equation is not balanced ($U_y + S \neq 0$) and the aircraft is actually accelerating to the left. This condition, which can be readily attained in flight if insufficient rudder is used in the $\phi = 0$ condition, is difficult to see visually but can be recognized by a displacement of the ball to the right. If additional right rudder is applied until the ball returns to the bottom of the race, the sideslip will return to the bad engine side.

The relationship between ϕ, β, and δ_r revealed above is summarized in figure 8.15.

FIGURE 8.15


(0 - ϕ) → INCREASING BANK INTO THE GOOD ENGINE
− β
+ β
FROM BAD ENGINE SIDE ←
(β = 0)
→ FROM GOOD ENGINE SIDE
VOLUNTARY OPERATION UNLIKELY IN THIS AREA
VOLUNTARY OPERATION UNLIKELY IN THIS AREA
INCREASING RUDDER REQUIREMENT ← (δ, = 0)
MIL SPEC
180#
THE ACTUAL QUANTITIVE VALUES ON THIS CHART VARY WITH AIRCRAFT DESIGN AND THE MAGNITUDE OF THE SECONDARY EFFECTS AND ASSUMPTIONS
MIL SPEC
MAX 5°

CHAPTER 9

REVISED JULY 1975

DYNAMIC STABILITY

9.1 PURPOSE

The purpose of the dynamic stability flight test is to investigate an aircraft's primary modes of motion. This investigation will ascertain the acceptability of these modes - frequency and damping being the characteristics of primary importance.

9.2 AIRCRAFT MODES OF MOTION

The characteristic modes of motion of a modern aircraft are becoming of more interest as flight regions expand. An aircraft that has its mass primarily distributed along the fuselage and is designed for high speed flight could foster undesirable conditions during certain flight regions. The dynamic response of an aircraft to various pilot control inputs is important in evaluating its handling qualities. The aircraft may be statically stable yet its dynamic response could be such that a dangerous or impossible flight characteristic results. The aircraft must have dynamic qualities that will permit the design mission to be accomplished. One of the test pilot's prime responsibilities is to evaluate these handling qualities with respect to the expected mission.

An airplane usually has five major modes of free motion. (Phugoid, short period, rolling, Dutch roll and spiral.) This chapter will deal with two longitudinal modes first (phugoid and short period) then two lateral-directional modes (Dutch roll and sprial). The rolling mode is covered in Chapter VII.

There are several different forms that the modes of motion may take. Figure 9.1 shows four possibilities for aircraft free motion; a pure divergence, a pure convergence, a damped or an undamped oscillation. The aircraft, being a rather complicated dynamic system, will move in a manner that is a combination of several different modes at the same time. One of the problems of flight testing is to initiate the excitation input so that the various individual modes can be picked out and analyzed on an individual basis.

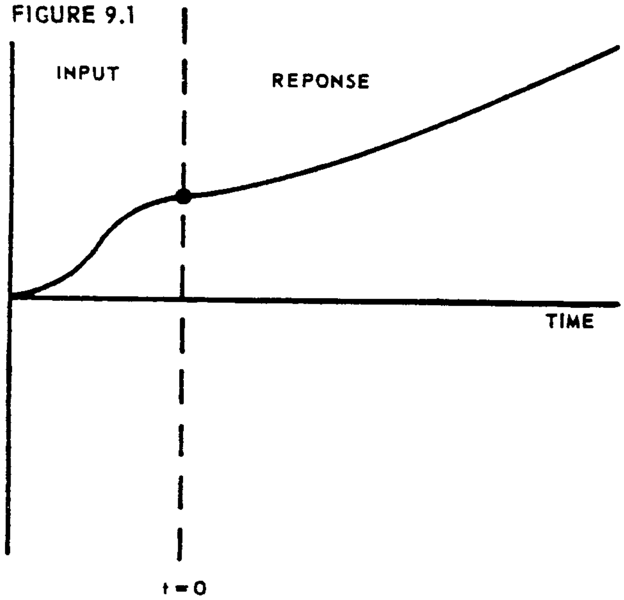
An airplane usually has two major longitudinal modes of free motion. One is the long period mode or "phugoid" which is essentially a variation in airspeed and pitch angle at nearly constant angle of attack. Its period is of the order of 20 seconds to 2 minutes. The other mode is of short period and is characterized by an oscillation of angle of attack and pitch angle at nearly constant airspeed. Its period is usually less than four seconds.

The phugoid mode is generally not considered an important flying quality because its period is usually of sufficient duration that the pilot has little difficulty in controlling it. However, under certain conditions it is possible for the damping to degenerate sufficiently so that the phugoid mode becomes important. The phugoid is characterized by airspeed, altitude, pitch angle, and rate variations while at essentially constant angle of attack.

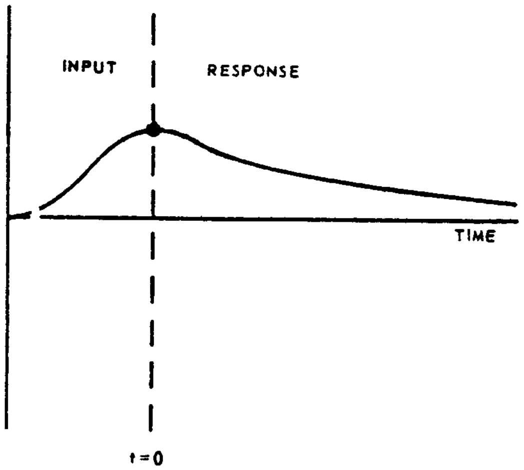
The short period mode is an important flying quality because its period can approach the limit of pilot reaction time and it is the mode which a pilot uses for longitudinal maneuvers in normal flying. The period and damping may be such that the pilot may induce an unstable oscillation if he attempts to damp the motion with control movements. Hence, heavy damping of this mode is desirable. In most airplanes the short period mode is sufficiently damped, but some airplanes must be fitted with artificial damping devices. These airplanes should be flight tested with dampers on and off. The short period is characterized by pitch angle, pitch rate, and angle of attack change while essentially at constant airspeed and altitude.

Damping is described in terms of damping ratio or number of cycles to damp to a specified fraction of initial amplitude. Although heavy damping of the short period mode is desired, investigations have shown that damping alone is insufficient for good flying qualities. In fact, very high damping may result in poor handling qualities. It is the combination of damping and frequency of the motion that is important.

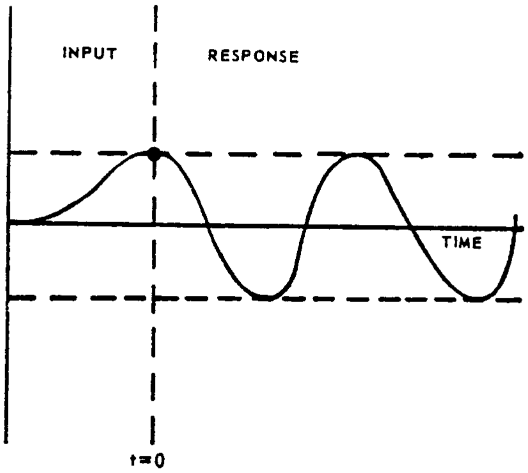
FIGURE 9.1



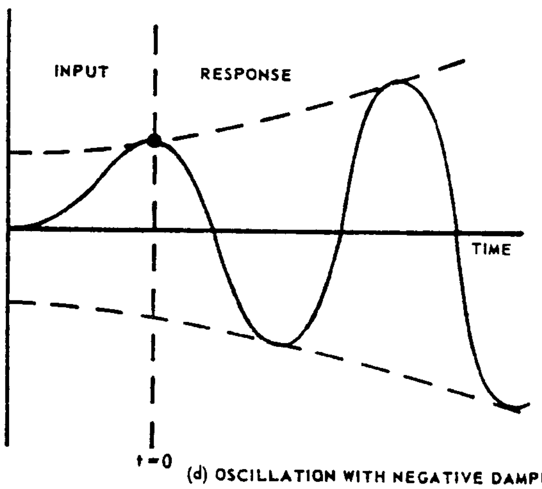
(a) PURE DIVERGENCE



(b) PURE CONVERGENCE



(c) OSCILLATION WITH ZERO DAMPING



(d) OSCILLATION WITH NEGATIVE DAMPING

The longitudinal modes should be flight tested for open-loop as well as closed-loop stability, since open-loop longitudinal modes can also be important. In open-loop motion, the elevator and control system is free to move (pilot does not hold the control) so that its motion is coupled with the longitudinal stick-fixed modes. The influence of the free elevator depends upon the magnitude, frequency and phase of elevator motion.

Tests for short period stability should be conducted from level flight at several altitudes and Mach numbers. Closed loop short period stability tests should be made also at various normal accelerations in maneuvering flight. This stability, when coupled to the pilot, is especially important to tracking and formation flying.

9.3 MILITARY SPECIFICATION REQUIREMENTS

MIL-F-8785B specifies that an aircraft's short period response, controls fixed or free, shall meet the requirements of frequency damping and acceleration sensitivity established in para. 3.2.2.1, 3.2.2.1.1 and figure 1. Residual oscillations shall not be greater than 0.05g at the pilot's station nor more than ±3 mils of pitch excursion for category A Flight Phase tasks.

9.4 EXAMPLE TEST METHODS

The phugoid mode may be examined by stabilizing the airplane at the desired flight conditions and trimming the control forces to zero. Increase or decrease the airspeed by some small increment by the proper control pressure. For stick-fixed stability return the control to neutral and hold it fixed. For stick-free stability, return the control to neutral and then release it. After the control is released or returned, it may be necessary to maintain wings level by light lateral or slight directional pressure. Damping and frequency of phugoid motion may be changed appreciably by the presence of small bank angles (5 to 15 degrees). It may be very difficult to return the control to its trimmed position if the aircraft control system has a very large friction band. In such a case, the airspeed increment may be obtained by an increase or decrease in power and returning it to its trim setting or extending a drag device. In either case the aircraft configuration should be that of the trim condition at the time the data measurements are made.

To examine the short period mode, stabilize the airplane at the desired flight condition, (altitude, airspeed, normal acceleration). Trim the control forces to zero (for one g normal acceleration). Abruptly deflect the longitudinal control to obtain a change in normal acceleration of about one-half g. For stick-fixed stability, return the control to neutral and hold fixed. For stick free stability, release the control after it is returned to neutral (normally conducted only from one g flight). The aircraft response should be examined for positive and negative changes in normal acceleration. If the aircraft is equipped with artificial stabilization devices the test should be conducted with this device off as well as on. A note of caution: <u>The abruptness and magnitude of the control input must be approached with due care!</u> Use very small inputs until it is determined that the response is not violent. A suggested technique is to apply a longitudinal control doublet (a small positive displacement followed immediately by a negative displacement of the same magnitude followed by rapidly returning the control to the trimmed position). Start with small magnitudes and gradually work up to the desired excitation.

An input that is too sharp or too large could very easily excite the aircraft structural mode or produce a flutter that might seriously damage the airplane and/or injure the pilot.

Data Required:

For trim conditions, pressure altitude, airspeed, weight, cg position, and configuration should be recorded.

The test variables of concern are, airspeed, altitude, angle of attack, normal acceleration, pitch angle, pitch rate, control surface position, and control position.

Reduction and Presentation of Data:

Time histories of stick-fixed and stick-free oscillations should be presented. A complete analysis would present damping ratio and frequency as a function of flight condition. If the motion were non-oscillatory divergent, the instability could be represented by the time required to attain a certain parameter value from a trimmed condition.

Short period mode investigations have shown that frequency as well as damping is important in a consideration of flying qualities. This is so because at a given frequency, damping alters the phase angle of the closed-loop system (which consists of a pilot coupled to the airframe system). Phase angle of the total system governs the dynamic stability.

A. Phugoid:

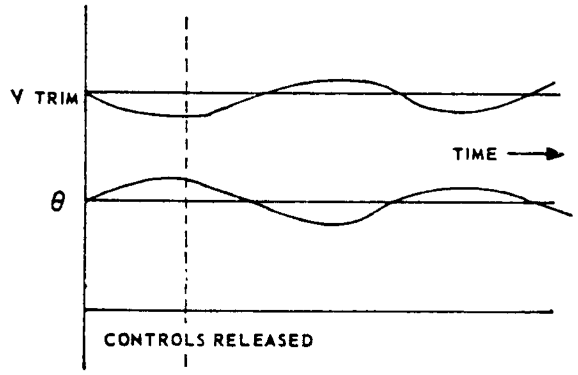
Stabilize the aircraft at the test altitude and the test airspeed. Smoothly increase the pitch angle until the airspeed reduces 10 to 15 knots below the trim airspeed. Very smoothly return the control column to the trim position and release all pressure. **When the pitch angle reverses start timing.** Record the maximum airspeed as the nose comes through level flight. The nose of the aircraft will continue to come up, reverse, and start down. Record the minimum velocity as the nose again passes through level flight. As the pitch angle reverses again mark the time. Continue the maneuver through 3 cycles.

Slight turbulence or imperfect lateral trim may result in wing roll during the pitch oscillations. If this should occur, then control the bank with smooth and light rudder pressures being careful not to excite aircraft Dutch roll.

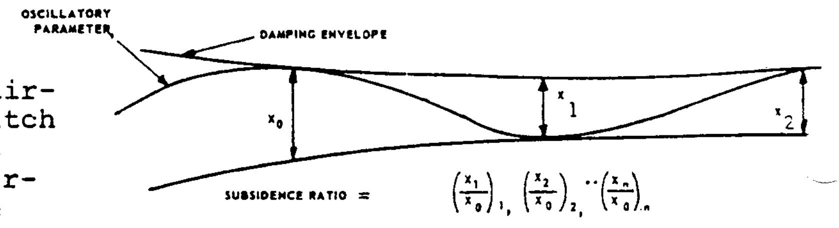
Data Reduction, Phugoid.

1. Plot a time history to include 3 cycles of the phugoid. Label airspeed, pitch angle and angle of attack.

FIGURE 9.2



2. Determine the frequency of the oscillation. Plot cycles versus time in a working plot.



3. Determine the phugoid damping ratio (ζ). Sketch the damping envelope on the working plot of airspeed versus time. Measure the width of the envelope at the peak values of the oscillation. Form the subsidence ratios (X_m/X_0). From figure 9.4 or 9.5 find the damping ratio for each subsidence ratio. Average these damping ratios. If the subsidence ratio is greater than 1.0 then use the inverse of that subsidence ratio. The damping ratio thus determined will be negative, and the mode divergent.

4. Determine the phugoid undamped nautral frequency (ω).

$$\omega_n = \frac{2\pi f}{\sqrt{1 - \zeta^2}}$$

$$f = \frac{\Delta \text{ cycles}}{\Delta \text{ time}}$$ from figure 9.3

FIGURE 9.3

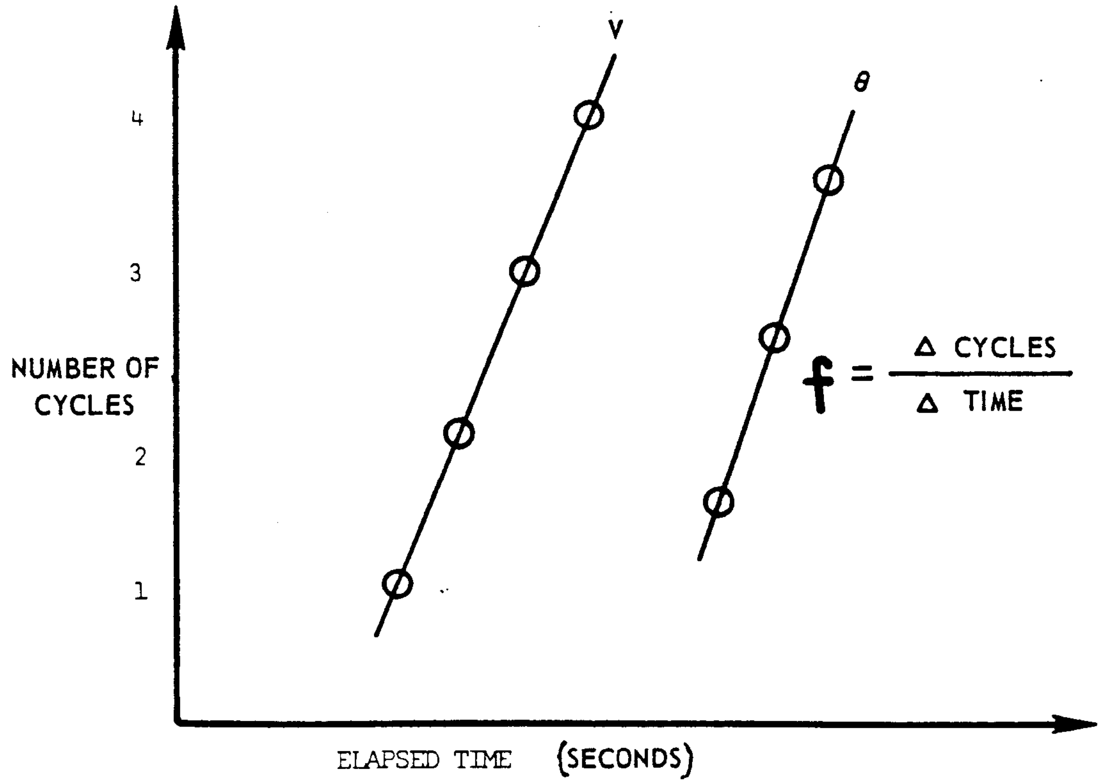
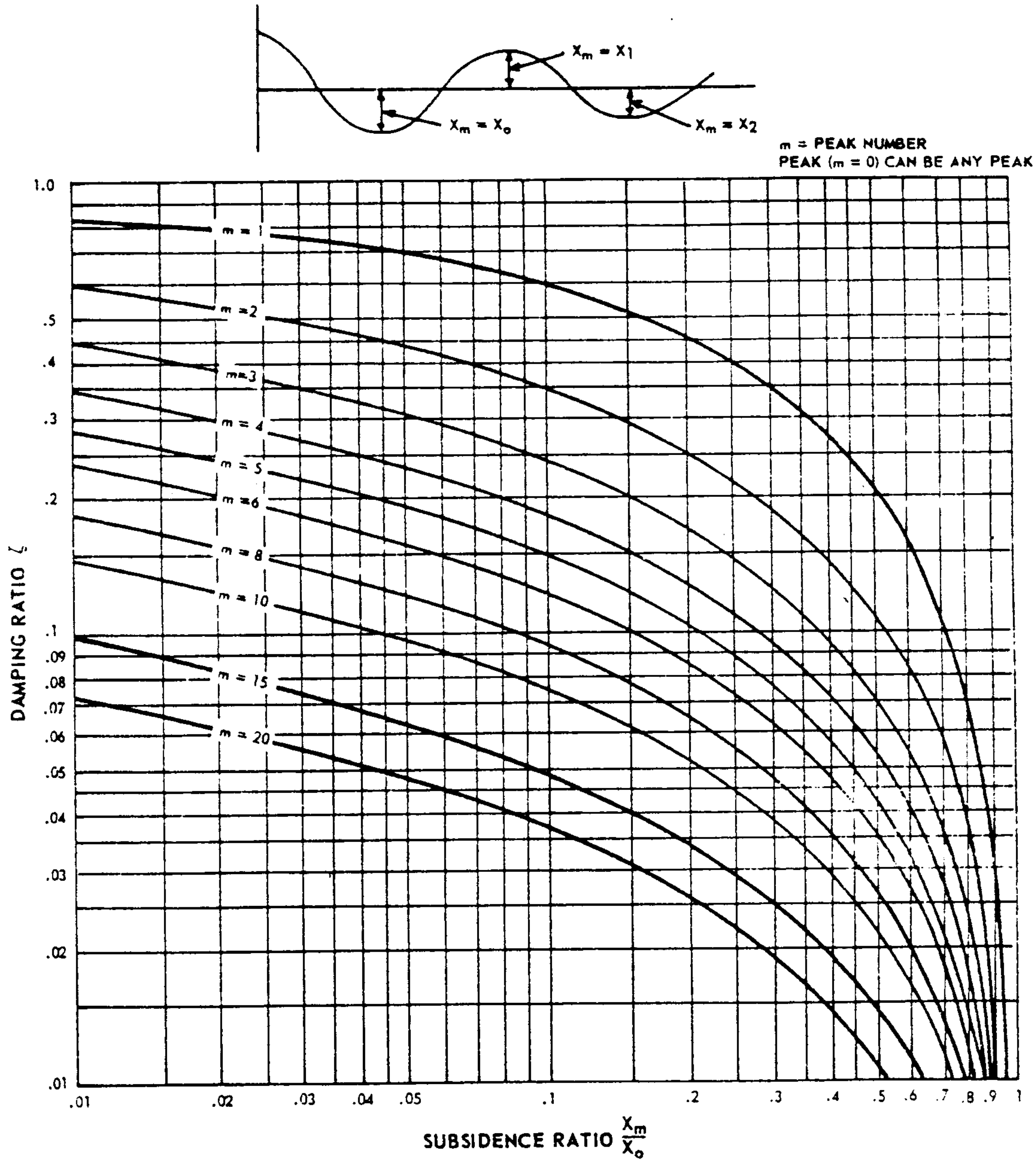
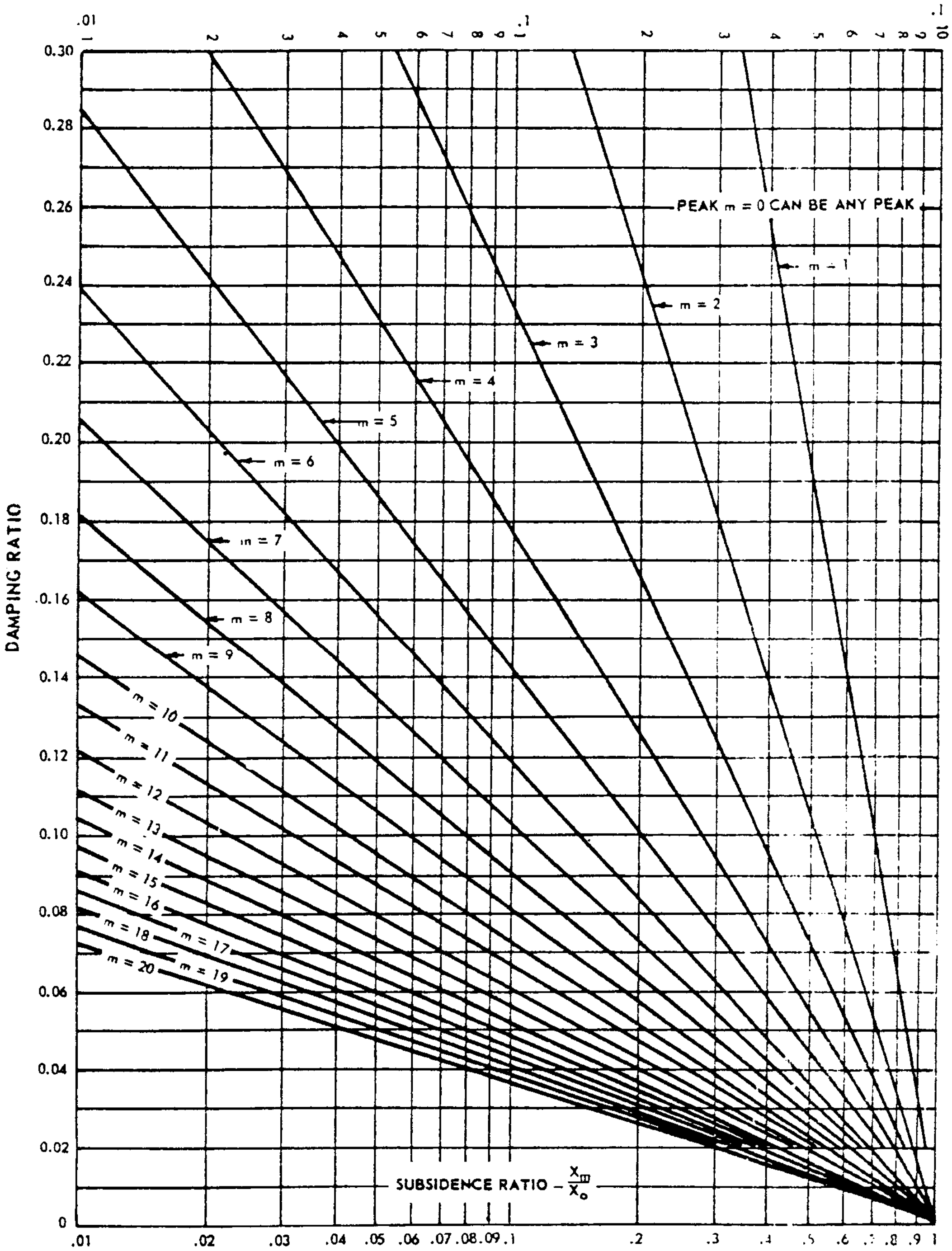


FIGURE 9.4



$X_m = X_1$
$X_m = X_o$
$X_m = X_2$
m = PEAK NUMBER
PEAK (m = 0) CAN BE ANY PEAK
m = 1
m = 2
m = 3
m = 4
m = 5
m = 6
m = 8
m = 10
m = 15
m = 20
1.0
.5
.4
.3
.2
.1
.09
.08
.07
.06
.05
.04
.03
.02
.01
DAMPING RATIO ζ
.01
.02
.03
.04
.05
.1
.2
.3
.4
.5
.6
.7
.8
.9
1
SUBSIDENCE RATIO $\frac{X_m}{X_o}$

FIGURE 9.5



DAMPING RATIO AS A FUNCTION OF SUBSIDENCE RATIO

5. Plot phugoid frequency and damping ratio versus Mach number.

FIGURE 9.6

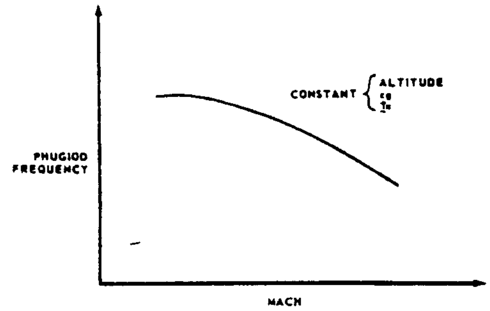
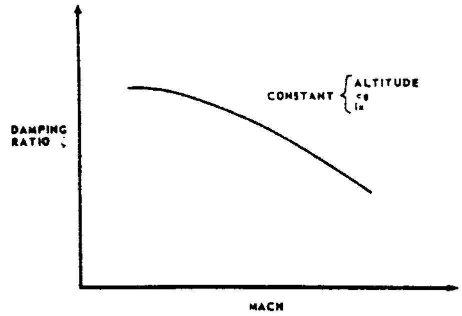
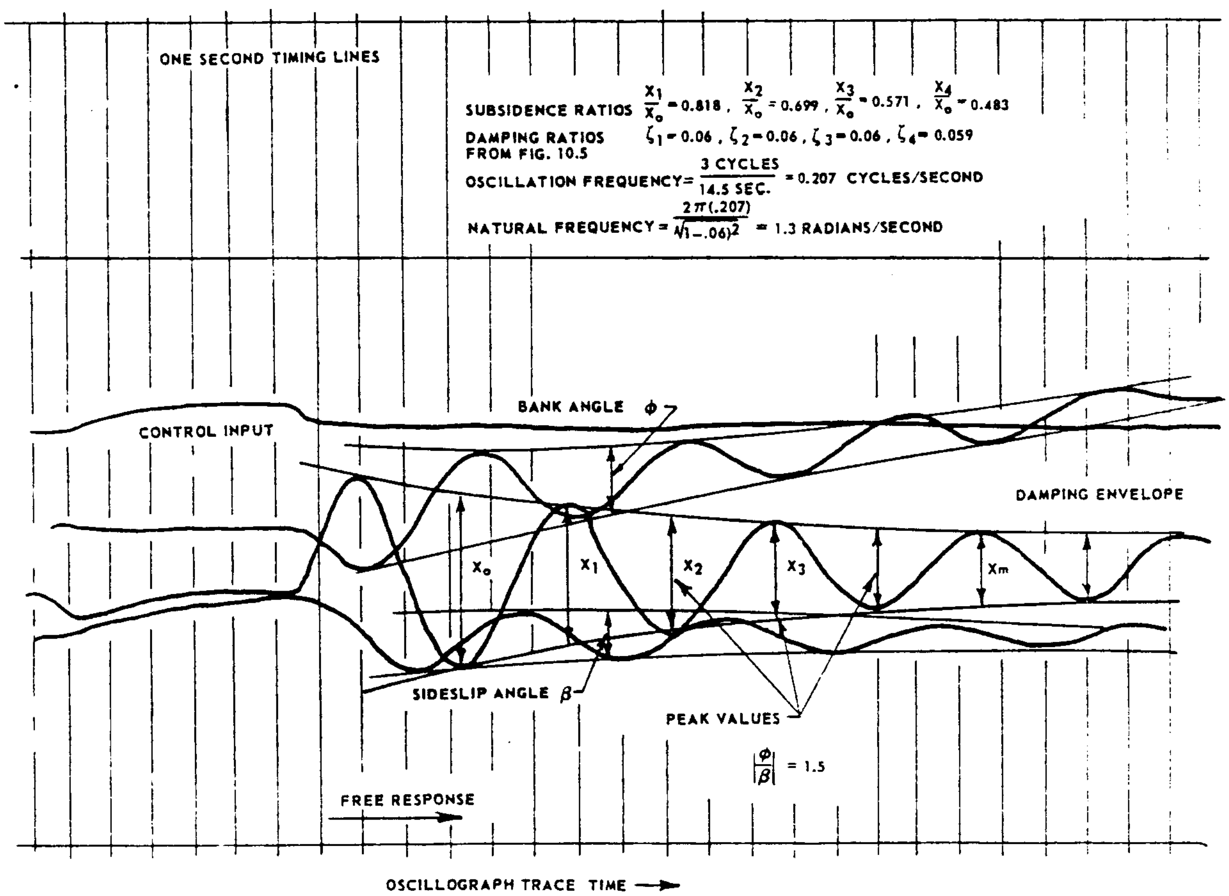


FIGURE 9.7



EXAMPLE NO. 1 DATA REDUCTION LIGHTLY DAMPED OSCILLATION



6. In the test results include a short discussion on the effect the phugoid mode has on the aircraft handling qualities. This discussion should be presented with respect to an intended mission for the aircraft. Compare the damping ratios with the requirements of MIL-F-8785B.

B. Short Period:

Stick-Fixed.

Stabilize the aircraft at the test altitude on the test airspeed. Select oscillograph on and start recording. Smoothly but abruptly pull back on the control column; push it forward, and then rapidly return it to the trimmed position and hold it there. When the aircraft transient motion stops, stop recording data. The airspeed and altitude should remain essentially constant during this maneuver. This input pulse should be started small and gradually increased as the pilot's technique improves and if the aircraft response is satisfactory.

Stick-Free.

Stabilize the aircraft at the test altitude on the test airspeed. Select oscillograph on and start recording. Smoothly but abruptly pull back on the control column, push it forward, return it to approximately neutral and release. When the aircraft transient motion stops, stop recording data.

Data Reduction, Short Period.

1. Plot a time history of the aircraft response for closed-loop and open-loop. Label elevator deflection, control deflection, load factor, and angle of attack.

2. Determine the short period damping ratio (ζ). If the short period response is oscillatory and the damping ratio 0.5 or less proceed as outlined for the phugoid mode. If the damping ratio is between 0.5 and 2.0 then use figure 9.9. Select the point on the response curve at which the response is free. Divide the amplitude into the values 0.736, 0.406, and 0.199. Measure time values t_1, t_2 and t_3. Form the time ratios t_2/t_1, t_3/t_1, and $(t_3 - t_2)/(t_2 - t_1)$. Enter figure 9.9 at the Time Ratio side and find a damping ratio for each time ratio. For this damping ratio find a frequency time product for $(\omega_n t_1)$ $(\omega_n t_2)$ and $(\omega_n t_3)$. Average the damping ratios.

FIGURE 9.8

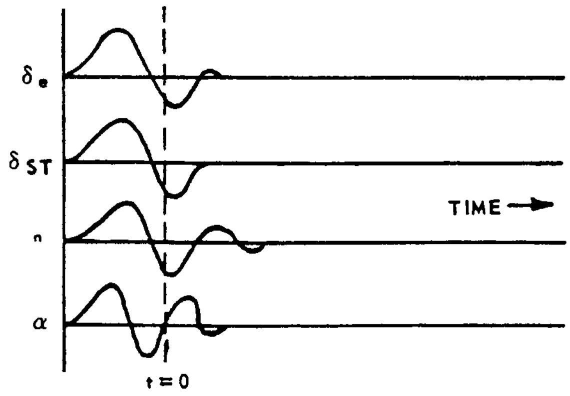
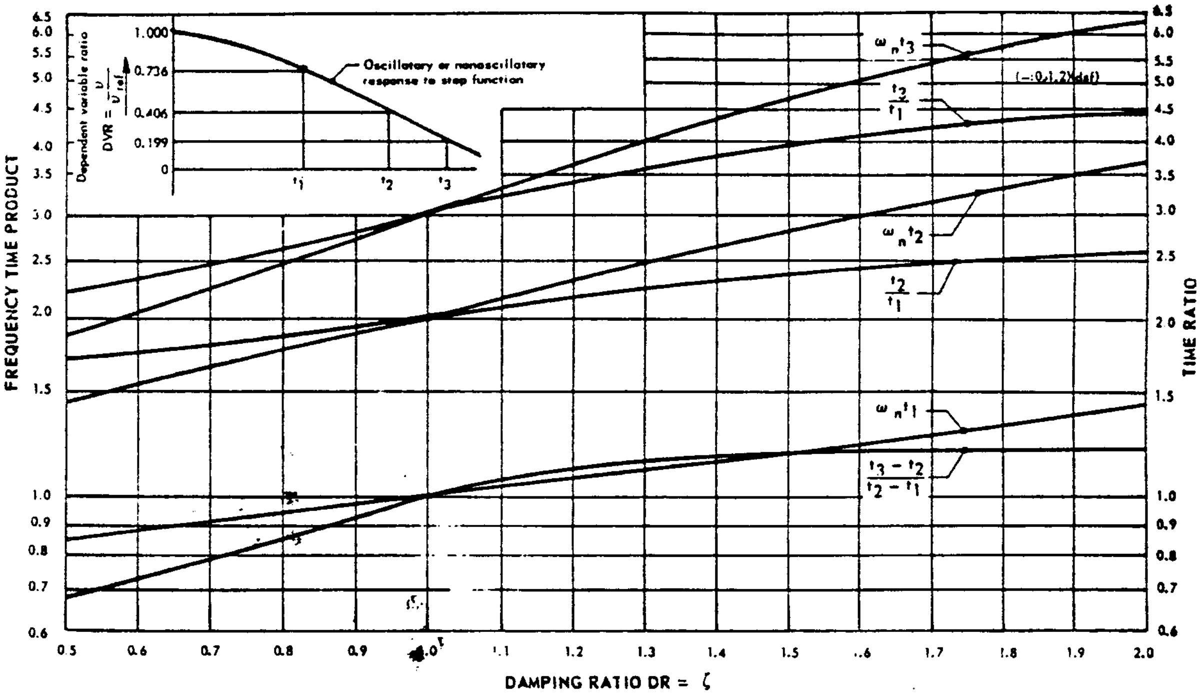


FIGURE 9.9



3. Determine the short period natural frequency (ω_n)

$$\omega_n = \frac{\omega_n t_3}{t_3}$$

$$\omega_n = \frac{\omega_n t_2}{t_2}$$

$$\omega_n = \frac{\omega_n t_1}{t_1}$$

Average the natural frequency.

4. Plot short period natural frequency and damping ratio versus Mach number.

FIGURE 9.10

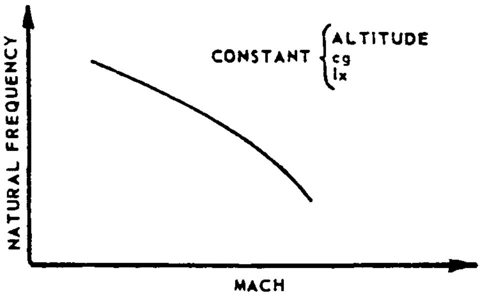
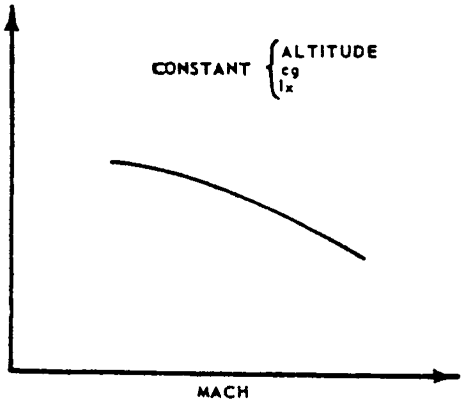


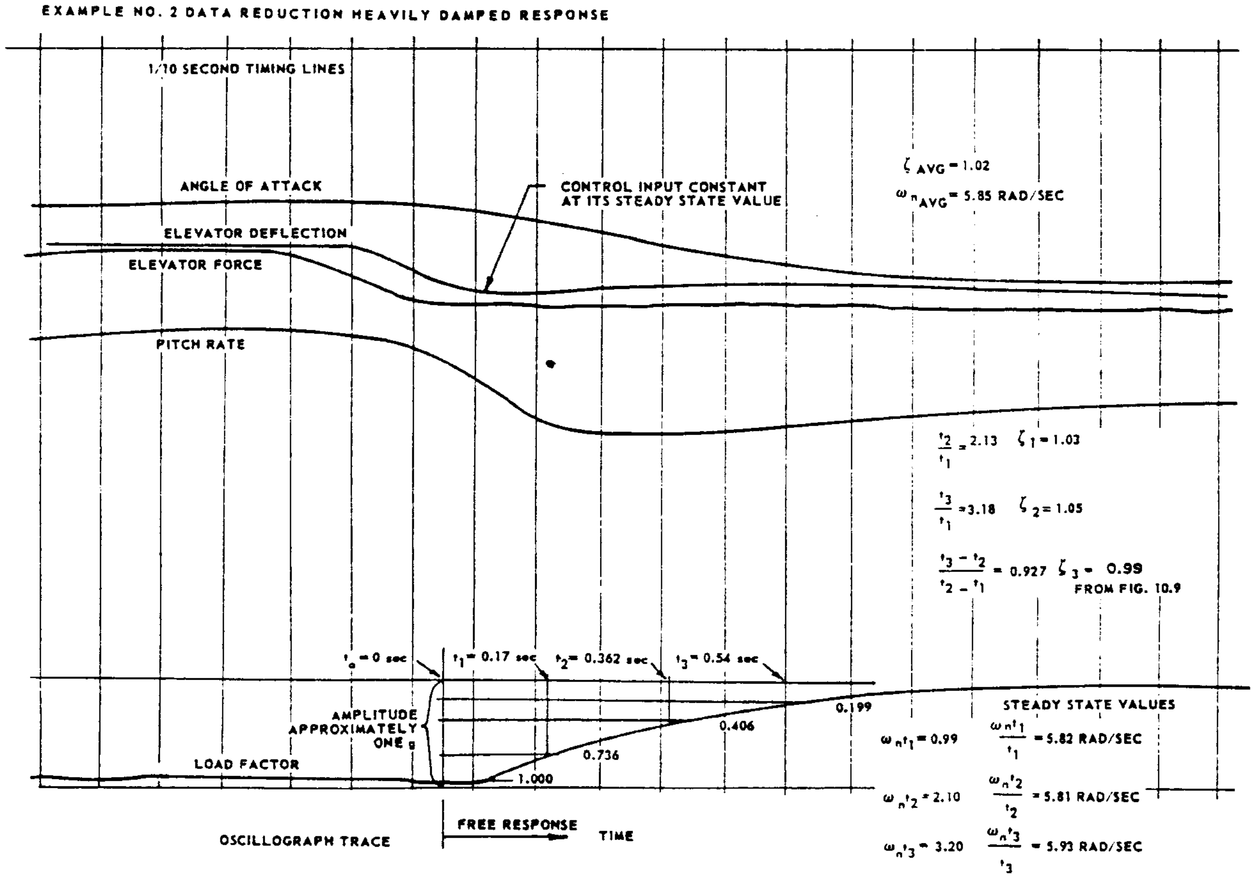
FIGURE 9.11



5. From the oscillograph trace, form the ratio, $\Delta n_z/\Delta\alpha$. This is determined from the peak angle of attack which produced the peak "g." See where ω_n versus n_z/α is located in figure 1, MIL-F-8785B.

6. Compute ω_n as shown for the phugoid mode on Page 9.5.

FIGURE 9.12



• 9.5 LATERAL-DIRECTIONAL DYNAMIC STABILITY

Dutch Roll:

The lateral-directional oscillations involve roll, yaw, and sideslip. The stability of the Dutch roll mode varies with airplane configuration, angle of attack, Mach number, and damper configuration. Dynamic stability of the lateral-directional modes is governed primarily by the static lateral and directional stabilities ($C_{\ell\beta}$ and $C_{n\beta}$), damping in roll and yaw ($C_{\ell p}$ and C_{n_r}), and moments of inertia. The presence of a lightly damped oscillation adversely affects aiming accuracy during bombing runs, firing of guns and rockets and precise formation work such as in-flight refueling.

Stability of the oscillations is represented by the damping ratio; however, the frequency of an oscillation is also important in order to correlate the motion data with the pilot's opinion of handling qualities.

Military Specification Requirements:

Section 3.3 of MIL-F-8785B specifies the requirements for lateral directional handling qualities.

Example Test Methods:

1. Release from Steady Sideslip.

Stabilize the airplane in level flight at test flight conditions and trim forces to zero. Establish a steady straight-path sideslip angle. Rapidly neutralize controls. Either hold controls for control-fixed or release controls for control free response. Start with small sideslip in case the aircraft diverges.

2. Rudder Pulse (Doublet).

Stabilize the airplane in level flight at test flight conditions and trim. Rapidly depress the rudder in each direction and neutralize. Hold at neutral for control-fixed or release rudder for control free response. For aircraft which require excessive rudder force in some flight conditions, the rudder pulse may be applied through the augmented directional flight control system.

3. Aileron Pulse.

Stabilize the airplane in level flight at test flight conditions and trim. Hold aircraft in a steady turn of 10 to 30 degrees of bank. Roll level at a maximum rate reducing the roll rate to zero at level flight. CAUTION . . . Such a test procedure must be monitored by an engineer who is thoroughly familiar with the inertial coupling of that aircraft and its effect upon structural loads and non-linear stability.

Data Required.

For trim condition, pressure altitude, airspeed, weight, cg position, and aircraft configuration should be recorded. The test variables of concern are: bank angle, sideslip angle, yaw rate, bank rate, control positions, and control surface positions.

Reduction and Presentation of Data.

Flight test data will be obtained as time histories. When determining the damping ratio the roll rate parameter usually presents the best trace.

Nonlinearities in the aircraft response may hinder the extraction of the necessary parameters. These can be induced by large input conditions. Small inputs balanced with instrument sensitivity give the best result.

Moderately damped high frequency oscillation may be less satisfactory than a lightly damped low frequency oscillation. If the frequency is higher than pilot reaction time, the pilot cannot control the oscillation, and in some cases may reinforce the oscillation to an undesirable amplitude. Since it is the damping frequency combination which influences pilot opinion more than damping alone, some effort should be made to correlate this combination with pilot opinion of the lateral-directional oscillation.

At supersonic speeds, directional stability often decreases with increased Mach number and altitude for constant g. An evaluation should proceed cautiously to avoid possible divergent responses that can result from nonlinear aerodynamics.

Control-Free Dutch Roll.

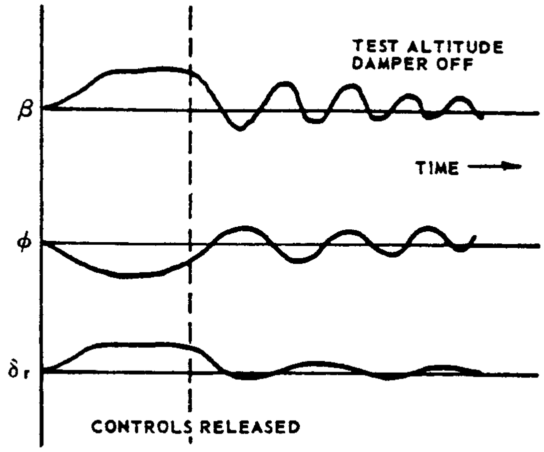
Stabilize the aircraft on test altitude and airspeed. Select oscillograph and start recording. Smoothly establish a steady straight sideslip using rudder and aileron. Release the controls. Start counting and timing oscillations when the aircraft nose reaches its extreme position from where it was released. Stop recording when the oscillation stops or after 5 to 8 cycles. Use caution and avoid any excessive sideslip angles.

Restabilize the aircraft with the yaw damper off. Establish a steady straight sideslip and release controls. Start counting and timing cycles when the nose reaches its extreme position from the point of release. Stop recording after 5 to 8 cycles. Use caution and avoid any excessive sideslip angles.

Data Reduction.

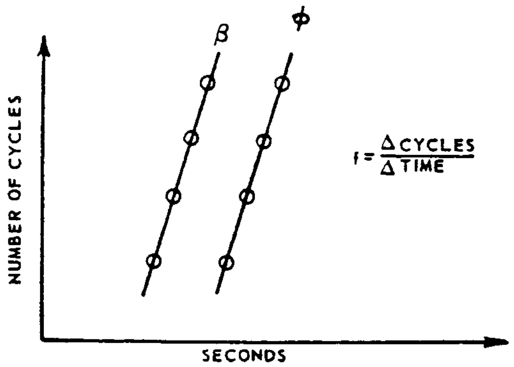
1. Sketch 5 cycles of the Dutch roll in each configuration, Label sideslip, bank angle and rudder deflection.

FIGURE 9.13



2. Determine the frequency of the oscillation. Plot cycles versus time on a working plot.

FIGURE 9.14



3. Determine the Dutch roll damping ratio (ζ) and natural frequency (ω_n) in the same manner as the phugoid mode.

4. Plot Dutch roll damping ratio and natural frequency versus Mach number for each configuration (damper on, damper off, rudder power off).

FIGURE 9.15

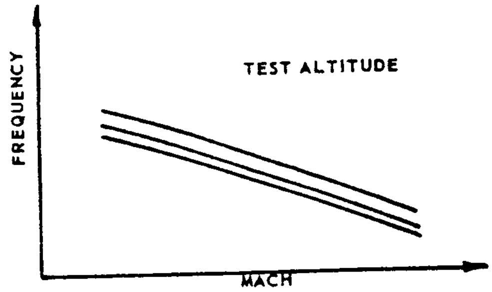
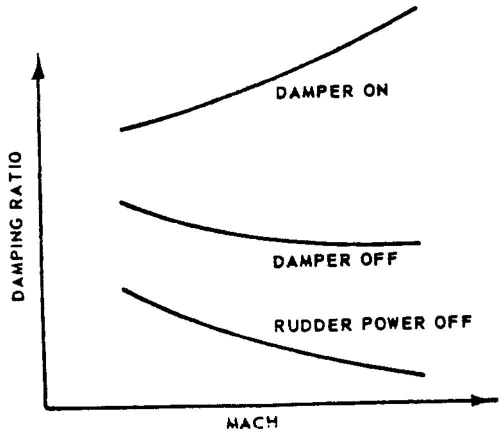
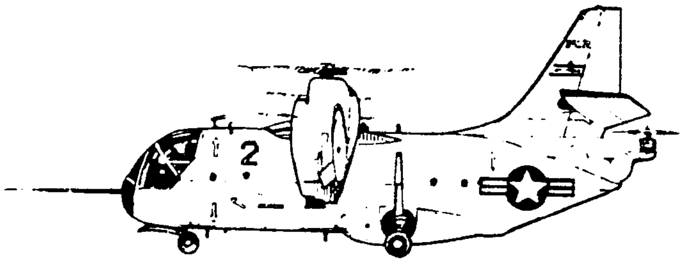


FIGURE 9.16



5. Compare the ω_n, ζ and $\zeta\omega_n$ so obtained with table IV of MIL-F-8785B and determine compliance.



9.6 SPIRAL MODE

The spiral mode is, in **general**, relatively unimportant as a flying quality. However, a combination of spiral instability and lack of precise lateral trimmability may be bothersome to the pilot. This problem will be evaluated as a whole due to the difficulty in separating the effects.

The divergent motion is non-oscillatory, and is most noticeable in the bank and yaw responses. If an airplane is spirally divergent, it will, when disturbed and not checked, go into a tightening spiral dive. This divergence can be easily controlled by the pilot if the divergence is not too fast.

Military Specification Requirements:

Spiral Stability is specified in MIL-F-8785 in table VIII. This table established limiting terms to double amplitude when the aircraft is put into a bank, up to 20 degrees, and the controls freed.

Example Test Methods:

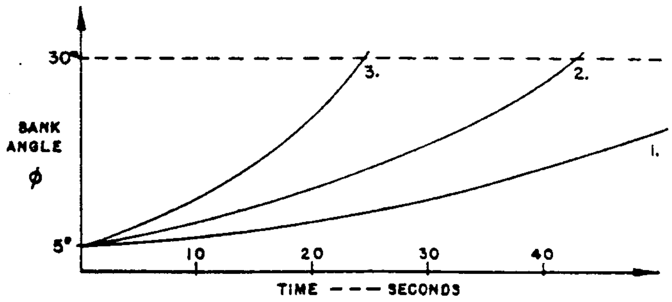
Trim the aircraft for hands-off flight, insuring that particular attention is given to lateral control and the ball being centered. Roll into a 20-degree bank in one direction, release the controls and measure the bank angle after 20 seconds. Repeat the maneuver in a bank to the opposite side.

Data Required:

Aircraft configuration, weight, cg position, altitude and airspeed should be recorded. The test variables are bank angle, sideslip angle, control position, and control surface position.

Excitation of the spiral mode only is difficult because of its relatively large time constant. Any practical input using control surfaces would usually excite other modes as well. If a deficiency in lateral trim control exists, it is often difficult to determine what portion of the resultant motion following a disturbance is caused by the spiral mode. This flight test is used to determine if a combined problem of lateral trim and spiral stability exists. If test results show a definite divergence in hands-off flight, the problem exists.

FIGURE 9.17



3. Briefly discuss the spiral mode characteristics with respect to an intended mission.

Spiral divergence, on its own, is of little importance as a flying quality, which is well within the control capability of the pilot. The ability to hold lateral trim in hands-off flight for 10 to 20 seconds is important.

Spiral Mode:

Stabilize the aircraft at test altitude and airspeed. Establish a 20-degree bank and release controls. Time the motion for 20 seconds and read bank angle. Establish an opposite 20-degree bank and repeat.

Data Reduction:

Average the time to double amplitude for right and left banks at each test condition. Compare with table V of MIL-F-8785B.

QUALITATIVE FLIGHT TESTING

REVISED FEBRUARY 1977

10.1 PURPOSE

The purpose of the qualitative flight test is to determine the maximum amount of information in the minimum amount of flying time in order to evaluate an aircraft with respect to its entire mission or some specific area of interest.

Qualitative flight testing has essentially the same purpose as quantitative flight testing, i.e., to determine how well the aircraft flies and how well it will perform its design mission. To accurately evaluate an aircraft from quantitative data requires analysis of large amounts of precisely measured data. The best a pilot can hope to do on a qualitative evaluation is to measure a limited amount of quantitative data. Thus, the test pilot's opinion on the acceptability of the aircraft is the important result and measured quantitative data when available is used primarily to support this opinion. Quantitative values of stick forces measured with a hand gage, for example, should be included in the report to support the pilot's opinion of acceptability. Estimations of values of stick force can be made if no reliable measurements are available or, qualifying terms such as "heavy", "medium", or "light" can be used to describe the forces. The point is that the difference in evaluating an aircraft qualitatively and quantitatively is a matter of degree. "Use what you've got." Pilot opinion supported by measured data is primary in qualitative testing while the reverse is true in quantitative testing. The general rule is to first decide how well the aircraft does its job and then use the quantitative data you can get to support your opinion.

10.2 PILOT OPINION

Naturally, all pilots will not have exactly the same opinion regarding the acceptability or unacceptability of a particular aircraft characteristic. No two people think exactly alike. However, the opinions of pilots with similar experience and background will usually not differ greatly, particularly with respect to the capability of an aircraft to perform a specific mission. In other respects, such as cockpit arrangements, the opinions may vary more markedly. For this reason, it is important for the qualitative test pilot to be as objective as possible in his evaluation. Guides which specify military requirements, such as MIL SPEC 203E and MIL-F-8785(ASG), should be used wherever possible to establish acceptability. However, it should be kept in mind that mere compliance with a set of requirements does not necessarily yield a satisfactory aircraft. The primary question is "will it do the job?", not "does it meet the specifications."

10.3 MISSION PREPARATION

A very limited amount of flight time is normally available for a qualitative evaluation. To acquire the information necessary to write an accurate and comprehensive report on an aircraft in this limited time requires a great deal of pre-flight study and planning.

The pre-flight preparation for a qualitative test is extremely important. It is almost impossible to put in too much time in planning for the flights. The amount of information acquired in the air will be directly proportional to the amount of preparation put in on the ground. A pilot who doesn't know what he is looking for is not likely to find it, and to know exactly what to look for in the evaluation requires considerable knowledge of the aircraft and its mission.

The precise mission of the aircraft is important in determining what specific investigations should be made in the evaluation. All fighters, for instance, do not have the same mission, and the characteristics of particular importance may not be the same. The roll characteristics of an air superiority fighter would be more important than for a long range strategic fighter, and the specific test plan should take this fact into account. Expected outstanding characteristics or weaknesses should also receive particular emphasis. Of course, the evaluation must be conducted within the cleared flight envelope of the aircraft, and the amount of flight time available may limit the number of altitudes, airspeeds and tests that can be investigated. However, concentration on the extremes of altitudes, airspeeds, etc., and the areas dictated by the primary mission will provide the best approach to the test planning.

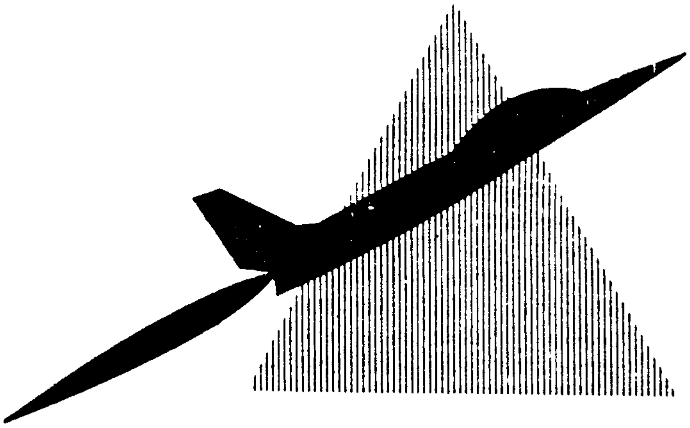
An outline of the test to be conducted and the various altitudes, airspeeds, and configurations to be used will aid in organizing the flights and planning the flight data cards. The points included in the outline should be compatible with the time available for the evaluation but it is always wise to overplan the flight and include more than seems possible to accomplish in the allotted time. Leave yourself the option of skipping the less important parts of your plan if time or fuel run short. The sequence of tests should be such that as little time as possible is wasted. With proper planning a continuous flow from one investigation to the next is possible.

• 10.4 FLIGHT DATA CARDS

Before planning the flight data cards, as much as possible should be learned about the aircraft. Study the pilot's handbook if one is available, discuss the aircraft with the engineers, or with other pilots who have flown it, and get adequate cockpit time. The more the pilot knows about the aircraft and the more comfortable he is in it, the more thorough will be the evaluation. A pilot who doesn't know the aircraft procedures, both normal and emergency or who has to spend most of his time in the air looking for controls or switches will not be able to do much evaluating.

The flight data cards should be self explanatory and should include all the points it is desirable to investigate during the flight. They should be designed so that a minimum of writing is required in the air because time will not be available to write down more than a word or two about each point. Remember, however, to provide places in the flight plan to write down these necessary comments. Numerous forms for the data cards are possible but completeness and legibility are essential.

Some possible formats for planning check lists and flight data cards are contained at the end of this chapter.



10.5 GENERAL TECHNIQUES

The cockpit evaluation can normally be made while getting cockpit time prior to the first flight. MIL SPEC 203E specifies the standard cockpit arrangement for the various types of aircraft in considerable detail and should be used as a guide in making the cockpit evaluation. However, a summary of some of the points to note may prove helpful. These include: ease of entry, comfort, adjustment of seat and controls, location of basic flight instruments, size and legibility of instruments, accessibility of switches and controls, ease of identification of switches and controls, location and identification of emergency switches and controls, methods of escape (both on the ground and airborne), and general impression of cockpit layout.

Several points should be observed and recorded during the start and while preparing the aircraft for flight. These should be weighed against the aircraft's mission requirements. An all-weather interceptor, for example, should be capable of fast, uncomplicated starts to meet its alert and scramble requirements. Starts for other types may not be so critical; however, no starting procedure should be unnecessarily complex or confusing. Evaluation of the start should include: complexity of start, time to prepare for start, time to start, external power and ground support equipment required, ground personnel required, and time from start to taxi. The system checks and normal procedure requirements from start to taxi should also be evaluated.

An evaluation of the ground handling characteristics can be made while taxiing. How much power is required to start moving and to taxi at the desired speed? Is braking action required to prevent taxiing too fast? Is the visibility adequate? Is the directional control satisfactory? Is the braking action satisfactory? What is the turning radius of the aircraft? Does the aircraft require any auxiliary equipment such as removable wheels, escape ladders, etc? Is there any problem with clearing obstacles with any part of the aircraft?

The takeoff distance may be difficult to determine without assistance from outside personnel, but an estimate should be made using whatever aid is available such as runway distance markers. Use the recommended takeoff procedure, don't try to make a maximum performance takeoff. The normal ground roll will be of more interest than the minimum possible. Some of the other points to note in the takeoff include: ability of brakes to hold in military power, directional control during ground roll, rudder effective speed, nose lift-off speed, visibility after nose up and during initial acceleration and climb, force required to raise nose, any over-controlling tendencies, airborne speed, adequacy of recommended takeoff trim settings, time to retract gear and flaps, trim changes with retraction of gear and flaps, any tendency to exceed gear or flap speed limitations, effectiveness of trimming action during acceleration, and any distracting noises or vibrations.

The in-flight techniques differ very little from the techniques used in flying quantitative tests. However, it generally is not necessary to be as precise in holding airspeeds and altitudes. To do so would only waste time because differences caused by variations of a few hundred feet in altitude or a few knots in airspeed will not be qualitatively discernible so far as qualitative information is concerned. This is not an indorsement for being lax in flying the aircraft. Just don't waste time with preciseness that will not contribute to the evaluation of the aircraft. If speeds are critical, such as in the climb or in the pattern, then hold them as closely as possible. Otherwise, use good judgment in determining how close to an aim condition it is necessary to be and fly accordingly.

If the climb rate of the aircraft is relatively slow, it may be possible to get some stability information in the climb, i.e., stick pulses, sideslips, etc. Most present day fighter aircraft climb so rapidly that this may not be practical. If so, just record climb performance information (time, fuel, and indicated speed) at intervals of approximately 5,000 feet. Start the time at brake release. Intercept the climb schedule at a comfortable altitude and attempt to fly the recommended schedule precisely. Continue the climb only as far as is compatible with the objective of the flight. Unless climb performance is of primary importance, this will probably be to the altitude selected for the first series of investigations. General aircraft characteristics should be observed during the climb. How difficult is it to maintain the recommended climb schedule? Are the control responses smooth?; too fast?; too slow?; compatible? Is visibility adequate? Is there any buffet?; vibration or excessive noise? Are the ventilation and pressurization systems satisfactory? Are the normal procedures required complicated or excessively distracting? If dampers or other artificial stability devices are provided, check the applicable characteristics with them "ON" and "OFF".

The altitude selected for the first series of stability investigations may be at the tropopause since this is where the aircraft will probably have its best performance. However, if the designed operating altitude is considerably above this level it may be advisable to select an altitude at or near the aircraft's operating altitude. The stability maneuvers performed will be essentially the same at all the altitudes and airspeeds selected. These should be sufficiently spaced to assure descernible qualitative differences in the aircraft's characteristics.

The stability characteristics investigated should include: longitudinal and directional static stability, longitudinal and directional dynamic stability, aileron rolls, and maneuvering flight at several different airspeeds and altitudes. An investigation of the transonic trim changes also should be made. All the dynamic characteristics should be checked with the stability augmentation devices, if any, both "ON" and "OFF". With proper planning these investigations can be made in a minimum amount of time. The longitudinal static stability can be checked while accelerating to V_{max}, for instance. Once at V_{max}, the aircraft can be trimmed for approximately hands-off flight and the static directional stability checked by entering a steady sideslip out to maximum rudder deflection (if the aircraft is cleared to that limit). The periods of the dynamic modes can be timed using a stop watch or counting the seconds. Estimate the cycles to damp completely or to one-half amplitude as the case may be for all the modes.

Approach the aileron rolls cautiously. Make several partial deflection rolls before making any full deflection rolls. The time to reach 90 degrees of roll and the time to roll 360 degrees can be estimated using a stopwatch or again by counting the seconds. It is advisable to make rapid reversals of ailerons and other rolling maneuvers if these can be expected in operational use of the aircraft. The rolling characteristics should also be checked in accelerated flight as well as 1 g flight.

After completion of investigations at V_{max}, a windup turn to limit load factor can be made to check the maneuvering stability of the aircraft. Then zoom back to the original altitude and repeat these investigations at the second airspeed. The other altitudes and airspeeds can be checked in the same manner. Any differences resulting from the altitude or speed changes should be noted.

Stalls should be approached with caution if the aircraft is cleared for such a maneuver, and investigated in all configurations and types of entry. Determine the approximate stall warning margin, what defines the warning and the stall, and the aircraft characteristics in the stall and the recovery. If possible, determine the best method of breaking the stall and altitude loss in recovery from several points in the stall.

If possible, check the tactical mission capability of the aircraft. Simulated dive bombing runs or labs maneuvers could be made for a strategic fighter; for example. All the information obtainable will be helpful in writing an accurate and comprehensive report.

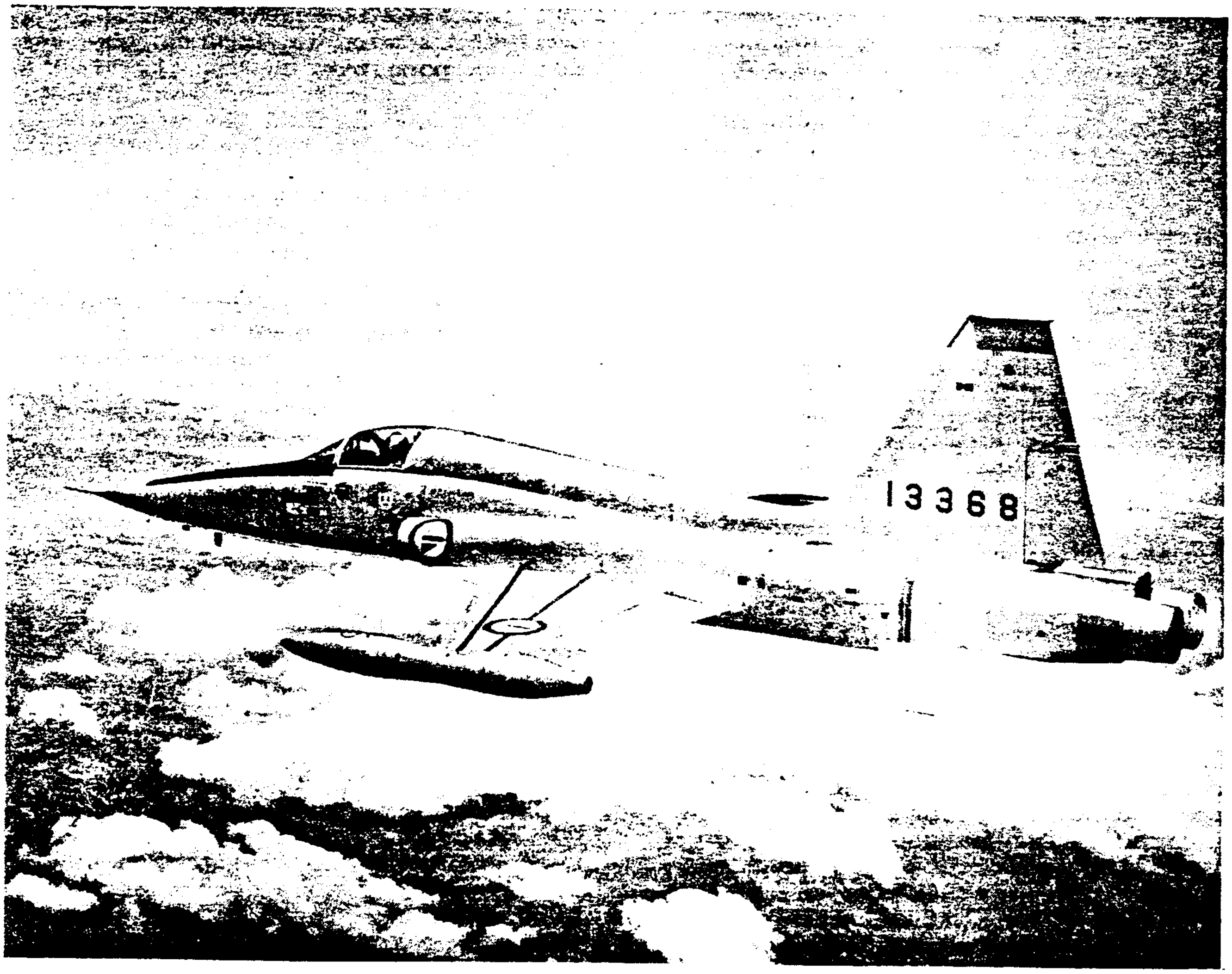
Fly the traffic pattern as recommended and, if fuel permits, make a go-around on the first pass. Note the power response, power required in the pattern, airspeed control and sink rate, trim changes with gear and flap extension, trimming action, buffet with gear extension, and general aircraft feel in the pattern. On the go-around, recheck the trim changes with gear and flap retraction and with drag device reaction. Don't forget to look at engine out characteristics if time and fuel permit. On the first landing in the aircraft it is probably not advisable to attempt to get the minimum landing roll. Make a normal touchdown and use normal braking action (use the drag chute if provided). Note the touchdown speed, the effects of any crosswind, directional control, nose lowering speed, etc. As with the takeoff, the normal landing roll is of more importance than the minimum possible.

While taxiing back to the parking area, review the flight, re-evaluate the cockpit, and attempt to determine whether the aircraft will perform its design mission and is safe and comfortable to fly. The opinion with everything fresh in mind is probably the most accurate possible. Continue this review of the flight immediately after leaving the aircraft. Put everything remembered about the flight and the impressions of the aircraft down on paper. Do this immediately and before talking to anyone about the airplane or the flight. Waiting or discussing points with other people may alter first hand impressions or cause important aspects of the flight to be forgotten.

10.6 DATA REDUCTION

10.6 The data reduction will consist of writing a comprehensive report on everything learned about the aircraft. A narrative form is normally used for qualitative reports. Comparisons with other aircraft can be used to assist in describing the aircraft. Care should be taken however, to insure that only aircraft familiar to most readers are used for comparison. Otherwise the comparison will mean nothing to them.

Keep in mind the purpose of the qualitative evaluation while writing the report. Mere figures are normally not enough to describe the stability of the aircraft, particularly on a qualitative evaluation since the data obtained are very limited. Analyze the aircraft characteristics in light of its ability to perform its design mission, give opinions of the aircraft's ability to do the job and support these opinions with the facts obtained on the evaluation flights. Comment on anything personally disliked but be objective in condemning any shortcomings. Recommendations for specific changes in the aircraft are to be included in the report. The exact manner in which the aircraft should be fixed should not be specified or recommended. The test pilot's job is to evaluate the existing hardware and state what should be changed. It is then the manufacturer's responsibility to determine how to make the necessary changes.



TYPICAL LARGE AIRCRAFT QUALITATIVE EVALUATION FLIGHT CARDS

I. Location____________ Test Crew: Pilot____________

Date____________ Co-Pilot____________

Aircraft____________ Flight Mechanic____________

Runway Length____________ Observers____________

Weather____________

Runway Temperature____________

Press. Altitude____________

Surface Winds____________

5M____________ Climb Wind____________

10M____________ Freezing Level____________

15M____________

20M____________

25M____________

Weight and Balance: Operating Weight____________

Fuel Weight____________

MAC____________% Gross Weight____________

T.O. Gross Weight____________

Est. Fuel Used____________

MAC____________% Est. Landing Weight____________

Performance

T.O. Distance____________ Refusal Speed____________ T.O. Speed____________

Minimum Control Speed____________ Abort Landing Distance____________

Climb Schedule: 4M________ 6M________ 8M________ 10M________

12M________ 14M________ 16M________ 18M________

20M________ 22M________ 24M________ 26M________

28M________ 30M________

Cruise (Max Range) Ht________ Vi________ Pwr________

Operating Limitations

Gear Down________ Flaps: 10 %________ 50 %________ 100 %________

Landing Light________ Cargo Doors________ Para-defl________

Dive Speeds: 30M________ 25M________ 20M________ 15M________

10M________ 5M________

Load Factor________ Weight________

Engine: Sync RPM________ Slave RPM________ Overspeed________

TIT: T.O.________ (MRP) Normal________ Other________

Airstart Vi________ Torque________ Max.________ Cont.________

Remarks:

Systems Operation:

DC Generators________ Eng. AC Generators________ Eng.________

Booster Hyd________ Eng. Utility Hyd________ Eng.________

Other:

Auxiliary Equipment Operation:

Auto-pilot________________________

Other:

NOTES:

A. Support Equipment
   1. Power Unit
      Type
      Capacity
   2. Other

B. Cargo Compartment
   1. Entrance
   2. Egress
   3. Systems Accessibility
   4. Other

C. Flight Deck
   1. Crew Stations
      a. Pilot
         Seat Adjustment
         Clearance
         Vision
         Rudder
         Pedal Adjustment
         Restrictions
         Other

      b. Copilot

      c. Flight Mechanic

      d. Navigator

   2. Instrument Panel
      a. Flight Instruments
         Grouping
         Readability
         Adequacy

      b. Engine Instruments
         Grouping
         Readability
         Adequacy

      c. Warning Lights
         Placards
         Switches
         Controls

   3. Pedestal
      a. Engine Controls
         System Controls
         Switches
         Guards
         Placards
         Lights
         Feel Identification
         Accessibility
         Confusion Factor
         Arrangement

      b. Remarks

   4. Overhead Panel
      a. Engine Controls
         System Controls
         Switches
         Guards
         Lights
         Placards
         Accessibility
         Feel Identification
         Confusion Factor
         Arrangement

      b. Remarks

   5. Side Panels
      a. Switches
         CSs
         Lights

      b. Remarks

   6. Flight Controls
      a. Rudder
         Break-out Force
         Travel
         Adjustment
         Clearance
         Slop
         Friction

      b. Elevator
         Break-out Force
         Travel
         Slop
         Friction
         Clearance

      c. Control Wheel
         Aileron Break-out Force
         Travel
         Slop
         Friction
         Clearance
         Grip
         Switches

   7. General Comments

5. Vibration
   a. Noise
   b. Air vent deflectors
   c. Ventilation/heating
6. Control Required To Maintain Proper Taxi Speed
7. Remarks

D. Pre-Take-Off (line up at even 1,000 feet and check W/V)
1. Flight Control Check With Boost Operating
   a. b/o force
   b. rate
   c. deflection
   d. slop
   e. friction
2. Flaps Set __________ Trim Set __________
3. Engine Power Check
   a. Acceleration
      Idle __________ to __________ (MRP) ______ Sec.
      Asymmetric __________
      Overshoot __________
   b. Stabilized conditions: OAT __________

Eng	%RPM	Torque	TIT	Throttle Pos
1				
2				
3				
4				

4. Brakes Hold At MIL PWR
5. Fuel reading __________ lbs. W/V __________ __________ kts.

E. Take-Off. (Use flight data on knee board)
1. Start Time Form BRAKE RELEASE TO START CLIMB __________
2. Brake Release Action
3. Directional Control. Rudder Effective __________ kts.
4. Elevator Effective (nose wheel off) __________ kts.
5. Aileron Control __________ kts.
6. T.O. Distance __________ ft. Lift-Off Speed __________ kts.
   Time ______ sec.
7. Control Force __________ Pitch __________ Trim __________
8. Trim-Out - Raise Gear
   Time __________ sec.
   Yaw __________
   Trim __________
9. Trim-Out - Raise Flaps
   Time __________ sec.
   Trim __________
10. Acceleration to MINIMUM CONTROL SPEED
11. Acceleration to Climb Speed (1,000 ft)
12. Visibility and Pitch Angle __________
13. Remarks:

F. Climb (M N________°, 90° to W/V).

1. Visibility________
   Pitch Angle________
2. Record: FUEL at START CLIMB__________

TIME	Hi	Vi	R/C	Ti	%RPM	TORQUE	TPT	Wf
	4M							
	6M							
	8M							
	10M							
	12M							
	14M							
	16M							
	18M							
	20M							
	22M							
	24M							
	26M							
	28M							
	30M							
	32M							

FUEL at LEVEL-OFF______________

3. Check Cabin Pressurization:
   10M______________
   15M______________
   20M______________
   25M______________
   30M______________ Note any fluctuations or surges.
4. Cabin Heat Adequacy
   a. Nesi glass
5. Remarks

G. Cruise

1. V_{max}
   a. Hi________
   b. Vi________
   c. OAT________
   d. Flt. Controls________
   e. RPM________
   f. Torque________
   g. TIT________
   h. Wf________
   i. FUEL________

2. Dynamics (Hi__________ Vi__________) Note Control Position
   a. Phugoid
      1. Trim__________ $V_{i_{in}}$__________ V_{max}__________ V_{min}__________
      2. Sec/cyc________•___ ______Damping__________
   b. Porpoise Mode. input__________cycls__________ampl. ______
   c. Spiral stability
      1. RT Ø 10°__________°/__________sec.
      2. LFT " "__________°/__________sec.
      3. Remarks:
   d. Dutch Roll
      1. RT sideslip s/c______Roll__________Yaw__________
         Damping__________(1)__________(2)__________(3)__________
      2. LFT sideslip s/c______Roll__________Yaw__________
         Damping__________(1)__________(2)__________(3)__________
      3. (1) Norm (2) Damper Off (3) Rudder Power Off.
   e. Short Period
      1. Fixed (1.0g) Damping__________
      2. Fixed (-1.0g) " __________
      3. Free (1.0g) " __________
      4. Free (-1.0g) " __________
      5. Remarks:

3. Maximum Range Data
   a. Hi__________Vi__________OAT__________FUEL__________
   b. RPM__________Torque__________TPT__________Wf__________
   c. Remarks:

4. Systems Check: Hi__________Vi__________
   a. Engine shut-down, No.__________
      1. Time to feather__________Control force__________
      2. Procedure, etc:
   b. Engine restart
      1. Time to Normal power__________Surge__________Trim______
      2. Procedure, etc:
   c. Anti-icing/de-icing system
      1. Full operation effect on engines__________
      2. Nesi glass
         Other
      3. Remarks:
   d. GTU/ATM operation__________
   e. Pressurization/heating__________
   f. Other:

5. Emergency Descent, Hi__________Vi__________(Initial)
   a. Time from cruise to start descent__________
   b. Procedure: G and F__________Clean__________Pressurization______
   c. Time__________from CR to Hi__________at Vi__________
   d. Visibility__________Pitch__________Control__________
   e. Remarks:

6. Static Longitudinal Stability and Performance Hi ____________
   a. Acceleration check Trim at Max Range Vi ____________
      1. Decel to Vi ______ Control force *(Trim setting) ____________
      2. Speed/Pwr Vi ______ Rpm ______ Tq ______ TIT ______ OAT ______
         Speed/Pwr Vi ______ Rpm ______ Tq ______ TIT ______
      3. Acceleration, (RESET TRIM), Time/10 kts (MRP) Initial Vi ______
         10 ______
         20 ______
         30 ______
         40 ______
         50 ______
         60 ______
         70 ______
         80 ______
         V/S ______ ft/min. Control forces/gradient
      4. Remarks: FUEL ____________
   b. Trim Changes: Hi ______ Vi ______
      1. Control boost off ______ on ______
      2. Runaway Trim: Elev ______ Ail ______ Rud ______
         5 sec delay (build-up)
   c. Turning Performance and Aileron Rolls. Cruise. (Build-up).
      FULL DEFLECT
      1. 60° Ø, Time 360° ______ Vmax ______ Hi ______
      2. 45° Lft - 45°Rt (FIX) Time for 90° ______
      3. 45° Rt - 45° Lft (FIX) Time for 91° ______
      4. 60° Ø, Time 360° ______ Vi ______ Hi ______
      5. 45° Lft - 45°Rt (FIX) Time for 90° ______
      6. 45° Rt - 45° Lft (FIX) Time for 90° ______
      7. 60° Ø, Time 360° ______ Vi ______ Hi ______
      8. 45° Lft - 45° Rt ______ (FIX) Time for 90° ______
      9. 45° Rt - 45° Lft (FIX) Time for 90° ______
         POWER APPROACH
      10. 45° Lft - 45° Rt (FIX) Time for 90° ______
      11. 45° Rt - 45° Lft (FIX) Time for 90° ______
   d. Spiral Stability PA Hi ______ Vi ______ Pwr ______
      1. Rt Ø 10° ______ o/ ______ sec. (1/2 - 2).
      2. Lft 10° ______ o/ ______ sec. (1/2 - 2).
   e. Phugoid (Hi CL) ______
   f. Sideslips, TRIM (L) Hi ______ Vi ______
      1. Rt ______ °, Fr ______ Fa ______ Fs ______ dr ______ da ______ de ______
      2. Lft ______ °, Fr ______ Fa ______ Fs ______ dr ______ da ______ de ______
         TRIM (CR) Hi ______ Vi ______
      3. Rt ______ °, Fr ______ Fa ______ Fs ______ dr ______ da ______ de ______
      4. Lft ______ °, Fr ______ Fa ______ Fs ______ dr ______ da ______ de ______
      5. D.E. with rudder (Pick up wing) ______
      6. Remarks: FUEL ______

7. Stalls, Gross Weight ______ Hi Trim ______
   a. CR 1.0g TRIM Vi ______ Vw ______ Vs ______ Hi ______
   b. CR 2.0g TRIM Vi ______ Vw ______ Vs ______ Hi ______
   c. Remarks:
   d. PA 1.0g TRIM Vi ______ Vw ______ Vs ______ Hi ______
   e. PA 1.5g TRIM Vi ______ Vw ______ Vs ______ Hi ______

8. Asymmetric Power - Hi ______________
   a. Climb configuration (MRP, Climb Vi, Trimmed-out)
      NTC ________ Feather ____________ No. 1 Eng. Rudder Free, 2 sec.
      Decel to 1.4 Vsl ______________ kts. Ø and sideslip
      (Cond. permitting check 2 out on one side)
   b. T.O. Configuration at V_{max} Gear and T.O. Flaps (168 kts.)
      Fail 1 and 2 and decelerate holding Ø = ZERO.
      Vi_{min} __________. Check Ø = 5° and SIDESLIP =, ZERO.
   c. AT Min control speed fail 3 and 4, Fr ____________ Fa ________
      Fs __________ TRIM OUT HANDS OFF AT 1.2 Vsl ______________
   d. Remarks:

9. Boost OFF Operation Hi ____________ Vi ______________ Pwr ________
   a. Asymmetric Control 1 and 2 idle, 3 and 4 MRP
   b. Response __________ Fr __________ Fa __________ Fs ____________
   c. Remarks:

10. Descent
   a. CR Configuration Vi ______________ V/S ______________
      1. Visibility ______________ Attitude ______________
      2. Engine operation at idle ______________
      3. Pressurization, systems, etc. ______________
      4. Remarks:

   b. L Configuration Vi ______________ V/S ______________
      1. Visibility ______________ Attitude ______________
      2. Engine operation at idle ______________
      3. Remarks:

11. Trim Changes Trim at Placard Speed, PLF
   a. Flaps to 50 % Vi ______________ Hi ______________ PLF/Trim
   b. Gear DOWN Vi ______________ Hi ______________ PLF/Trim
   c. Flaps to 100 % Vi ______________ Hi ______________ PLF/Trim
   d. Power to IDLE Vi ______________ Hi ______________ Trim
   e. Idle to HRP Vi ______________ Att ______________ Trim
   f. Gear UP Vi ______________ V/S ______________ Trim
   g. Flaps UP Vi ______________ V/S ______________ Trim

12. Asymmetric Power Go-around
   a. ________ Out, Pa Vi ________ Hi ________ Pwr ________
   b. Fr ________ Fa ________ Fe ________ Response and Control
   c. Remarks:

13. General Comments Prior to Completion of Flying.

H. Approach and Landing
   1. Pre-landing check: Operating Weight ____________
      Alt Setting ____________ Fuel Weight ____________
      W/V ____________ Landing GR WT ____________
      Runway ____________ Best Flare Speed ____________
      (Pilot Pwr and Steer) Touchdown speed ____________
      (Copilot Ailerons) Vs_L ____________

   2. Traffic pattern:
      a. Visibility ____________ Control
      b. Power response ________
      c. Remarks:

   3. Landing:
      a. Flare ____________ Response ____________ Control ____________
      b. Float ____________ Characteristics in ground effect ____________
      c. Touchdown ____________ Nose-wheel off ____________ Grd Idle ____________
         Reverse ____________ Brakes ____________ Steering ____________
      d. Directional control with ailerons ______________
      e. Stopping distance ______________

   4. Remarks:

I. Post-flight and Shut-down
   1. Normal procedures. Ease and time to accomplish ______________
   2. Coordination ______________
   3. Fuel ______________
   4. Flight Time ______________
   5. Squawks ______________

J. Re-evaluate Cockpit and A/C in General

Typical fighter aircraft qualitative evaluation flight cards (2 hour flight).

EXTERNAL INSPECTION TOD START________
TOD FINISH________

Remarks:

COCKPIT EVALUATION

1. Ease of Entry ladder________________
Steps________________
2. Location of Instruments and Controls
3. Adjustment of Seat and Controls
4. Comfort
5. Ease of Identification of:
   Switches
   Controls
   Emergency Devises
   Warning Lights
6. Egress - ground and Airborne

BEFORE STARTING CHECKS TOD________________

Remarks

Complexity:

STARTING ENGINES Fuel______ TOD______________

Complexity:

Ground Support:

Equipment______________

Personnel______________

BEFORE TAXI CHECKS TOD______________

Estimated Break-out Force

Longitudinal +______# - ________#

Lateral +______# - ________#

Directional +______# - ________#

Trim rate (Longitudinal) Aft________Sec

Fore________Sec

Flap Extension________sec Retraction__________sec

TAXIING Fuel________ TOD______________

RPM req to move______________

Visibility

Steering N.W.S.

Brakes

Visibility

Power required______rpm, fuel/flow__________pph

Runway temp________°F. P.A.______________ft.

TAKEOFF Fuel________# TOD______________

Do brakes hold in MIL PWR Yes No

Symmetry of brake release

Directional control

Rudder effective speed__________knots

Ease of rotoation

Lift-off speed_____________knots

Estimated T/O distance__________feet

Gear up time______sec Flaps up time______________sec

Trim changes Landing gear + - ________#

Flaps + - ________#

Are placards hard to exceed? Yes No

Visibility during T/O and Initial Climb

Adequacy of T/O trim setting:

Speed stability during acceleration:

CLIMB Fuel________# TOD______________

Control during climb

Longitudinal

Directional

Lateral

Climb Schedule

5000 ft.	.89IMN	550
10000 ft.	.89IMN	510
15000 ft.	.90IMN	470
20000 ft.	.905IMN	430
25000 ft.	.910IMN	390
30000 ft.	.915IMN	360
35000 ft.	.92 IMN	320
39000 ft.	.92 IMN	

LEVEL OFF FUEL __________ # TOD __________

EASE

Attitude Change __________________ o

****

CRUISE 90 % RPM .86IMN (recommended cruise)

Start Fuel __________ # TOD __________

					Linear	
Sideslip:	Cl_{β}	Hvy	Med	Lt	Yes	No
	$C_{n_{\beta}}$	Hvy	Med	Lt	Yes	No

Dutch Roll Period __________ sec

Damping Hvy Med Lt

Cycles to Damp __________

CRUISE cont. 39,000 ft. .86IMN

PIO Tendency Yes No

Short Period Cycles to Damp __________

Period __________________ sec

Do controls have dynamic tendency?

Yes No

Aileron Rolls: t_{90}

	R	L	Adv. Yaw
1/2 deflection	_____ sec	_____ sec	
Full deflect.	_____ sec	_____ sec	

************************DAMPERS OFF************************

					Linear?	
Sideslip:	$C\ell_{\beta}$	Hvy	Med	Lt	Yes	No
	Cn_{β}	Hvy	Med	Lt	Yes	No

Dutch Roll: Period __________ sec

Damping Hvy Med Lt

Cycles to damp __________

PIO Tendency Yes No

Short Period: Cycles to damp __________

Period __________ sec

**********************************DAMPERS ON********************

Finish: Fuel________# TOD________________

Speed brake trim change Hvy Med Lt

Extend Push Pull

Retract Push Pull

MANEUVERING FLIGHT .90 IMN 39-35,000 ft.

Fuel________#

Initial buffet____________g

Heavy buffet____________g n_{max}____________g

Stick force Hvy Med Lt

Linear Yes No

****

ACCELERATION TO 1.20 IMN at 35,000 ft. (trim .90 IMN)

Start: Fuel________# TOD________________

NB Light L________sec R________sec

NB Trim Change______# Push Pull

Stick force gradient__________________

Transonic trim change__________________

Finish fuel____________# TOD________________

****

CRUISE 1.15 TMN 35,000 ft.

Start: Fuel____________# TOD________________

					Linear?	
Sideslips:	C_{ℓ_β}	Hvy	Med	Lt	Yes	No
	C_{n_β}	Hvy	Med	Lt	Yes	No

Dutch Roll: Period____________sec

Damping Hvy Med Lt

Cycles to Damp____________

PIO Tendency Yes No

CRUISE cont 1.15 IMN 35,000 ft.

Short Period: Cycles to Damp______________

Period____________________sec

******************************DAMPERS OFF************************

					Linear	
Sideslip	C_{l_β}	Hvy	Med	Lt	Yes	No
	C_{n_β}	Hvy	Med	Lt	Yes	No

Dutch Roll:

Period______________sec

Damping Hvy Med Lt

Cycles to Damp________________

PIO Tendency Yes No

Short Period: Cycles to Damp_______________

Period__________________sec

*****************************DAMPERS ON*************************

Aileron Rolls	t_{90} R	t_{90} L	Adverse Yaw
1/2 deflection	______sec	______sec	______
Full deflect	______sec	______sec	______

Finish Fuel_____________ TOD_________________

****

SPEED BRAKE TRIM CHANGE 1.15-1-1.10 IMN

Hvy Med Lt

Extend Push Pull

Retract Push Pull

MANEUVERING FLIGHT 1.10 IMN 35-30,000 ft.

Fuel________#

Initial buffet_________g Heavy buffet_________g

n_{max}_________g

Stick force Hvy Med Lt

Linear? Yes No

DECELERATION TO 210 knots 30,000 ft. (Long Stat)

Stick Force gradient ____________________

****

CRUISE 210 knots 30,000 ft.

Start: Fuel __________ # TOD ______________

						Linear?	
Sideslips:	C_{ℓ_β}	Hvy	Med	Lt		Yes	No
	C_{n_β}	Hvy	Med	Lt		Yes	No

Dutch Rolls Period ____________ sec

Damping Hvy Med Lt

Cycles to Damp ______________________

PIO Tendency Yes No

Short period: Cycles to Damp ______________________

Period ____________ sec

¤¤¤¤¤¤¤¤¤¤¤¤¤¤¤¤¤¤¤¤¤¤¤¤¤¤¤¤¤DAMPERS OFF¤¤¤¤¤¤¤¤¤¤¤¤¤¤¤¤¤¤¤¤¤¤¤¤¤¤

						Linear?	
Sideslips:	C_{ℓ_β}	Hvy	Med	Lt		Yes	No
	C_{n_β}	Hvy	Med	Lt		Yes	No

CRUISE 210 knots at 30,000 ft.

Dutch Roll: Period ____________ sec

Damping Hvy Med Lt

Cycles to Damp ______________________

PIO Tendency Yes No

Short Periods: Cycles to Damp ______________________

Period ______________________

Finish: Fuel __________ # TOD ______________

****************************DAMPERS ON**************************

AILERON ROLLS t90 Adverse Yaw

1/2 Deflection R________sec L________sec ____________________

Full deflect R________sec L________sec ____________________

****

MANEUVERING FLIGHT at 210 knots

Fuel____________#

Initial Buffet____________g Heavy Buffet__________g

"max" _________________g

Stick force gradient: Hvy Med Lt

****

STALLS Cruise Configuration 25,000 ft.

Fuel___________#

Cr V_w____________knots V_s____________knots

GLIDE v_w____________knots V_s_____________knots

Remarks

POWER APPROACH CONFIGURATION

Gear extension____________sec

Flap extension____________sec

Asymmetric power at 155 knots

MIL RWR Rudder Force Hvy Med Lt

MAX TWR Rudder Force Hvy Med Lt

Trimability MIL MAX

STALLS: Fuel___________

V_w___________knots V_s ____________knots

Remarks:

Trim at 160 knots

					Linear?	
Sideslip:	C_{n_β} Hvy	Med	Lt		Yes	No
	C_{ℓ_β} Hvy	Med	Lt		Yes	No

Dutch Roll Period ____________ sec

Damping Hvy Med Lt

Cycles to Damp________

PIO Tendency Yes No

Short Period Cycles to Damp________

Period____________ sec

**************************DAMPERS OFF**********************

Dutch Roll Period____________ sec

Damping Damping Hvy Med Lt

Cycles to Damp________

PIO Tendency? Yes No

Short Period: Cycles to Damp________

Period____________ sec

**************************DAMPERS ON**************************

AILERON ROLLS t_{90} Adverse Yaw

1/2 Deflection R_____ sec L_____ sec ___________

Full Deflect R_____ sec L_____ sec ___________

****

ACROBATICS

Loop

Immelman

Barrel Roll

INSTRUMENTS

Holding at 20,000 Ft.		250 knots	90-92 %
Penetration	S/B	270 knots	90 %
Initial	Clean	220 knots	94 %

Low Cone gear, 86 %, flaps, 155 knots

LANDING

Normal traffic pattern 60 % flaps

Single engine go-around closed pattern

Full stop Full flaps

Touchdown speed________knots' marker______________

****

TAXIING Fuel__________# TOD__________________

Engine acceleration Idle to mil___________sec

Turning radius____________feet

Re-evaluate cockpits

ENGINE SHUTDOWN

Check servicing for turn-around

Time__________

Oil____________qts

Hydraulic fluid________qts

LOX_____________liters

Typical aircraft qualitative evaluation for a pilot training mission (1 hour flight).

TOD____________________ beside A/C

START Procedure

F Flow__________ RPM__________ F Flow__________

Before Taxi Check

TOD________________

TAXI

Power to roll____________ Brakes S NS

Nosewheel steering Turn Rad.__________

NWS Off Brake turn____________

Canopy Operation

Visibility

TOD____________________

LINE UP

Brakes Mil Pwr__________

Pump one brake

Engine Acc Time__________

RPM________ EGT__________ FF________

Throttle friction S NS

FUEL L________ R__________

TOD______________

TAKEOFF

Brake release

A/B light

NWS rel at Rudder Eff A/S__________

CONTROL FORCES L M H__________lbs

NW LIFT OFF__________

T.O. ROLL__________ft A/S__________

GEAR UP__________sec. FLAPS UP__________sec

Trim Changes__________

Noises

Press. Sys

Accelereleration Rotation

CLIMB

Schedule .9 to 35M

Control

Trim

Visibility

Dampers

35M Time__________ Fuel L__________R__________

Throttle Mil Level Off

TOD__________

SUPERSONIC

A/B Light time____________

TRIM CHANGES

STABILITY

DAMPERS	PULSE	CYCLE	TIME
ON	Elev		
	Rud		
OFF	Elev		
	Rud		

45' Roll

ONE ENGINE IDLE

Wind Up Turn to g Max.

A/S____________ "g"____________

Stick force gradient

Buffet

FUEL L____________ R____________

TOD____

TURNING PERFORMANCE 300 Kts________________sec

Zoom to Slow A/C

PWR STALL WARN__________ STALL_____________

230 Kts. Flight Roll

STABILITY

DAMPERS	PULSE	CYCLE	TIME
ON	Elev		
	Rud		
OFF	Elev		
	Rud		
Sideslip	6' Apx.		

CUT ONE ENGINE

EMERGENCY GEAR EXTENSION___________________sec

AIRSTART

170 knots Flaps Down

Aileron Power

Cycle gear Flaps up TRIM

FUEL L____________ R_________________________

TOD____

DIVE 450 Kts 12M

CLOVERLEAF

BARREL ROLL

IMMELMAN

Level at 20 M inbound to VOR

200 Kts F FLOW______________

250 Kts F FLOW______________

300 Kts F FLOW_ . ___________

HIGH CONE

240 Kts. Gear Flaps . Dive Brakes

1 g stall

200 Kts.

STABILITY Check

STALL RIGHT TURN 190 Kts

Clean up A/C 275 Kts. turn to ILS

350 Kts. Speed Brakes Decelerate

ISL Gear, Flaps, D/C 170 Kts

TOD____

SINGLE ENGINE GO-AROUND

SINGLE ENGINE TOUCH AND GO

RE-ENTER

PITCH OUT

NO FLAP LANDING

TRIM CHANGES

TAXI

AFTER LANDING CHECK

SHUTDOWN